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# CONVERGENCE: NEW POST-WIDDER OPERATORS

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ABSTRACT. The concern of present article is to study approximation properties of generalized form of Post-Widder operators. The modified operators conserve polynomial function $\alpha_s(v) = v^s,\ s \in \mathbb{N}$. we find some error in estimation of these operators with help of different approximation tools like usual, weighted and exponential modulus of continuity. The rate of convergence of these operators is also shown by numerical table and graphically using mathematica.
Keywords: Post-Widder operators, Peetre's K-functional , Exponential modulus of continuity, weighted modulus of continuity.
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## 1. Introduction

May [10] addressed the Post-Widder operators defined in [14] as

$$\mathbf{P}_l(\mathbf{h}; y) = \frac{1}{(l-1)!}\left(\frac{l}{y}\right)^l \int_0^\infty \exp\left(-\frac{l}{y}v\right) v^{(l-l)}\mathbf{h}(v)dv, \tag{1.1}$$

for $\mathbf{h} \in C[0,\infty),\ l \in \mathbb{N}$ and $y \in [0,\infty)$. Rempulska and Sporupka [11] addressed reformed of the operators $\mathbf{P}_l(.;y)$, upholding the polynomial function $\alpha_2 = y^2$, procure better approximation over original. These operators preserve linear functions. Along with this thought, we set up a new modification of Post-Widdder operators, upholding the test function $\alpha_s = y^s, s \in \mathbb{N}$, as

$$\mathfrak{P}_{l,s}(\mathbf{h}; y) = \frac{1}{l!}\left(\frac{((l+1)_s)^{1/s}}{ly}\right)^{(l+1)} \int_0^\infty \exp\left(-\frac{((l+1)_s)^{1/s}}{ly}v\right) v^l \mathbf{h}\left(\frac{v}{l}\right)dv. \tag{1.2}$$

Or, we can redefine the above operators in kernel form as follows

$$\mathfrak{P}_{l,s}(\mathbf{h}; y) = \int_0^\infty \mathrm{G}_{l,s}(y,v)\mathbf{h}\left(\frac{v}{l}\right)dv,$$

Where

$$\mathrm{G}_{l,s}(y,v) = \frac{1}{l!}\left(\frac{((l+1)_s)^{1/s}}{ly}\right)^{(l+1)} \exp\left(-\frac{((l+1)_s)^{1/s}}{ly}v\right)v^l,$$

for bounded $\mathbf{h} \in \mathbb{R}^+,\ l \in \mathbb{N}$ and $y \in \mathbb{R}^+$.
Where $(l+1)_s = (l+1)(l+2)\ldots(l+1+s), (l+1)_0 = 1$ is rising factorial.
It is obvious $\mathfrak{P}_j(1; y) = 1, \mathfrak{P}_j(v; y) = y,$

and

$$\begin{aligned}\mathfrak{P}_{l,s}(v^s;y) &= \frac{1}{l!}\bigg(\frac{((l+1)_s)^{1/s}}{ly}\bigg)^{(l+1)}\int_0^\infty \exp\bigg(-\frac{((l+1)_s)^{1/s}}{ly}v\bigg)v^l\frac{v^s}{l^s}dv.\\ &= \frac{1}{l!}\bigg(\frac{((l+1)_s)^{1/s}}{ly}\bigg)^{(l+1)}\frac{\Gamma(l+s+1)}{l^s\bigg(\frac{((l+1)_s)^{1/s}}{ly}\bigg)^{(l+s+1)}},\\ &= y^s.\end{aligned}$$

For more details reader can prefer [c.f [4],[2],[3], [6], [7],[8], [9] and [12]] and references therein. The work of present article is sort as:

- In section 2, we represent some lemmas for operators defined by 1.2 which will be useful throughout the article.
- In section 3, we represent error of estimation of operators defined by 1.2 via usual modulus of continuity and Peetre's K-functional. After that we show this error of estimation in form of numerical table and some graphical examples.
- In section 4, we introduce exponential modulus of continuity and find error of approximation of operators 1.2 using exponential modulus of continuity.
- In last section, we focus to error in estimation via weighted modulus of continuity.

## 2. Some Result

The underneath lemma represent p-th order moment in light of moment generating function (M.G.F).

