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Periodic Solutions of Functional Difference Equations

Shilpee Srivastava Saxena

Assistant Professor, Department of General Sciences(Former), BITS Pilani Dubai Campus, UAE

Abstract - *In this paper, the existence of at least two positive periodic solutions for the following functional difference equation*

$$\Delta y(n) = a(n)g(y(n))y(n) - \lambda b(n)f(n, y(h(n)))$$

has been studied. An application of the equation has been given to study the periodic solutions of model of allee effect.

Keywords - *Allee effect, Fixed point theorem, Functional difference equation, Periodic solution, Positive solution*

I. INTRODUCTION

The difference equation plays a vital role in applied sciences and real world cases where the functions are considered to be discrete. These are significantly used to model discrete dynamic process. From past many years researchers have been studying stability, global attractivity, periodicity, maximal regularity, asymptotic behaviour etc of the functional difference equations, which could further applied to various models in economics, neural networks, ecological and biological systems.

In this paper, the existence of multiple positive periodic solutions has been studied for the first order functional difference equation of the form

$$\Delta y(n) = a(n)g(y(n))y(n) - \lambda b(n)f(n, y(h(n))), \tag{1}$$

where $a(n)$, $b(n)$ and $h(n) \in C(Z, R+)$ are T-periodic positive sequences with $T \geq 1$. λ is a positive parameter. $f \in C(Z \times R_+, R_+)$ is T-periodic with respect to the variable n. Here R_+ denote the set of positive reals, T and $n \in Z(set\ of\ 1\ integers)$ and $\Delta y(n) = y(n+1) - y(n)$. Let $p, q \in Z$ and $[p, q] = \{p, p+1, \ldots, q\}$ for $p < q$, $\prod_{n=p}^{q} y(n)$ is the product of $y(n)$ from $n = p$ to $n = q$.

Eq.(1) is discrete form of the following first order functional differential equation

$$y'(t) = a(t)g(y(t))y(t) - \lambda b(t)f(t, y(h(t))). \tag{2}$$

In literature, sufficient work has been done on Eq.(2), e.g. ([1], [6], [7] [10], [22]). Very few work [21] is presented on discrete form of above equation. A large number of papers ([2], [4], [5], [8], [9], [13] - [17], [19], [20], [23] and references cited therein) are in literature to study the periodic solutions of discrete equations. In ([14], [16], [17], [20], [21], [23]) the authors have studied the existence of periodic solutions of the following or similar functional difference equations

$$\Delta y(n) = a(n)y(n) - \lambda b(n)f(n, y(h(n))) \tag{3}$$

and

$$y(n+1) = a(n)y(n) + \lambda b(n)f(y(n - \tau(n))). \tag{4}$$

Eqs.(3) and (4) are the particular form of Eq.(1), where $\tau(n)$, $a(n)$ and $b(n)$ are T-periodic functions, λ, $a(n)$ and $b(n)$ are nonnegative with $0 < a(n) < 1$ for all $n \in [0, T-1]$. Authors used Upper lower solution method and Krasnoselskii fixed point theorem [3] to prove their results. The outcomes in ([17], [20], [23]) can be contemplated for further study of periodic solutions of (4).

In number of papers, Legett-Williams Theorem ([12], see Theorem 3.3, Theorem 3.5) has been used to study the existence of at least two as well as three periodic solutions for differential or difference equations. Theorem 3.3 can be applied to the unimodel function f, whereas Theorem 3.5 of [12] is applicable to study various ecological models for example logistic equation, Michaelis Menton single growth model, Richard species growth model, model with allee effects, where the nature of the function is different from the unimodel functions.

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This paper is concerned with the sufficient conditions for the existence of two positive periodic solutions of the Eq.(1). Here L.W. Fixed point theorem (Theorem 3.5, [12])) has been used to find the outcomes of the functional difference equations by using the bounds on Green's kernel.

