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Discrete Optimal Control of a Class of Match-Stick Puzzles

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Abstract *- This article formulated and obtained unprecedented analytic solutions to a class of relevant problems in a two-person match-stick games. The proofs, which were accomplished using well-crafted mnemonically efficient notations, set-theoretic notions, the greatest and the least integer functions, established the certainty of victory for the starting player if and only if* $(N-1)$ *is not a multiple* of $(M+1)$, *where* M *and* N *are arbitrary maximum match-stick pick size and match-stick availability, respectively, provided the specified optimal strategy is adopted. The article also proved robustly that the condition* $(N-2)\neq M \pmod{(M+1)}$ *is imperative for first-pick feasibility, based on the optimal policy. Finally the winning strategy was illustrated for some problem instances.*

Keywords - *Availability, Feasibility, Imperative, Optimal, Pick-size, Policy, Recursions.*

I. INTRODUCTION

The match-stick problem is used for the most part, in motivating backward dynamic programming recursions in optimal control/operations research. This puzzle was presented and resolved by Winston [1], for the particular case of $M=3$ and $N=30$ and then given as group problem for the pair $M=4$ and $N=40$. There was no attempt at general formulation and generalization to arbitrary pertinent game input parameter pair (*M, N*). A flurry of literature review undertaken by the author does not reveal any such generalization either.

In an effort to fill the above yawning gap, this paper makes a positive contribution to the body of knowledge by posing the following general match-stick puzzle of practical interest and providing a trail-blazing mathematical solution to the puzzle, based on backward dynamic programming recursive reasoning.

II. RESEARCH QUESTION

Suppose there are N match sticks on a table. I begin by picking up 1, 2, $\cdots, M-1$ or M match sticks; then my opponent must pick up 1, 2, $\cdots, M-1$ or M match sticks. We continue in this fashion until the last match stick is picked up. The player who picks up the last match stick is the loser. Assume that $(N-1)$ is not a multiple of $(M+1)$. Can I be sure of victory? If so, how? Can the assumption that $(N-1)$ is not a multiple of $(M+1)$ be waived ?

III. THEORETICAL UNDERPINNING OF THE WINNING STRATEGY FOR THE PUZZLE

The solution lies in using the working-backward strategy and modulo operation. I realize that the only way for me to win the game is to have only 1 match stick remaining after my last pick. Backtracking one step to the penultimate stage, an optimal strategy is feasible if the number of match sticks remaining just before my opponent's turn is equal to 1 (modulo some

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appropriate positive integer), where upon I can devise an optimal strategy that will force one match to remain just before my opponent's last turn. This backward continuation terminates successfully in stage 1 just before my opponent's first pick, provided my first pick is consistent with the modulo reasoning. The problem then is for me to devise the unique optimal first pick size and the optimal strategy for the rest of the game.

Notations

Let $j \in \{1,2,\cdots,T\}$, for some finite T; let n_j be the number of match sticks remaining before my opponent's j^{th} pick; let m_j be the size of my j^{th} pick and let O_j be the size of my opponent's j^{th} pick. Let $[[[.]]]$ denote the least integer function; thus for real x, $[[[x]]]$ is the least integer greater than or equal to x. Then, my optimal strategy is encapsulated in the following theorem:

IV. THEOREM ON OPTIMAL MATCH-STICK PICK STRATEGY

Under the standing hypotheses of the match-stick puzzle, my optimal pick sizes and the remaining match-stick levels are given as follows:

$$m_1 = (N-1) - p(M+1),\ m_1 \in \{1, 2, \cdots, M\}$$

$$m_j = (M+1) - O_{j-1},\ j \geq 2.$$

$$n_j = (p+1-j)(M+1)+1 \Rightarrow n_j = 1\ (\operatorname{mod}(M+1)),\ j \in \{1, 2, \cdots, p+1\}$$

where

$$p = \left[\left[\left[\frac{N-(M+1)}{M+1}\right]\right]\right],$$

provided that

$$(N-2) \neq M\ (\operatorname{mod}(M+1));\ \text{that is, } N-2 \neq k(M+1)+M,\ \text{for any positive}$$

integer, k; in other words, the quotient $\dfrac{N-2}{(M+1)}$ does not produce the remainder M.

The assumption that $(N-1)$ is not a multiple of $(M+1)$ cannot be waived.

Proof

I am assured of victory if I initiate the game by picking m_1 such that $N - m_1 = p(M+1)+1$, $1 \leq m_1 \leq M$, for some positive integer p, followed by the strategy $m_j = (M+1) - O_{j-1},\ j \geq 2,$ which guarantees that $n_j = 1\ (\operatorname{mod}(M+1)),\ j \geq 1.$ The problem then reduces to the determination of p such that $m_1 = (N-1) - p(M+1)$ and $1 \le m_1 \le M$. Clearly, $1 \le (N-1) - p(M+1) \le M$, if $p \ge \frac{N-(M+1)}{M+1}$ and $p \le \frac{N-2}{M+1}$. Therefore $1 \le m_1 \le M$, if $\left[\left[\left[\frac{N-(M+1)}{M+1}\right]\right]\right] \le p \le \left[\left[\frac{N-2}{M+1}\right]\right]$, for a uniquely defined p.

