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Explicit Global Analytic Solutions to Scalar Single-delay Autonomous Initial Function Problems

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Abstract *- This research article devised an optimal computational strategy for the variation of constants formula for systems of autonomous linear delay initial-function problems, with a view to devising global analytic functional forms for the solutions of single-delay autonomous linear delay scalar initial-function problems. In the sequel the article obtained the unique solutions to all associated constant initial function problems, using a sequence of internally established claims, integration by parts and mathematical inductive principle. These results are unprecedented. An illustrative example followed.*

Keywords - *Computational, Initial-Function, Problems, Matrices, Optimal*

I. INTRODUCTION

Transition and solution matrices are key imperatives for the computations of the unique solutions to linear delay systems with specified feasible initial functions [1]. However, there has not been any known result from other authors, on the functional forms of solution matrices for various classes of linear functional differential systems. The usual approach has been the forward continuation scheme, starting from the left-most interval of a fixed length and proceeding to at most three contiguous intervals, with no attempt to address the issue of structure, as dictated by practical exigencies, due to inherent computational intractability.

II. THEORETICAL UNDERPINNING

With a view to filling in the existing gaps for associated scalar problems, [2] considered the class of double – delay scalar differential equations:

$$\dot{x}(t) = ax(t) + bx(t-h) + cx(t-2h),\ t \in \mathbf{R}, \qquad (1.1)$$

where a, b and c are arbitrary real constants.

By exploiting ingenious combinations of summation notations, multinomial distribution, greatest integer functions, change of variables techniques, multiple integrals, as well as the method of steps, the article obtained the following functional form for the solution matrices:

$$Y(t) = \begin{cases} e^{at},\ t \in J_0; \\ e^{at} + \sum_{i=1}^{k} b^i \frac{(t-ih)^i}{i!} e^{a(t-ih)} + \sum_{j=1}^{\left[\left[\frac{k}{2}\right]\right]} \sum_{i=0}^{k-2j} \frac{b^i c^j}{i!\,j!} \left(t-[i+2j]h\right)^{i+j} e^{a(t-[i+2j]h))};\ t \in J_k, k \ge 1 \end{cases} \qquad (1.2)$$

where $J_k = [kh, (k+1)h], k \in \{0,1,\cdots\}, \left[\left[.\right]\right]$ denotes the greatest integer function, and $Y(t)$ denotes a generic solution matrix of the above class of equations for $t \in \mathbf{R}$.

In the sequel, [3] derived the following theorem on the computational structure and disposition of solution matrices of the single-delay linear neutral scalar differential equations:

$$\dot{x}(t) = a_{-1}\dot{x}(t-h) + a_0 x(t) + a_1 x(t-h), \qquad (1.3)$$

on the interval $J_{k-i} = [(k-i)h, (k+1-i)h], k \in \{0,1,\cdots\}, i \in \{0,1,\ 2\}$, where

$Y_{k-i}(t-ih)$ is a solution matrix of

$$\dot{x}(t) = a_{-1}\dot{x}(t-h) + a_0 x(t) + a_1 x(t-h), \qquad (1.4)$$

on the interval $J_{k-i} = [(k-i)h, (k+1-i)h], k \in \{0,1,\cdots\}, i \in \{0,1,\ 2\}$, such that

$$Y(t) = \begin{cases} 1, t = 0 \\ 0, t < 0 \end{cases} \tag{1.5}$$

Theorem on the optimal computational structure of the solution matrices [3]

Let $t \in J_k, i, j \in \{1, 2, \cdots, k\}$ and let $c_{0j} = \frac{1}{j!}, c_{i1} = 1$. Suppose that $a_{-1}(a_{-1}a_0 + a_1) \neq 0$. Then Suppose that

$$Y(t) = e^{a_0 t} + \left(\sum_{i=1}^{k} \frac{(a_{-1}a_0 + a_1)^i}{i!} (t - ih)^i e^{a_0(t-ih)} \right) \mathrm{sgn}(\max\{0, k\})$$
$$+ \sum_{i=1}^{k-1} a_{-1}^i (a_{-1}a_0 + a_1)(t - [i+1]h) e^{a_0(t-[i+1]h)} \mathrm{sgn}(\max\{0, k-1\})$$
$$+ \sum_{i=1}^{k-2} \sum_{j=2}^{k-i} c_{ij} a_{-1}^i (a_{-1}a_0 + a_1)^j (t - [i+j]h)^j e^{a_0(t-[i+j]h)} \mathrm{sgn}(\max\{0, k-2\}) \tag{1.6}$$

for some real positive constants c_{ij} such that $c_{1j} = \frac{1}{(j-1)!}$, $c_{i2} = \frac{1}{2}(i+1)$, $i \in \{2, 3, \cdots k-2\}$

and $c_{ij} - \frac{1}{j} c_{i\,j-1} - c_{i-1\,j} = 0, \forall i \in \{2, 3, \cdots k-2\}, j \in \{2, 3, \cdots k-i\}$.