The whole work is divided into five sections. Section 1 is introduction. In Section 2 preliminary results are given. Proof of the result for the existence of periodic solutions of Eq.(1) has been given in Section 3. In Section 4, an example and application to the model with allee effect are given. As a particular case of (1), some results for Eq.(3) has been given in section 5, with application to mathematical model and example.

II. PRELIMINARIES

In this section, initially some preliminary concepts are defined which will be used in Leggett-Williams fixed point theorem. Let K be a cone in Banach space X, where X is the set of all bounded periodic sequences and is defined as

$$\|y\| = \max_{n\in[0,T-1]} |y(n)|. \tag{5}$$

Next, let ψ be a concave nonnegative continuous functional on K, where function $\psi: K \to [0,\infty)$ is continuous and satisfies the inequality

$$\psi(\mu u + (1-\mu)v) \geq \mu\psi(u) + (1-\mu)\psi(v), \qquad u, v \in K, \mu \in [0,1].$$

On cone K, functional ψ is defined as

$$\psi(y) = \min_{n\in[0,T-1]} y(n). \tag{6}$$

Let a, b and c be positive constants such that $K_a = \{y \in K: \|y\| < a\}$, $\overline{K}_a = \{y \in K: \|y\| \leq a\}$, $K(\psi, b, c) = \{y \in K: b \leq \psi(y),\ \|y\| < c\}$.

Theorem 1. *(Leggett-Williams multiple fixed point Theorem,(Theorem 3.5,[12]))*: Let $c_3 > 0$ be a constant. Assume that $A: \overline{K}_{c_3} \to K$ is completely continuous, there exis concave nonnegative functional ψ with $\psi(y) \leq \|y\|, y \in K$ and numbers c_1 and c_2 with $0 < c_1 < c_2 < c_3$ satisfying the following conditions:

(i) $\{y \in K(\psi, c_2, c_3); \psi(y) > c_2\} \neq \phi$ and $\psi(Ay) > c_2$ if $y \in K(\psi, c_2, c_3)$;

(ii) $\|Ay\| < c_1$ if $y \in \overline{K}_{c_1}$

(iii) $\psi(Ay) > \frac{c_2}{c_3}\|Ay\|$ for each $y \in \overline{K}_{c_3}$ with $\|Ay\| > c_3$.

Then A has at least two fixed points $y_1,\ y_2$ in $\overline{K}_{c_3}$. Furthermore, $\|y_1\| \leq c_1 < \|y_2\| < c_3$.

III. MAIN RESULTS

In this section, sufficient conditions have been obtained for the existence of positive periodic solutions of Eq.(1) under some assumptions. Consider functional difference equation

$$\Delta y(n) = a(n)g(y(n))y(n) - \lambda b(n)f(n, y(h(n))), \tag{7}$$

where $a, b, h \in C(Z, (0,\infty))$, are T- periodic functions s.t. $a(n) = a(n+T)$, $b(n) = b(n+T)$ *and* $h(n) = h(n+T), n \in Z$ and T is a positive integer. λ is positive parameter. Assume f and g follow the conditions :

$(A1)$ $f \in C(Z \times R+, R+)$ is a nondecreasing function with respect to y and is T-periodic w.r.t. integer n.

$(A2)$ There exists constants g_1 and g_2 such that $0 < g_1 \leq g(y) \leq g_2 < \infty$ for all $y > 0$.