A. Assertion on Key Greatest and Least Integer Relations

$$\left[\left[\left[\frac{N-(M+1)}{M+1}\right]\right]\right] = \begin{cases} 1 + \left[\left[\frac{N-2}{M+1}\right]\right], & \text{if } \frac{N-2}{M+1} \text{ leaves the remainder } M, \\ \left[\left[\frac{N-2}{M+1}\right]\right], & \text{otherwise} \end{cases}$$

<u>Proof of Assertion 4.1</u>

Case 1:

$\frac{N-2}{M+1}$ is an integer. Then

$$\left[\left[\frac{N-2}{M+1}\right]\right] = \left[\left[\left[\frac{N-2}{M+1}\right]\right]\right] = \frac{N-2}{M+1}$$

and

$$\left[\left[\left[\frac{N-(M+1)}{M+1}\right]\right]\right] = \left[\left[\left[\frac{(N-2)}{M+1} - \frac{(M-1)}{M+1}\right]\right]\right] = \left[\left[\left[\frac{(N-2)}{M+1}\right]\right]\right] = \frac{N-2}{M+1} \left(\text{since } \frac{(M-1)}{M+1} < 1\right)$$

Therefore,

$$\left[\left[\left[\frac{N-(M+1)}{M+1}\right]\right]\right] = \left[\left[\frac{N-2}{M+1}\right]\right],$$

if $\frac{N-2}{M+1}$ is an integer, in which case $m_1 = 1$ is optimal.

Case 2:

$\frac{N-2}{M+1}$ is not an integer. Then $N-2=k(M+1)+r$, for some integer, $r\in\{1,2,\cdots,M\}$.

$$\Rightarrow k=\left[\left[\frac{k(M+1)+r}{M+1}\right]\right]=\left[\left[\frac{N-2}{M+1}\right]\right]$$

and

$$\left[\left[\left[\frac{N-(M+1)}{M+1}\right]\right]\right]=\left[\left[\left[\frac{(N-2)}{M+1}-\frac{(M-1)}{M+1}\right]\right]\right]=\left[\left[\left[k+\frac{r}{M+1}-\frac{(M-1)}{M+1}\right]\right]\right]$$

$$=\left[\left[\left[k+\frac{(r+1)-M}{M+1}\right]\right]\right]=\begin{cases}k, \text{ if } r\le M-1\\ k+1, \text{if } r=M\end{cases}=\begin{cases}\left[\left[\frac{N-2}{M+1}\right]\right], \text{ if } r\le M-1\\ \left[\left[\frac{N-2}{M+1}\right]\right]+1, \text{if } r=M\end{cases},$$

The feasibility condition $\left[\left[\left[\frac{N-(m+1)}{M+1}\right]\right]\right]\le p\le\left[\left[\frac{N-2}{M+1}\right]\right]$ precludes the case $r=M$.

Therefore p is uniquely defined as follows: $p=\left[\left[\left[\frac{N-(M+1)}{M+1}\right]\right]\right]=\left[\left[\frac{N-2}{M+1}\right]\right]$, proving the assertion.

Finally, we prove that the assumption that $(N-1)$ is not a multiple of $(M+1)$ cannot be waived

for the following reason:

If $(N-1)$ is a multiple of $(M+1)$, then $N-1=k(M+1)$, for some positive integer k. Now,

$$m_1=N-1-(M+1)\left[\left[\left[\frac{N-(M+1)}{M+1}\right]\right]\right]$$

$$\Rightarrow m_1=k(M+1)-(M+1)\left[\left[\left[\frac{k(M+1)+1-(M+1)}{M+1}\right]\right]\right]=k(M+1)-(M+1)\left[\left[\left[k-\frac{M}{M+1}\right]\right]\right].$$

Clearly, $\left[\left[\left[k-\frac{M}{M+1}\right]\right]\right]=k$, since $k-1<k-\frac{M}{M+1}<k$. Therefore $m_1=k(M+1)-(M+1)k=0$,

violating the rule of the game that each player must pick at least one match stick in equal turns; it is unbecoming and unacceptable for a player to pass on a turn. It stands to reason that the first player is forced to adopt a suboptimal strategy by initiating the play with picking at least one match stick, thereby losing control of the game. So, $(N-1)$ must not be a multiple of $(M+1)$, for success guarantee for the first player.