[4] went further to obtain the ensuing result on the single-delay autonomous control system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t); t \geq 0 \tag{1.7}$$

$$x(t) = \phi(t), t \in [-h, 0], h > 0 \tag{1.8}$$

where A_0, A_1 are $n \times n$ constant matrices with real entries; B is an $n \times m$ constant matrix with real entries. The initial function ϕ is in $C([-h, 0], \mathbf{R}^n)$, the space of continuous functions from $[-h, 0]$ into the real *n*-dimension Euclidean space, $\mathbf{R}^n$ with norm defined by $\|\phi\| = \sup_{t \in [-h,0]} |\phi(t)|$, (the sup norm). The control u is in the space $L_\infty([0, t_1], \mathbf{R}^n)$, the space of essentially bounded measurable functions taking $[0, t_1]$ into $\mathbf{R}^n$ with norm $\|\phi\| = ess\sup_{t \in [0, t_1]} |u(t)|$. Any control $u \in L_\infty([0, t_1], \mathbf{R}^n)$ will be referred to as an admissible control:

III. METHODS

A. Theorem 1: Solution Matrices for autonomous single-delay linear systems (1.1), [4].

Let $Y(t)$ be a generic solution matrix for the uncontrolled part of (1.1) such that

$$Y(t) = \begin{cases} I_n, t = 0 \\ 0,\ t < 0 \end{cases}$$

Let $J_k = [kh, (k+1)h], k = 0, 1, \cdots$. Then

$$Y(t)=\begin{cases} e^{A_0 t}, t\in J_0 \\ e^{A_0 t}+\int_h^t e^{A_0(t-s)}A_1 e^{A_0(s-h)}ds, t\in J_1 \\ e^{A_0 t}+\int_h^t e^{A_0(t-s)}A_1 e^{A_0(s-h)}ds \\ +\left[\sum_{j=2}^{k}\int_{jh}^{t} e^{A_0(t-s_j)}\prod_{i\in\{j,j-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})}\right]A_1 e^{A_0(s_1-h)}\prod_{\lambda=1}^{j}ds_\lambda, t\in J_k, k\geq 2 \end{cases} \tag{1.9}$$

See [4] for proof. This is a solution characterizing matrix function for system (1.1), with $B=0$.

Observe that the above expression for $Y(t)$ is a piece-wise function that is integral-intensive.

The variation of constants formula for the unique solution of system (1.1) with initial function specification (1.2) is

$$x(t,\phi,u)=Y(t)\phi(0)+\int_{-h}^{0}Y(t-s-h)A_1\phi(s)ds+\int_0^t Y(t-\sigma)Bu(\sigma)d\sigma,\ t\geq 0 \tag{1.10}$$

[4].

There is no direct straight-forward application of the variation of constants formula to any problem in one fell swoop- this fact is hardly emphasized by field practitioners.

This article provides a straight-forward application of the variation of constants formula and makes a positive contribution to the body of knowledge by executing the following new tasks:

(i) Redefining $Y(t)$ as $Y_k(t)$ for any $t\in[0,\infty)$ and $k\in\mathbf{Z}^+$..

(ii) Redefining $Y_k(t)$ as an explicit piece-wise function of $t\in[0,\infty)$, for $k\in\mathbf{Z}^+$, with proof.

(iii) Developing the decomposition of the identified solution matrices in the variation of constants formula, into the appropriate intervals of application, thereby computing the variation of constants formula without any ambiguity.

(iv) Devising global integral functional forms for the solutions of single-delay autonomous linear delay scalar initial-function problems.

(v) Deriving the unique solutions to all associated constant initial function problems

(vi) Furnishing the optimal computations of the variation of constants formula for given problem instances.

The above tasks are accomplished through the following sequence of theorems and corollaries.

B. Theorem 2.1: On Redefinition of $Y(t)$ in terms of $Y_k(t)$ for any $t\in[0,\infty)$ and $k\in\mathbf{Z}^+$..

Let J_k be the domain of any *n*-by-*n* matrix function component $f_k(.)$, of the solution matrix $Y(t)$ of the free part of (1.1). Then $t\in J_k \Rightarrow Y(t)=Y_k(t)$ and $Y_k(t)$ can be expressed in the form:

$$Y_k(t)=\begin{cases} e^{A_0 t},\ k=0 \\ e^{A_0 t}+\int_h^t e^{A_0(t-s)}A_1 e^{A_0(s-h)}ds,\ k=1 \\ Y_1(t)+\left[\sum_{j=2}^{k}\int_{jh}^{t} e^{A_0(t-s_j)}\prod_{i\in\{j,j-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})}\right]A_1 e^{A_0(s_1-h)}\prod_{\lambda=1}^{j} ds_\lambda,\ k\ge 2\end{cases} \tag{2.1}$$

The proof is immediate, since $Y_k(t)$ is equivalent to $Y(t),\ \text{for}\ t\in J_k$.