It is easy to see that Eq.(7) can be rewritten as

$$y(n+1) = y(n)[a(n)g(y(n)) + 1] - \lambda b(n)f(n, y(h(n))) \tag{8}$$

and

$$\Delta(y(n)\prod_{\theta=0}^{n-1}\frac{1}{1+a(\theta)g(y(\theta))}) = -\prod_{\theta=0}^{n-1}\frac{1}{1+a(\theta)g(y(\theta))}\lambda b(n)f(n, y(h(n))). \tag{9}$$

Now summing the above equation from n to $n+T-1$ we obtain

$$y(n) = \lambda \sum_{u=0}^{T-1} G_{g_1,g_2}(n,u)b(u)f(u,y(h(u))), \tag{10}$$

define $G_{g_1,g_2}(n,u)$ as

$$G_{g_1,g_2}(n,u) = \frac{\prod_{\theta=u+1}^{n+T-1}(1+a(\theta)g(y(\theta)))}{\prod_{\theta=0}^{T-1}(1+a(\theta)g(y(\theta)))-1}, \qquad n \le u \le n+T-1,$$

satisfying the property

$$0 < \frac{1}{\sigma_m - 1} \le G_{g_1,g_2}(n,u) \le \frac{\sigma_m}{\sigma_l - 1},$$

where $\sigma_m = \prod_{u=0}^{T-1}(1+g_2\, a(u))$ and $\sigma_l = \prod_{u=0}^{T-1}(1+g_1\, a(u))$.

Now, if $y(n)$ is a T-periodic solution of Eq.(10) then equivalently, $y(n)$ will be T-periodic solution of Eq.(7).
Next, define cone K_a and operator A_λ in Banach space X as

$$K_a = \{y(n); y \in X, y(n) \ge \frac{\sigma_l - 1}{(\sigma_m - 1)\sigma_m}||y||, \ \forall n \in [0,T]\}$$

and

$$(A_\lambda y)(n) = \lambda \sum_{u=n}^{n+T-1} G_{g_1,g_2}(n,u)b(u)f(u,y(h(u)))$$

respectively. By simple calculations, it can be shown that $A(K) \subset K$ and $A : K \to K$ is completely continuous.
Now, we proceed to proof of the result for the existence of periodic solutions of (7).

Theorem 2. Let $(A1)$, $(A2)$ hold. Further, suppose that there are positive constants c_1, c_2 such that

$$\frac{\sigma_m \sum_{n=0}^{T-1} b(n)f(n,c_1)}{(\sigma_l - 1)c_1} < \frac{1}{\lambda} < \frac{\sum_{n=0}^{T-1} b(n)f(n,c_2)}{(\sigma_m - 1)c_2}.$$

Then Eq.(7) has at least two positive T-periodic solutions.

Proof. Let $\phi_0(n) = \phi_0 \in (c_2, c_3)$ and $c_3 = \frac{\sigma_m(\sigma_m - 1)c_2}{\sigma_l - 1}$, Then $\phi_0 \in \{y; y \in K(\psi, c_2, c_3), \psi(y) > c_2\}$. For $y \in K(\psi, c_2, c_3)$, using the above assumptions we have

$$\psi(A_\lambda y) = \min_{n\in[0,T-1]} \lambda \sum_{u=n}^{n+T-1} G_{g_1,g_2}(n,u)b(u)f(u,y(h(u)))$$

$$\ge \frac{1}{\sigma_m - 1}\lambda \sum_{u=0}^{T-1} b(u)f(u,c_2) > c2.$$

Next, for $y \in \overline{K}_{c_1}$, from the hypothesis in theorem, we obtain

$$||A_\lambda y|| = \max_{n\in[0,T-1]} \lambda \sum_{u=n}^{n+T-1} G_{g_1,g_2}(n,u)b(u)f(u,y(h(u)))$$

$$\le \frac{\sigma_m}{\sigma_l - 1}\lambda \sum_{u=0}^{T-1} b(u)f(u,||y||)$$

$$\le \frac{\sigma_m}{\sigma_l - 1}\lambda \sum_{u=0}^{T-1} b(u)f(u,c_1) < c_1.$$

Lastly, if $||A_\lambda y|| > c_3$ and $y \in \overline{K}_{c_3}$, using above two conditions, we have

$$\psi(A_\lambda y) \ge \frac{1}{\sigma_m - 1}\lambda \sum_{u=0}^{T-1} b(u)f(u,y(h(u))) \ge \frac{\sigma_l - 1}{\sigma_m(\sigma_m - 1)}||A_\lambda y||$$

this implies

$$\psi(Ay) \ge \frac{c_2}{c_3}||A_\lambda y||.$$

Since all the conditions of L.W. Theorem are satisfied. Therefore, proof of Theorem 2 shows that Eq.(7) has at least two positive T-periodic solutions.