**Lemma 1.** *For $s \in \mathbb{N}$, the p-th order moment is specified by following expression*

$$\mathfrak{P}_{l,s}(\alpha_p, y) = \frac{(l+1)_p}{((l+1)_s)^{p/s}}y^p \tag{2.1}$$

*Proof.* Suppose that $\mathbf{h}(v) = \mathbf{e}^{tv}, t \in \mathbb{R}$, then we find

$$\begin{aligned}\mathfrak{P}_{l,s}(\mathbf{e}^{tv}, y) &= \frac{1}{l!}\bigg(\frac{((l+1)_s)^{1/s}}{ly}\bigg)^{(l+1)}\int_0^\infty \exp\bigg(-\frac{((l+1)_s)^{1/s}}{ly}v\bigg)v^l\exp\bigg(\frac{tv}{l}\bigg)dv\\ &= \frac{1}{l!}\bigg(\frac{((l+1)_s)^{1/s}}{ly}\bigg)^{(l+1)}\int_0^\infty \exp\bigg[-\bigg(\frac{((l+1)_s)^{1/s}}{ly}-\frac{t}{l}\bigg)v\bigg]v^l dv\\ &= \bigg(1-\frac{yt}{((l+1)_s)^{1/s}}\bigg)^{-(l+1)}\\ &= \bigg[1+\frac{(l+1)}{1!}\frac{yt}{((l+1)_s)^{1/s}}+\frac{(l+1)(l+2)}{2!}\frac{y^2t^2}{((l+1)_s)^{2/s}}+\frac{(l+1)(l+2)(l+3)}{3!}\frac{y^3t^3}{((l+1)_s)^{3/s}}\\ &\quad+\frac{(l+1)(l+2)(l+3)(l+4)}{4!}\frac{y^4t^4}{((l+1)_s)^{4/s}}+\frac{(l+1)(l+2)(l+3)(l+4)(l+5)}{5!}\\ &\quad\frac{y^5t^5}{((l+1)_s)^{5/s}}+\frac{(l+1)(l+2)(l+3)(l+4)(l+5)(l+6)}{6!}\frac{y^6t^6}{((l+1)_s)^{6/s}}+\dots\bigg]\\ &= \bigg[1+\frac{(l+1)_1}{1!}\frac{yt}{((l+1)_s)^{1/s}}+\frac{(l+1)_2}{2!}\frac{y^2t^2}{((l+1)_s)^{2/s}}+\frac{(l+1)_3}{3!}\frac{y^3t^3}{((l+1)_s)^{3/s}}\\ &\quad+\frac{(l+1)_4}{4!}\frac{y^4t^4}{((l+1)_s)^{4/s}}+\frac{(l+1)_5}{5!}\frac{y^5t^5}{((l+1)_s)^{5/s}}+\frac{(l+1)_6}{6!}\frac{y^6t^6}{((l+1)_s)^{6/s}}+\dots\bigg]\end{aligned} \tag{2.2}$$

The above expression represent M.G.F for operator $\mathfrak{P}_{l,s}(.;y)$ and the coefficient of $\frac{t^p}{p!}$ gives p-th order of moment. Hence the proof of lemma is immediate. □

**Lemma 2.** *For $p \in \mathbf{N}\bigcup\{\mathbf{0}\}, s \in \mathbf{N}$, the central moments $\nu_p^{\mathfrak{P}_{l,s}}(y) = \mathfrak{P}_{l,s}((v-y)^p, y)$ are specified as*

*(i)* $\nu_1^{\mathfrak{P}_{l,s}}(y) = \left(\frac{(l+1)}{((l+1)_s)^{1/s}} - 1\right)y$

*(ii)* $\nu_2^{\mathfrak{P}_{l,s}}(y) = \left(\frac{(l+1)_2}{((l+1)_s)^{2/s}} - \frac{2(l+1)}{((l+1)_s)^{1/s}} + 1\right)y^2$

*(iii)* $\nu_4^{\mathfrak{P}_{l,s}}(y) = \left(\frac{(l+1)_4}{((l+1)_s)^{4/s}} - \frac{4(l+1)_3}{((l+1)_s)^{3/s}} + \frac{6(l+1)_2}{((l+1)_s)^{2/s}} - \frac{4(l+1)}{((l+1)_s)^{1/s}} + 1\right)y^4$