The above expressions define $Y_k(t)$ as an explicit piece-wise function of $t\in[0,\infty),\ \text{for}\ k\in\mathbf{Z}^+$..

Straight-forward representation of $Y(t)$ incorporating (1.3) and devoid of explicit piece-wise formulation is given as follows:

C. Corollary 2.1

Let $k\in\mathbf{Z}=\{\cdots,-2,-1,0,1,\cdots\}$ and $r\in\mathbf{Z}^+$. Then

(i) $t\in J_k \Rightarrow t-rh\in J_{k-r}$

(ii) $Y_k(t)=e^{A_0 t}\operatorname{sgn}(\max\{1-k^2\operatorname{sgn}|t|,0\}+\left[e^{A_0 t}+\int_h^t e^{A_0(t-s)}A_1 e^{A_0(s-h)}ds\right]\operatorname{sgn}(\max\{k,0\}$

$$+\left[\sum_{j=2}^{k}\int_{jh}^{t} e^{A_0(t-s_j)}\prod_{i\in\{j,j-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})}\right]A_1 e^{A_0(s_1-h)}\prod_{\lambda=1}^{j} ds_\lambda\left(\operatorname{sgn}(\max\{k-1,0\}\right) \tag{2.2}$$

<u>Proof of (i)</u>

$t\in J_k\Rightarrow t\in[kh,\ (k+1)h]\Rightarrow t-rh\in -rh+[kh,\ (k+1)h]=[-rh+kh,\ -rh+(k+1)h]$

$\Rightarrow t-rh\in[(k-r)h,\ (k+1-r)h]=J_{k-r}\Rightarrow t-rh\in J_{k-r}$, as claimed.

<u>Proof of (ii)</u>

$k\le -2\Rightarrow \operatorname{sgn}\left(\max\{1-k^2\operatorname{sgn}|t|,0\}\right)=0,\operatorname{sgn}\left(\max\{k,0\}\right)=0,\operatorname{sgn}\left(\max\{k-1,0\}\right)=0$

$\Rightarrow Y_k(t)=Y(t)=0,\ \text{for}\ t\in(-\infty,-h]$

$k=-1\Rightarrow \operatorname{sgn}\left(\max\{1-k^2\operatorname{sgn}|t|,0\}\right)=\begin{cases}0,\ t\in[-h,0)\\ 1,\ t=0\end{cases},\ \operatorname{sgn}\left(\max\{k,0\}\right)=0,\operatorname{sgn}\left(\max\{k-1,0\}\right)=0$

$\Rightarrow Y_{-1}(t)=\begin{cases}0,\ t\in[-h,0)\\ I_n,\ t=0\end{cases}=Y(t),\text{for}\ t\in[-h,\ 0]$. Therefore, $Y_k(t)=Y(t),\ \text{for}\ t\in(-\infty,\ 0]$

$k=0\Rightarrow \operatorname{sgn}\left(\max\{k,0\}\right)=0,\operatorname{sgn}\left(\max\{k-1,0\}\right)=0\Rightarrow Y_0(t)=e^{A_0 t}$, with $Y_0(0)=I_n$. In like manner,

$k=1\Rightarrow Y_1(t)=\left[e^{A_0 t}+\int_h^t e^{A_0(t-s)}A_1 e^{A_0(s-h)}ds\right];k\ge 2\Rightarrow \operatorname{sgn}\left(\max\{1-k^2,0\}\right)=0,\operatorname{sgn}\left(\max\{k,0\}\right)=1,$

$\operatorname{sgn}\left(\max\{k-1,0\}\right)=1,\Rightarrow Y(t)=Y_k(t),t\in J_k,k\ge 2,$ completing the proof of (ii).

D. Theorem 2.2: Appropriate Intervals For $Y(t)$ In The Variation Of Constants Formula

The computable variation of constants formula is given as follows:

$$x_k(t,\phi,u) = Y_k(t)\phi(0) + \int_{-h}^{t-(k+1)h} Y_k(t-s-h)A_1\phi(s)ds + \int_{t-(k+1)h}^{0} Y_{k-1}(t-s-h)A_1\phi(s)ds$$

$$+\int_{t-(i+1)h}^{t-ih} \sum_{i=0}^{k} Y_i(t-\sigma)Bu(\sigma)d\sigma; t \in J_k \qquad (2.3)$$

<u>Proof</u>

The proof will be achieved through rigorous reasoning regarding the application of the method of steps as follows:

$$s \in [-h,0], t \in J_k \Rightarrow t-s-h \in [(k-1)h,\ (k+1)h];$$

$$t-s-h \geq kh \ \text{ iff } s \leq t-(k+1)h \Rightarrow t-s-h \in [-h,\ t-(k+1)h];$$

$$t-s-h \leq kh \ \text{ iff } s \geq t-(k+1)h \Rightarrow t-s-h \in [t-(k+1)h,\ 0].$$