IV. APPLICATION TO MODEL WITH ALLEE EFFECTS

In Eq.(7), if function $g(y)$ is not bounded, i.e. $g(y) \to \infty$ as $y \to \infty$, then $\exists$ constant $y_0 > 0$ s.t. $g(y) = g_{y_0}(y)$ for all $0 \le y < y_0$ and $g(y) = g_{y_0}(y_0)$ $\forall$ $y \ge y_0$. Let $g_{y_0}(0) \le g(y) \le g_{y_0}(y_0)$ for $0 < ||y|| \le y_0$. We denote $g_1 = g_{y_0}(0)$ and $g_2 = g_{y_0}(y_0)$, then we can write $0 < g_1 \le g(y) \le g_2 < \infty$. Next, consider the discrete model with allee effect

$$\Delta y(n) = a(n)y(n)(y(n) - b(n))(c(n) - y(n)), \tag{11}$$

where $a(n)$, $b(n)$ and $c(n)$ are T-periodic positive functions. An equivalent equation can be written in the following form

$$\Delta y(n) = A(n)y(n)y(n) - B(n)(y^2(n) + b(n)c(n))y(n),$$

where $A(n) = a(n)(b(n) + c(n))$, $g(y(n)) = y(n)$, $\lambda = 1$, $B(n) = a(n)$, $f(n, y) = (y^2(n) + b(n)c(n))y(n)$, which is equivalent to Eq.(7). Now applying hypothesis of Theorem 2, we have the following result:

$$\frac{\sigma_m \sum_{n=0}^{T-1} B(n)f(n, c_1)}{(\sigma_l - 1)c_1} < \frac{1}{\lambda} < \frac{\sum_{n=0}^{T-1} B(n)f(n, c_2)}{(\sigma_m - 1)c_2}; \quad 0 < c_1 < c_2$$

$$\frac{\sigma_m T \max_{n\in[0,T-1]} a(n)[c_1^2 + b(n)c(n)]c_1}{(\sigma_l - 1)c_1} < 1 < \frac{T \min_{n\in[0,T-1]} a(n)[c_2^2 + b(n)c(n)]c_2}{(\delta_m - 1)c_2}$$

$$\frac{\sigma_m T a^*[c_1^2 + b^*c^*]}{(\sigma_l - 1)} < 1 < \frac{T a_*[c_2^2 + b_*c_*]}{(\sigma_m - 1)}$$

$$0 < \frac{[(\sigma_l - 1) - \sigma_m T a^*(c_1^2 + b^*c^*)](\sigma_m - 1)}{T a_*[c_2^2 + b_*c_*](\sigma_l - 1) - \sigma_m T a^*[c_1^2 + b^*c^*](\sigma_m - 1)} < 1.$$

If above condition holds then Eq.(11) will have at least two positive periodic solutions. The result of the paper is different from the results in literature. The results can be applied to various models in real world. Now, we give example to illustrate Theorem 3.1.