*(iv)* $\nu_6^{\mathfrak{P}_{l,s}}(y) = \left(\frac{(l+1)_6}{((l+1)_s)^{6/s}} - \frac{6(l+1)_5}{((l+1)_s)^{5/s}} + \frac{15(l+1)_4}{((l+1)_s)^{4/s}} - \frac{20(l+1)_3}{((l+1)_s)^{3/s}} + \frac{15(l+1)_2}{((l+1)_s)^{2/s}} - \frac{6(l+1)}{((l+1)_s)^{1/s}} + 1\right)y^6$

*In general, we find* $\nu_m^{\mathfrak{P}_{l,s}}(y) = O((l+1)^{-[(p+1)/2]})$

*Proof.* Alongside Lemma 1 and expanding in powers of $t$, we find

$$\begin{aligned}
\exp(-ty)\mathfrak{P}_{l,s}(\mathbf{e}^{tv}, y) &= e^{-ty}\left(1 - \frac{yt}{(l+1)_s^{1/s}}\right)^{-(l+1)} \\
&= \left[1 - \frac{(ty)}{1!} + \frac{(ty)^2}{2!} - \frac{(ty)^3}{3!} + \frac{(ty)^4}{4!} - \frac{(ty)^5}{5!} + \frac{(ty)^6}{6!} - \dots\right]\left[1 + \frac{(l+1)_1}{1!}\right. \\
&\quad \frac{yt}{((l+1)_s)^{1/s}} + \frac{(l+1)_2}{2!}\frac{y^2t^2}{((l+1)_s)^{2/s}} + \frac{(l+1)_3}{3!}\frac{y^3t^3}{((l+1)_s)^{3/s}} + \frac{(l+1)_4}{4!} \\
&\quad \left.\frac{y^4t^4}{((l+1)_s)^{4/s}} + \frac{(l+1)_5}{5!}\frac{y^5t^5}{((l+1)_s)^{5/s}} + \frac{(l+1)_6}{6!}\frac{y^6t^6}{((l+1)_s)^{6/s}} + \dots\right] \\
&= \left[1 + \left(\frac{(l+1)_1}{((l+1)_s)^{1/s}} - 1\right)\frac{yt}{1!} + \left(\frac{(l+1)_2}{((l+1)_s)^{2/s}} - \frac{2(j+1)_1}{((l+1)_s)^{1/s}} + 1\right)\frac{y^2t^2}{2!}\right. \\
&\quad + \left(\frac{(l+1)_3}{((l+1)_s)^{3/s}} - \frac{3(l+1)_2}{((l+1)_s)^{2/s}} + \frac{3(l+1)_1}{((l+1)_s)^{1/s}} - 1\right)\frac{y^3t^3}{3!} \\
&\quad + \left(\frac{(l+1)_4}{((l+1)_s)^{4/s}} - \frac{4(l+1)_3}{((l+1)_s)^{3/s}} + \frac{6(l+1)_2}{((l+1)_s)^{2/s}} - \frac{4(l+1)_1}{((l+1)_s)^{1/s}} + 1\right)\frac{y^4t^4}{4!} \\
&\quad + \left(\frac{(l+1)_5}{((l+1)_s)^{5/s}} - (\frac{5(l+1)_4}{((l+1)_s)^{4/s}} + \frac{10(l+1)_3}{((l+1)_s)^{3/s}} - \frac{10(l+1)_2}{((l+1)_s)^{2/s}}\right. \\
&\quad \left. + \frac{5(l+1)_1}{((l+1)_s)^{1/s}} - 1\right)\frac{y^5t^5}{5!} + \left(\frac{(l+1)_6}{((l+1)_s)^{6/s}} - \frac{6(l+1)_5}{((l+1)_s)^{5/s}} + \frac{15(l+1)_4}{((l+1)_s)^{4/s}}\right. \\
&\quad \left.\left. - \frac{20(l+1)_3}{((l+1)_s)^{3/s}} + \frac{15(l+1)_2}{((l+1)_s)^{2/s}} - \frac{6(l+1)_1}{((l+1)_s)^{1/s}} + 1\right)\frac{y^6t^6}{6!} + \dots\right] \qquad (2.3)
\end{aligned}$$