Therefore

$$t-s-h \in J_k \ \text{ iff } \ s \in [-h,\ t-(k+1)h]; t-s-h \in J_{k-1} \ \text{ iff } \ s \in [t-(k+1)h, 0].$$

$$t \in J_k \ \text{ and } \sigma \in [0,t] \Rightarrow \sigma \in [0,(k+1)h] \Rightarrow \sigma \in [ih,(i+1)h], \text{for some } i \in \{0,1,\cdots k\}$$

$$\Rightarrow \sigma \in \bigcup_{i=0}^{k} J_i. \text{ Furthermore, } t-\sigma \in J_i \ \text{ iff } \ \sigma \in [t-(i+1)h,\ t-ih].$$

$t \geq 0 \Rightarrow t \in J_k$, for some nonnegative integer k. Thus, the relations $Y(t) = Y_k(t), x(t) = x_k(t); t \in J_k$ are well defined; thus the proof is accomplished. This efficient expression constitutes the master stroke for the successful prosecution of the variation of constants formula.

The last integral is a formidable expression involving $k+1$ component integrals. There is hardly any easy way out of this inherent intractability.

IV. RESULTS

The main results of this article are as follows:

A. Theorem 2.3: Optimal Representations of Unique Solutions to Scalar Initial-Function Problems

For a given continuous function ϕ with domain $[-h,0]$, the unique solution to the initial-function problem:

$$\dot{x}(t) = ax(t) + bx(t-h) \text{ on } [0,\ \infty)$$

$$x(t) = \phi(t),\ t \in [-h,0] \qquad (2.4)$$

is given by

$$x_0(t) = e^{at}\phi(0) + b\int_{-h}^{t-h} e^{a(t-s-h)}\phi(s)ds;$$

$$x_k(t) \ = \left(e^{at} + \sum_{i=1}^{k} b^i \frac{(t-ih)^i}{i!} e^{a(t-ih)} \right)\phi(0)$$

$$+b\int_{-h}^{0}\left(e^{a(t-s-h)}+\sum_{i=1}^{k-1}b^i\frac{(t-s-[i+1]h)^i}{i!}e^{a(t-s-[i+1]h)}\right)\phi(s)ds$$

$$+\int_{-h}^{t-(k+1)h}b^{k+1}\frac{(t-s-[k+1]h)^k}{k!}e^{a(t-s-[k+1]h)}\phi(s)ds,\ \text{for } k\geq 1 \qquad (2.5)$$

Proof

By an appeal to the computable form of the variation of constants formula,

$$x_k(t)=y_k(t)\phi(0)+b\int_{t-(k+1)h}^{0}y_{k-1}(t-s-h)\phi(s)ds+b\int_{-h}^{t-(k+1)h}y_k(t-s-h)\phi(s)ds,$$

where $y_p(.)=0, \text{for } p<0$

$$\Rightarrow x_k(t)=y_k(t)\phi(0)+b\int_{-h}^{0}y_{k-1}(t-s-h)\phi(s)ds+b\int_{-h}^{t-(k+1)h}z_k(t-s-h)\phi(s)ds$$

$$\Rightarrow x_0(t)=y_0(t)\phi(0)+b\int_{-h}^{t-h}z_0(t-s-h)\phi(s)ds=e^{at}\phi(0)+b\int_{-h}^{t-h}e^{a(t-s-h)}\phi(s)ds$$

$$x_k(t)=\left(e^{at}+\sum_{i=1}^{k}b^i\frac{(t-ih)^i}{i!}e^{a(t-ih)}\right)\phi(0)$$

$$+b\int_{-h}^{0}\left(e^{a(t-s-h)}+\sum_{i=1}^{k-1}b^i\frac{(t-s-[i+1]h)^i}{i!}e^{a(t-s-[i+1]h)}\right)\phi(s)ds$$

$$+b\int_{-h}^{t-(k+1)h}b^k\frac{(t-s-[k+1]h)^k}{k!}e^{a(t-s-[k+1]h)}\phi(s)ds, k\geq 1 \qquad (2.6)$$

The proof is concluded. Note that $\sum_{i=p}^{q}(.)=0, \text{ if } q<p.$

B. Theorem 2.4: Explicit Integral-Free Form Unique Solution To Constant Scalar Initial-Function Problems (CSIFP)

Consider the constant initial function problem (CIFP):

$$\dot{x}(t)=ax(t)+bx(t-h) \text{ on } [0,\infty)$$

$$x(t)=\phi(t)=x_0,\ t\in[-h,0] \qquad (2.7)$$

Let $x_k(t)$ be the unique solution of the CIFP on the interval $J_k=[kh,(k+1)h]$, for $k\in\{0,1,\cdots\}$. Then

$$x_0(t)=\frac{b}{a}\left(e^{at}-1\right)x_0;$$

$$x_k(t)=\left(e^{at}\left[1+\frac{b}{a}-e^{-ah}\right]+\sum_{i=1}^{k}b^i\frac{(t-ih)^i}{i!}e^{a(t-ih)}\right)x_0+\left(\sum_{i=1}^{k-1}I_i+\hat{I}_k\right)x_0, k\in\mathbf{N}, \qquad (2.8)$$