Example 1. Consider the equation

$$\Delta y(n) = a(n)g(y(n))y(n) - \lambda b(n)(y^2(n) + b(n)c(n))y(n), \tag{12}$$

where $a(n) = 0.01 + \frac{\cos n\pi/2}{1000}$, then $a^* = 0.011$, $a_* = 0.009$
b(n) = $b(n) = 0.01 + \frac{\sin n\pi/2}{1000}$, then b^* = 0.011, b_* = 0.009
$c(n) = 0.02 + \frac{\cos n\pi/2}{1000}$, then c^* = 0.021, c_* = 0.019
let $\lambda = 1/2$ and $g(y) = 2y + \sin y$, then for $0 < \pi/6 < y < \pi/3, 0 < 1.55 < g(y) < 2.96$.
Now using Theorem 2, we conclude the following result:

Theorem 3. If $A1$, $A2$ hold and there exist positive numbers $c_1 < c_2$ such that

$$\frac{\sigma_m T b^*[c_1^2 + b^*c^*]}{(\sigma_l - 1)} < 2 < \frac{T b_*[c_2^2 + b_*c_*]}{(\sigma_m - 1)},$$

then there exist at least two positive T-periodic solutions of (12). Here $a(n), b(n)$ and $c(n)$ are periodic functions of same period 4. Since $\sigma_m = (\prod_{n=0}^{3}(1 + g_2 a(n)))$ and $\sigma_l = (\prod_{n=0}^{1}(1 + g_1 a(n)))$. By calculation we obtain that $\sigma_m = 1.124$ and $\sigma_l = 1.0634$. Now one can choose constant $c_1 < 1.59$ and $c_2 > 2.62$ so that above condition would satisfy. Hence Eq.(12) has two positive T-periodic solutions with period $T = 4$.

V. PARTICULAR CASE

As a particular case, similar result of (7) is applied to the equation

$$\Delta y(n) = a(n)y(n) - f(n, y(h(n))), \tag{13}$$

where $a,\ h \in C(Z,(0,\infty))$, are T- periodic functions s.t. $a(n) = a(n+T)$, and $h(n) = h(n+T)$, $n \in Z$ and T is a positive integer. Let $f \in C(Z \times R_+, R_+)$ be a nondecreasing function with respect to y and is T-periodic w.r.t. integer n. Following is the equivalent form of (13) in terms of Green's function

$$y(n) = \sum_{u=n}^{n+T-1} G(n,u) f(u, y(h(u))),$$

where

$$G(n,u) = \frac{\prod_{\theta=u+1}^{n+T-1}(1+a(\theta))}{\prod_{\theta=0}^{T-1}(1+a(\theta)) - 1}, \qquad n \le u \le n+T-1$$

and satisfies the property:

$$0 < G_l = \frac{\sigma}{1-\sigma} \le G(n,u) \le \frac{1}{1-\sigma} = G_m, \tag{14}$$

where $\sigma = \frac{1}{\prod_{n=0}^{T-1}(1+a(n))} < 1$.

Operator $A : X \to X$ is defined as

$$(Ay)(n) = \sum_{s=n}^{n+T-1} G(n,u) f(u, y(h(u))). \tag{15}$$

From bounds on Green's kernel

$$\|Ay\| \le G_m \sum_{u=n}^{n+T-1} f(u, y(h(u)))$$

therefore

$$(Ay)(n) \ge G_l \sum_{u=n}^{n+T-1} f(u, y(h(u))) \ge \frac{G_l}{G_m} \|Ay\|.$$

In view of this, define a cone K in Banach space X as $K = \{y \in X; y(n) \ge \sigma\|y\|, y(n) > 0, n \in Z\}$. Now, we have the following theorem:

Theorem 4. Assume that function f is nondecreasing w.r.t. y and there are constants $0 < c_1 < c_2$ such that $\frac{G_m \sum_{n=0}^{T-1} f(n,c_1)}{c_1} < 1 < \frac{G_l \sum_{n=0}^{T-1} f(n,c_2)}{c_2}$ holds. Then (13) has at least two positive T-periodic solutions.