□

## 3. Usual modulus of continuity

The class of real valued continuous bounded functions denoted by $C_B[0,\infty)$ and $||\mathbf{h}|| = \sup_{y\in[0,\infty)}|\mathbf{h}(y)|$ denotes the norm for normed linear space $C_B[0,\infty)$. Now,

$$\varpi_p(\mathbf{h},\gamma) = \sup_{0\le k\le\gamma}\sup_{y\in[0,\infty)}|\triangle_k^p\mathbf{h}(y)|, \text{ for } \mathbf{h}\in C_B[0,\infty) \text{ and } \gamma > 0,$$

is characterized as $p$-th order modulus of continuity and the forward difference is denoted by $\triangle$. For $p = 1$, we indicate $\varpi(\mathbf{h},\gamma)$ as the usual modulus of continuity.

**Theorem 1.** *For* $\mathbf{h}\in C_B[0,\infty), y\in[0,\infty)$

$$|\mathfrak{P}_{l,s}(\mathbf{h};y) - \mathbf{h}(y)| \le \mathcal{M}\varpi(\mathbf{h}, er(s))$$

*where* $\mathcal{M}$ *is a positive constant and* $er(s) = \sqrt{\nu_2^{\mathfrak{P}_{l,s}}(y)}$ *is the error function for* $s = 0, 1, 2, 3, \ldots$ .

*Proof.* Addressing (7) of Rempulska and Sporupka [11], we find for every $\mathbf{h} \in C_B[0, \infty)$, there holds

$$|\mathfrak{P}_{l,s}(\mathbf{h}; y) - \mathbf{h}(y)| \leq \mathcal{M}\varpi(\mathbf{h}, \sqrt{\nu_2^{\mathfrak{P}_{l,s}}(y)}),$$

where $\mathcal{M} > 0$ is a constant. □

For family of preserving operators, i.e. $s = 1, 2, 3, \ldots$, we find

$$\begin{aligned}
|\mathfrak{P}_{l,1}(\mathbf{h}; y) - \mathbf{h}(y)| &\leq \mathcal{M}_1\varpi\left(\mathbf{h}, \frac{y}{\sqrt{(l+1)}}\right), \\
|\mathfrak{P}_{l,2}(\mathbf{h}; y) - \mathbf{h}(y)| &\leq \mathcal{M}_2\varpi\left(\mathbf{h}, y\sqrt{2\frac{\sqrt{l+2} - \sqrt{l+1}}{\sqrt{(l+2)}}}\right), \\
|\mathfrak{P}_{l,3}(\mathbf{h}; y) - \mathbf{h}(y)| &\leq \mathcal{M}_3\varpi\left(\mathbf{h}, y\sqrt{\frac{(l+1)_2}{((l+1)_3)^{2/3}} - \frac{2(l+1)}{((l+1)_3)^{1/3}} + 1}\right),
\end{aligned}$$

where $\mathcal{M}_i, i = 1, 2, 3$ are constant.

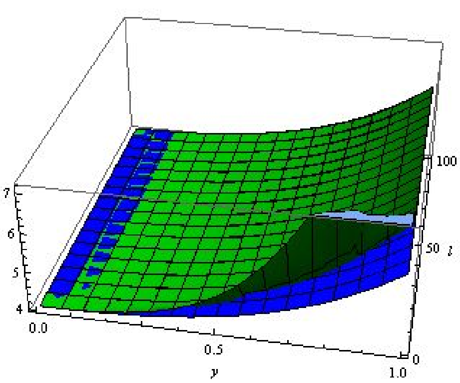


Figure 1

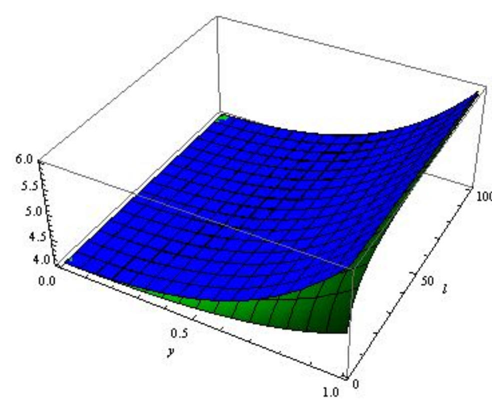


Figure 2

**Figure 1** and **Figure 2** show the convergence of $\mathfrak{P}_{l,1}(\mathbf{h}; y)$ and $\mathfrak{P}_{l,2}(\mathbf{h}; y)$ (represented by green curve) for $\mathfrak{h}(y) = y^4 + y^2 + 4$, (represented by blue curve).