where

$$I_i = \sum_{j=1}^{i+1} I_{ij}; I_{i1} = \frac{-b^{i+1}}{a(i!)}\Big[\big(t-s-[i+1]h\big)^i e^{a(t-s-[i+1]h)}\Big]_{-h}^{0};$$

$$I_{ij} = \frac{(-1)^j b^{i+1}}{a^j (i!)}\left[\prod_{\lambda=i+2-j}^{i} \lambda\big(t-s-[i+1]h\big)^{i+1-j} e^{a(t-s-[i+1]h)}\right]_{-h}^{0}, \text{for } i \in \{1,\cdots,k-1\}, j \in \{2,\cdots,i+1\},$$

$$\hat{I}_k = \sum_{j=1}^{k+1} \hat{I}_{kj}, \text{ for } k \geq 1;$$

$$\hat{I}_{kj} = \frac{(-1)^j b^{k+1}}{a^j (k!)}\left[\prod_{\lambda=k+2-j}^{k} \lambda\big(t-s-[k+1]h\big)^{k+1-j} e^{a(t-s-[k+1]h)}\right]_{-h}^{t-(k+1)h}, \text{ for } j \in \{2,\cdots,k+1\}. \quad (2.9)$$

Proof

By a direct appeal to theorem 2.2, the unique integral-form solution to the constant initial-function problem (CIFP) is given by

$$x_0(t) = \frac{b}{a}\left(e^{at}-1\right)x_0;$$

$$x_k(t) = \left(e^{at} + \sum_{i=1}^{k} b^i \frac{(t-ih)^i}{i!} e^{a(t-ih)}\right)x_0$$

$$+ bx_0 \int_{-h}^{0}\left(e^{a(t-s-h)} + \sum_{i=1}^{k-1} b^i \frac{(t-s-[i+1]h)^i}{i!} e^{a(t-s-[i+1]h)}\right)ds$$

$$+ x_0 \int_{-h}^{t-(k+1)h} b^{k+1} \frac{(t-s-[k+1]h)^k}{k!} e^{a(t-s-[k+1]h)} ds, \text{ for } k \geq 1 \quad (2.10)$$

$$\Rightarrow x_0(t) = \frac{b}{a}\left(e^{at}-1\right)x_0;$$

$$x_k(t) = \left(e^{at}\left[1+\frac{b}{a}-e^{-ah}\right] + \sum_{i=1}^{k} b^i \frac{(t-ih)^i}{i!} e^{a(t-ih)}\right)x_0$$

$$+ x_0 \int_{-h}^{0}\left(\sum_{i=1}^{k-1} b^{i+1} \frac{(t-s-[i+1]h)^i}{i!} e^{a(t-s-[i+1]h)}\right)ds$$

$$+ x_0 \int_{-h}^{t-(k+1)h} b^{k+1} \frac{(t-s-[k+1]h)^k}{k!} e^{a(t-s-[k+1]h)} ds, \text{ for } k \geq 1 \quad (2.11)$$

Let us examine the integral expression $\int_{-h}^{t-(k+1)h} \frac{(t-s-[k+1]h)^k}{k!} e^{a(t-s-[k+1]h)} ds$: (2.12)

$$k=1\Rightarrow\int_{-h}^{t-2h} b^2(t-s-2h)e^{a(t-s-2h)}ds=b^2\left[\left(-\frac{1}{a}(t-s-2h)+\frac{1}{a^2}\right)e^{a(t-s-2h)}\right]_{-h}^{t-2h}$$
$$=b^2\left[\frac{1}{a^2}+\left(\frac{1}{a}(t-h)-\frac{1}{a^2}\right)e^{a(t-h)}\right]$$

$k=2\Rightarrow i=1$

$$\Rightarrow\int_{-h}^{0} b^2(t-s-2h)e^{a(t-s-2h)}ds=b^2\left[\left(-\frac{1}{a}(t-s-2h)+\frac{1}{a^2}\right)e^{a(t-s-2h)}\right]_{-h}^{0}$$

Next, we examine the integral $\int_{-h}^{0}\frac{(t-s-[i+1]h)^i}{i!}e^{a(t-s-[i+1]h)}ds$ (2.13)

Indeed

$$\int_{-h}^{0}(t-s-[i+1]h)^i e^{a(t-s-[i+1]h)}ds$$
$$=\left[\left(-\frac{1}{a}\left(t-s-[i+1]h\right)^i+\frac{1}{a^2}i\left(t-s-[i+1]h\right)^{i-1}\right)e^{a(t-s-[i+1]h)}\right]_{-h}^{0}$$
$$-\frac{1}{a^3}i(i-1)\left[\left(t-s-[i+1]h\right)^{i-2}e^{a(t-s-[i+1]h)}\right]_{-h}^{0}$$
$$-\frac{1}{a^3}i(i-1)(i-2)\int_{-h}^{0}\left(t-s-[i+1]h\right)^{i-3}e^{a(t-s-[i+1]h)}ds \tag{2.14}$$