Proof. Setting $c_3 = c_2/\sigma$ and $\phi_0(n) = \phi_0 \in (c_2, c_3)$, for $y \in K(\psi, c_2, c_3)$ from the assumption above, we have

$$\psi(Ay) = \min_{n\in[0,T-1]} \sum_{u=n}^{n+T-1} G(n,u) f(u, y(h(u))) \ge G_l \sum_{u=0}^{T-1} f(u, c_2) > c_2.$$

Next, for $y \in \overline{K}_{c_1}$, we have

$$\|Ay\| = \max_{n\in[0,T-1]} \sum_{u=n}^{n+T-1} G(n,u) f(u, y(h(u))) \le G_m \sum_{u=0}^{T-1} f(u, \|y\|) \le G_m \sum_{u=0}^{T-1} f(u, c_1) < c_1.$$

Now from the above two inequalities, for $y \in \overline{K}_{c_3}$ and $\|Ay\| > c_3$, we obtain

$$\psi(Ay) \ge G_l \sum_{u=0}^{T-1} f(u, y(h(u))) \ge \frac{G_l}{G_m} \|Ay\| > \sigma\|Ay\|$$

and then $\psi(Ay) > \frac{c_2}{c_3}\|Ay\|$. Since all the conditions of L.W. theorem are satisfied. Hence (13) has at least two positive T-periodic solutions.

The next theorem gives another condition in terms of period T.

Theorem 5. Let function f be nondecreasing w.r.t. y and there exist constants $0 < c_1 < c_2$ such that $\frac{G_M T \max_{n\in[0,T-1]} f(n,c_1)}{c_1} < 1 < \frac{G_l T \min_{n\in[0,T-1]} f(n,c_2)}{c_2}$ holds, where G_l and G_m are defined in (14). Then (13) has at least two positive T-periodic solutions. The following example illustrates the result of Theorem 5.

Example 2. Consider the functional difference equation

$$\Delta y(n) = (1.5 + \cos n\pi)y(n) - \frac{1}{4}[y(n) + y^{\alpha}(h(n))], \quad (16)$$

where $T = 2, h \in C(Z, R+)$ with $h(n+2) = h(n)$ for any $n \in [0,1]$ and $\alpha > 1$ is a real number. Here $a(n) = 1.5 + cos\, n\pi$, $f(n,y) = \frac{1}{4}[y(n) + y^{\alpha}(h(n))]$, and $\sigma = (\prod_{n=0}^{1}(1 + a(n)))^{-1} = \frac{4}{21}$. Using Theorem 5, we obtain the following result.

Theorem 6. Let there are positive numbers $0 < c_1 < c_2$ such that $\frac{21[1+c_1^{\alpha-1}]}{17} < 2 < \frac{4(1+c_2^{\alpha-1})}{17}$, then there exist at least two positive T-periodic solutions of Eq.(16). Here for given α in (16) one can choose $c_1 < (\frac{13}{21})^{\frac{1}{\alpha-1}}$ and $c_2 > (7.5)^{\frac{1}{\alpha-1}}$ such that above condition holds then Eq.(16) has two positive T-periodic solutions, where $T = 2$.

Example 3. Consider the generalized logistic model of single species:

$$\Delta y(n) = y(n)a(n) - y(n)[b(n)y(n) + c(n)y(n - \tau(n))], \quad (17)$$

where the functions $a(n)$, $b(n)$, $c(n)$ and $\tau(n)$ are nonnegative continuous and periodic functions.
Here $f(n,y) = y(n)[b(n)y(n) + c(n)y(n - \tau(n))]$.

Now from Theorem 5, we conclude the following result:

Theorem 7. If there exist constants $0 < c_1 < c_2$ such that

$$G_m \max_{n\in[0,T-1]}[b(n) + c(n)]c_1 < \frac{1}{T} < G_l \min_{n\in[0,T-1]}[b(n) + c(n)]c_2,$$

where G_l and G_m are the lower and upper bound of Green's kernal. Then Eq.(17) has at least two positive T-periodic solutions.

Here the outcomes are obtained when $f(n,y)$ is nondecreasing function in y. These results may be applicable to various models in real world.

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