B. Remark on Key Modulo Relation

$$n_j = (p+1-j)(M+1)+1$$

is the optimal solution to the equation:

$$n_j = 1 \ (\operatorname{mod}(M+1)),\ j \in \{1,2,\cdots,p+1\}$$

$$\Rightarrow n_j \in \{1,\ (M+1)+1, 2(M+1)+1,\ ,\cdots,\ p(M+1)+1\}.$$

C. Assertion on m_1

(i) $m_1 = 1 \Leftrightarrow \dfrac{N-2}{M+1}$ is an integer and $m_1 = M \Leftrightarrow \dfrac{N-(M+1)}{M+1}$ is an integer.

(ii) Neither $\dfrac{N-2}{M+1}$ nor $\dfrac{N-(M+1)}{M+1}$ is an integer if $(N-2) = M \ \operatorname{mod}\ (M+1)$

Proof of (i)

The first part follows from the fact that $\dfrac{N-2}{M+1}$ is an integer if an only if

$$\frac{N-2}{M+1} = \left[\left[\frac{N-2}{M+1}\right]\right] = \left[\left[\left[\frac{N-(M+1)}{M+1}\right]\right]\right],$$

in which case

$$m_1 = N-1-\left[\left[\left[\frac{N-(M+1)}{M+1}\right]\right]\right](M+1) = N-1-\left(\frac{N-2}{M+1}\right)(M+1) = N-1-(N-2) = 1$$

The second part follows from the fact that $\dfrac{N-(M+1)}{M+1}$ is an integer if an only if

$$\frac{N-(M+1)}{M+1} = \left[\left[\left[\frac{N-(M+1)}{M+1}\right]\right]\right],$$

in which case

$$m_1 = N-1-\left[\left[\left[\frac{N-(M+1)}{M+1}\right]\right]\right](M+1) = N-1-\frac{N-(M+1)}{M+1}(M+1) = N-1-(N-M-1) = M$$

Proof of (ii)

If $(N-2) = M \ \operatorname{mod}\ (M+1)$, then $N-2 = k(M+1)$, for some positive integer k. Therefore

$\dfrac{N-2}{M+1} = k + \dfrac{M}{M+1}$, which is not integral-valued, since $0 < \dfrac{M}{M+1} < 1$. Also

$\dfrac{N-(M+1)}{M+1} = \dfrac{k(M+1)+1}{M+1} = k + \dfrac{1}{M+1}$, which is not an integer, since $0 < \dfrac{1}{M+1} < 1$.

Therefore a sufficient condition for the feasibility of the optimal strategy for player 1 is that either

$\dfrac{N-2}{M+1}$ or $\dfrac{N-(M+1)}{M+1}$ is an integer. This condition is not necessary, as already established.

D. Some Instances of the class of Analytical Puzzles

Solve the match stick puzzle for (i) $N=60,\ M=5$; (ii) $N=42, M=4$; (iii) $N=72, M=6$;

(iv) Is the $(M,N)=(5,61)$ a feasible pair?

Solutions to the Analytical Puzzles

(i) $N=60,\ M=5 \Rightarrow p=\left[\left[\left[\dfrac{60-6}{6}\right]\right]\right]=9 \Rightarrow m_1=59-54=5 \Rightarrow m_j=6-O_{j-1}$, for any

$O_{j-1}\in\{1,\ 2,\cdots,5\},\ j\geq 2$

(ii) $N=42, M=4 \Rightarrow p=\left[\left[\left[\dfrac{42-5}{5}\right]\right]\right]=8 \Rightarrow m_1=41-5(8)=1 \Rightarrow m_j=5-O_{j-1}$, for any

$O_{j-1}\in\{1,\ 2,3,4\},\ j\geq 2$

(iii) $N=72, M=6 \Rightarrow p=\left[\left[\left[\dfrac{72-7}{7}\right]\right]\right]=10 \Rightarrow m_1=71-7(10)=1 \Rightarrow m_j=7-O_{j-1}$, for any

$O_{j-1}\in\{1,\ 2,\cdots,6\},\ j\geq 2$

(iv) For the (N,M) pair $(61,5), m_1=60-6\left[\left[\left[\dfrac{61-6}{6}\right]\right]\right]=60-6(10)=0.$ Therefore, that pair

precludes an optimal first pick for the starting player, due to first-pick infeasibility, forcing

a sub-optimal strategy from the onset of the game.

V. CONCLUSION

This paper undertook a robust trail-blazing treatment of two-person match-stick problems, complete with necessary and sufficient conditions and relevant assertions for certainty of victory or infeasibility of the optimal strategy, for the starting player, based on appropriate relations between match-stick availability and maximum pick level. For elaboration, the work established the conditions for the existence and uniqueness of the optimal strategy for the starting player, on the one hand, and the nonexistence of such strategy, on the other, if certain conditions are not waived. The proofs were accomplished by the deployment of modulo algebra and the interplay of the greatest and least integer functions. In the sequel, the paper illustrated the optimal control strategy for three problem instances and the infeasibility of the optimal strategy for the starting player, in the fourth problem. This work is a generalization of the particular examples of the match-stick problem solved by [1].

REFERENCES

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