Clearly $i=2$

$$\Rightarrow\int_{-h}^{0}(t-s-3h)^2e^{a(t-s-3h)}ds$$
$$=\left[\left(-\frac{1}{a}\left(t-s-3h\right)^2+\frac{2}{a^2}\left(t-s-3h\right)\right)e^{a(t-s-3h}\right]_{-h}^{0}-\frac{2}{a^3}\left[e^{a(t-s-3h)}\right]_{-h}^{0} \tag{2.15}$$

Define

$$I_{i1}=\frac{-b^{i+1}}{a(i!)}\left[\left(t-s-[i+1]h\right)^i e^{a(t-s-[i+1]h)}\right]_{-h}^{0}$$
$$I_{ij}=\frac{(-1)^j b^{i+1}}{a^j(i!)}\left[\prod_{\lambda=i+2-j}^{i}\lambda\left(t-s-[i+1]h\right)^{i+1-j}e^{a(t-s-[i+1]h)}\right]_{-h}^{0}, \tag{2.16}$$

for $i\in\{1,\cdots,k-1\}$, $j\in\{2,\cdots,i+1\}$

Then

$$I_{11}+I_{12}=\frac{-b^2}{a}\left[\left(t-s-2h\right)e^{a(t-s-2h)}\right]_{-h}^{0}+\frac{b^2}{a^2}\left[e^{a(t-s-2h)}\right]_{-h}^{0}$$
$$=b^2\left[\left(\frac{-1}{a}\left(t-s-2h\right)+\frac{1}{a^2}\right)e^{a(t-s-2h)}\right]_{-h}^{0}=b^2\int_{-h}^{0}\left(t-s-2h\right)e^{a(t-s-2h)}ds \tag{2.17}$$

Claim 1:

Set

$$I_i = \int_{-h}^{0} b^{i+1} \frac{(t-s-[i+1]h)^i}{i!} e^{a(t-s-[i+1]h)} ds$$

Then

$$I_i = \sum_{j=1}^{i+1} I_{ij}.$$

Hence for $k \geq 2$,

$$\int_{-h}^{0} \left(\sum_{i=1}^{k-1} b^{i+1} \frac{(t-s-[i+1]h)^i}{i!} e^{a(t-s-[i+1]h)} \right) ds = \sum_{i=1}^{k-1} I_i$$

Proof of claim 1:

$$i=1 \Rightarrow I_1 = b^2 \int_{-h}^{0} (t-s-2h) e^{a(t-s-2h)} ds = b^2 \left[\left(\frac{-1}{a}(t-s-2h) + \frac{1}{a^2} \right) e^{a(t-s-2h)} \right]_{-h}^{0} = I_{11} + I_{12}$$

Therefore the claim is valid for $i=1$.

It was earlier proved that

$$I_2 = \frac{b^3}{2!} \int_{-h}^{0} (t-s-3h)^2 e^{a(t-s-3h)} ds$$

$$= \frac{b^3}{2!} \left[\left(-\frac{1}{a}(t-s-3h)^2 + \frac{2}{a^2}(t-s-3h) \right) e^{a(t-s-3h} \right]_{-h}^{0} - \frac{2}{a^3} \frac{b^3}{2!} \left[e^{a(t-s-3h)} \right]_{-h}^{0} = \sum_{j=1}^{3} I_{2j}$$

Therefore the claim is also valid for $i=2$. The rest of the proof is by the principle of mathematical induction: Assume that the claim is valid for $3 \leq i \leq p$, for some integer p. Then

$$I_p = \int_{-h}^{0} b^{p+1} \frac{(t-s-[p+1]h)^p}{p!} e^{a(t-s-[p+1]h)} ds = \sum_{j=1}^{p+1} I_{pj}, \text{ by the induction hypothesis.}$$

$$I_{p+1} = \int_{-h}^{0} b^{p+2} \frac{(t-s-[p+2]h)^{p+1}}{(p+1)!} e^{a(t-s-[p+2]h)} ds$$

$$= -\frac{b^{(p+1)+1}}{a(p+1)!} \left[\left((t-s-[(p+1)+1]h)^{p+1} \right) e^{a(t-s-[p+2]h)} \right]_{-h}^{0}$$

$$- \frac{b^{p+2}}{a(p+1)!} \int_{-h}^{0} (p+1)(t-s-[p+2]h)^p e^{a(t-s-[p+2]h)} ds$$

$$= -\frac{b^{(p+1)+1}}{a(p+1)!} \left[\left((t-s-[(p+1)+1]h)^{p+1} \right) e^{a(t-s-[p+2]h)} \right]_{-h}^{0}$$

$$+ \frac{b^{p+2}}{a^2 p!} \left[\left((t-s-[(p+1)+1]h)^{p} \right) e^{a(t-s-[p+2]h)} \right]_{-h}^{0} + \frac{b^{p+2}}{a^2 p!} \int_{-h}^{0} p(t-s-[p+2]h)^{p-1} e^{a(t-s-[p+2]h)} ds$$

$$= I_{(p+1),1} + I_{(p+1),2} + \frac{b^{p+2}}{a^2 (p-1)!} \int_{-h}^{0} (t-s-[p+2]h)^{p-1} e^{a(t-s-[p+2]h)} ds$$

Claim 2:

$$\frac{b^{p+2}}{a^2(p-1)!}\int_{-h}^{0}(t-s-[p+2]h)^{p-1}e^{a(t-s-[p+2]h)}ds=\sum_{j=3}^{p+2}I_{(p+1),j}$$

To establish this claim, observe that

$$\frac{b^{p+2}}{a^2(p-1)!}\int_{-h}^{0}(t-s-[p+2]h)^{p-1}e^{a(t-s-[p+2]h)}ds$$
$$=-\frac{b^{p+2}}{a^3(p-1)!}\Big[(t-s-[p+2]h)^{p-1}e^{a(t-s-[p+2]h)}\Big]_{-h}^{0}$$

$$-\frac{b^{p+2}}{a^2(p-1)!}\left[\frac{1}{a}\int_{-h}^{0}(p-1)(t-s-[p+2]h)^{p-2}e^{a(t-s-[p+2]h)}ds\right]$$
$$=I_{(p+1),3}-\frac{b^{p+2}}{a^3(p-2)!}\left[\int_{-h}^{0}(t-s-[p+2]h)^{p-2}e^{a(t-s-[p+2]h)}ds\right]$$
$$=I_{(p+1),3}+\frac{b^{p+2}}{a^4(p-2)!}\Big[(t-s-[p+2]h)^{p-2}e^{a(t-s-[p+2]h)}\Big]$$
$$+\frac{b^{p+2}}{a^4(p-3)!}\left[\int_{-h}^{0}(t-s-[p+2]h)^{p-3}e^{a(t-s-[p+2]h)}ds\right]$$
$$=I_{(p+1),3}+I_{(p+1),4}+\frac{b^{p+2}}{a^4(p-3)!}\left[\int_{-h}^{0}(t-s-[p+2]h)^{p-3}e^{a(t-s-[p+2]h)}ds\right]$$

The process continues until $j=p+1$, yielding

$$\frac{b^{p+2}}{a^2(p-1)!}\int_{-h}^{0}(t-s-[p+2]h)^{p-1}e^{a(t-s-[p+2]h)}ds$$
$$=\sum_{j=3}^{p+1}I_{(p+1),j}+(-1)^{p+1}\frac{b^{p+2}}{a^{p+1}(p-p)!}\left[\int_{-h}^{0}(t-s-[p+2]h)^{p-p}e^{a(t-s-[p+2]h)}ds\right]$$
$$=\sum_{j=3}^{p+1}I_{(p+1),j}+(-1)^{p+2}\frac{b^{p+2}}{a^{p+2}}\Big[e^{a(t-s-[p+2]h)}\Big]_{-h}^{0}=\sum_{j=3}^{p+1}I_{(p+1),j}+I_{(p+1),(p+2)}=\sum_{j=3}^{p+2}I_{(p+1),j},$$

establishing claim 2. Therefore,

$$I_i=\int_{-h}^{0}b^{i+1}\frac{(t-s-[i+1]h)^i}{i!}e^{a(t-s-[i+1]h)}ds=\sum_{j=1}^{i+1}I_{ij}.$$

Hence for $k\geq 2$,

$$\int_{-h}^{0}\left(\sum_{i=1}^{k-1}b^{i+1}\frac{(t-s-[i+1]h)^i}{i!}e^{a(t-s-[i+1]h)}\right)ds=\sum_{i=1}^{k-1}I_i$$

$$\Rightarrow \hat{I}_k = \int_{-h}^{t-(k+1)h} b^{k+1} \frac{(t-s-[k+1]h)^k}{k!} e^{a(t-s-[k+1]h)} ds = \sum_{j=1}^{k+1} \hat{I}_{kj}, \text{ for } k \geq 1,$$

where

$$\hat{I}_{k1} = \frac{-b^{k+1}}{a(k!)} \left[\left(t-s-[k+1]h\right)^k e^{a(t-s-[k+1]h)} \right]_{-h}^{t-(k+1)h}$$

$$\hat{I}_{kj} = \frac{(-1)^j b^{k+1}}{a^j (k!)} \left[\prod_{\lambda=k+2-j}^{k} \lambda \left(t-s-[k+1]h\right)^{k+1-j} e^{a(t-s-[k+1]h)} \right]_{-h}^{t-(k+1)h}. \quad (2.18)$$

for $j \in \{2, \cdots, k+1\}$.

The proof is accomplished.

C. An Illustrative Problem 1

Solve above constant initial-function problem on $[0,3h]$

Solution

Let the solution formulation notations be preserved. Then $a \neq 0 \Rightarrow$

$$x_0(t) = \frac{b}{a}\left(e^{at}-1\right)x_0;$$

$$x_1(t) = \left(e^{at}\left[1+\frac{b}{a}-e^{-ah}\right] + b(t-h)e^{a(t-h)} \right) x_0 + \hat{I}_1 x_0,$$

where

$$\hat{I}_1 = \hat{I}_{11} + \hat{I}_{12},$$

$$\Rightarrow x_1(t) = \left(e^{at}\left[1+\frac{b}{a}-e^{-ah}\right] + b(t-h)e^{a(t-h)} \right) x_0 + b^2 x_0 \left[\left(\frac{-1}{a}(t-s-2h) + \frac{1}{a^2} \right) e^{a(t-s-2h)} \right]_{-h}^{t-2h}$$

$$\Rightarrow x_1(t) = \left(e^{at}\left[1+\frac{b}{a}-e^{-ah}\right] + b(t-h)e^{a(t-h)} \right) x_0 + b^2 \left(\frac{1}{a^2} + \left[\frac{1}{a}(t-h) - \frac{1}{a^2} \right] e^{a(t-h)} \right) x_0$$

$$\Rightarrow x_1(t) = \left(e^{at}\left[1+\frac{b}{a}-e^{-ah}\right] + b(t-h)e^{a(t-h)} \right) x_0 + b^2 \left(\frac{1}{a^2} + \left[\frac{1}{a}(t-h) - \frac{1}{a^2} \right] e^{a(t-h)} \right) x_0$$

$$\Rightarrow x_1(t) = \left(\left(1+\frac{b}{a}\right)e^{at} + \left[\frac{b^2}{a^2} + b(t-h) - 1 + \frac{b^2}{a}(t-h) - \frac{b^2}{a^2} \right] e^{a(t-h)} \right) x_0$$

$$x_2(t) = \left(e^{at}\left[1+\frac{b}{a}-e^{-ah}\right] + \sum_{i=1}^{2} b^i \frac{(t-ih)^i}{i!} e^{a(t-ih)} \right) x_0 + \left(I_1 + \hat{I}_2 \right) x_0$$

$$= \left(e^{at}\left[1+\frac{b}{a}-e^{-ah}\right] + \sum_{i=1}^{2} b^i \frac{(t-ih)^i}{i!} e^{a(t-ih)} \right) x_0$$

$$+ \left(b^2 \left[\left(\frac{-1}{a}(t-s-2h) + \frac{1}{a^2} \right) e^{a(t-s-2h)} \right]_{-h}^{0} \right) x_0 + \left(\sum_{j=1}^{3} \hat{I}_{kj} \right) x_0$$

$$= \left(e^{at} \left[1 + \frac{b}{a} - e^{-ah} \right] + \sum_{i=1}^{2} b^i \frac{(t-ih)^i}{i!} e^{a(t-ih)} \right) x_0 + \left(b^2 \left[\left(\frac{-1}{a} (t-s-2h) + \frac{1}{a^2} \right) e^{a(t-s-2h)} \right]_{-h}^{0} \right) x_0$$

$$\frac{-b^3}{2a} \left(\left[1 - \frac{1}{a} + \frac{1}{a^2} \right] \left[(t-s-3h)^2 e^{a(t-s-3h)} \right]_{-h}^{t-3h} \right) x_0$$

$$= \left(e^{at} \left[1 + \frac{b}{a} - e^{-ah} \right] + \sum_{i=1}^{2} b^i \frac{(t-ih)^i}{i!} e^{a(t-ih)} \right) x_0$$

$$+ \left(b^2 \left[\left(\frac{-1}{a} (t-2h) + \frac{1}{a^2} \right) e^{a(t-2h)} + \left(\frac{1}{a} (t-h) - \frac{1}{a^2} \right) e^{a(t-h)} \right] \right) x_0$$

$$+ \frac{b^3}{2a} \left(\left[1 - \frac{1}{a} + \frac{1}{a^2} \right] \left[(t-2h)^2 e^{a(t-2h)} \right] \right) x_0$$

V. CONCLUSION

This article has enhanced the understanding of the intricate subject of solution matrices which constitute the only appropriate vehicle for the computations of variation of constants formula for linear delay systems, through the optimal reformulation of the standard solution matrices and the delineation of the integral boundaries for the determination of the unique solutions to the corresponding initial function problems. The article went further to determine the exact integral-free functional forms for all constant scalar initial-function problems, as well as furnished an illuminating example. The utility of the results is in the effective elimination of the tedium and computational drudgery inherent in the determination of the appropriate intervals for the computations of the solution matrices with multinomial linear arguments, as reflected in the standard variation of constants formula.

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