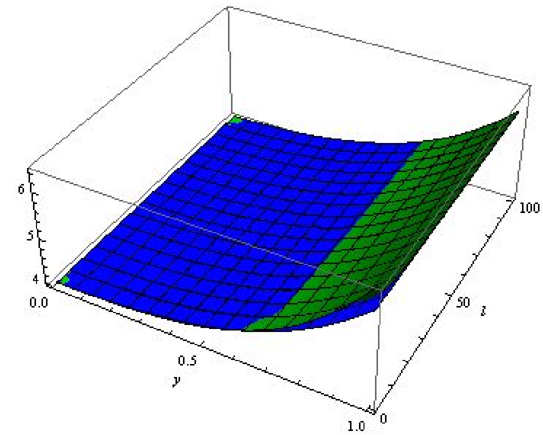


Figure 3

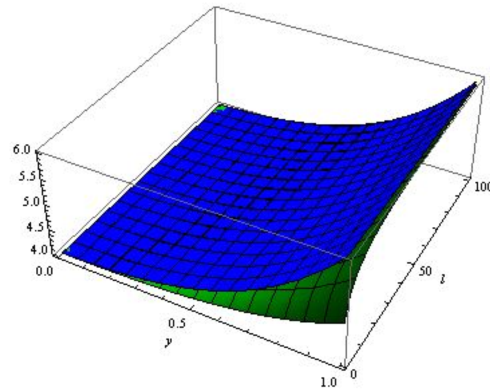


Figure 4

**Figure 3** and **Figure 4** show the convergence of $\mathfrak{P}_{l,3}(\mathbf{h};y)$ and $\mathfrak{P}_{l,6}(\mathbf{h};y)$ (represented by green curve) for $\mathfrak{h}(y) = y^4 + y^2 + 4$, (represented by blue curve).
In the following table, we represent numerical value of error of approximation of operators 1.2.

| l | er(1) | er(2) | er(3) | er(4) | er(5) | er(6) |
|---|---|---|---|---|---|---|
| 1 | 0.707107y | 0.605811y | 0.578275y | 0.582385y | 0.598959y | 0.619618y |
| 2 | 0.57735y | 0.517638y | 0.500355y | 0.504981y | 0.519975y | 0.539268y |
| 5 | 0.408248y | 0.385175y | 0.378018y | 0.381509y | 0.391646y | 0.405636y |
| 10 | 0.301511y | 0.291798y | 0.288684y | 0.29074y | 0.296656y | 0.305324y |
| 100 | 0.0995037y | 0.0991366y | 0.0990148y | 0.0991308y | 0.099476y | 0.10004y |
| 1000 | 0.031607y | 0.0315951y | 0.0315912y | 0.0315951y | 0.0316069y | 0.0316264y |
| 10000 | 0.0099995y | 0.00999913y | 0.009999y | 0.00999913y | 0.0099995y | 0.0100001y |

TABLE 1. **Table for error function** $er(s)$

**Remark 1.** *From the above table and graphs we conclude that error of estimation by the operators $\mathfrak{P}_{l,s}(.;y)$ decreases from $s=1$ to $s=3$ and after that it starts to increase. Therefore better approximation for the operators $\mathfrak{P}_{l,s}(.;y)$ occurs on $s=3$ for function $\mathfrak{h}(y) = y^4+y^2+4$. For $s>3$, we do not approach to better estimation.*

**Theorem 2.** *Suppose that $\mathbf{h} \in C_B[0,\infty)$, then*

$$|\mathfrak{P}_{l,s}(\mathbf{h};y) - \mathbf{h}(y)| = C\varpi_2(\mathbf{h}, \sqrt{\gamma_{l,s}}) + \varpi\left(\mathbf{h}, \left|\frac{(l+1)}{((l+1)_s)^{1/s}} - 1\right| y\right),$$

*where* $C>0$ *is a constant and* $\gamma_{l,s}$ *is defined by*

$$\gamma_{l,s} = \left[\frac{(l+1)(2l+3)}{((l+1)_s)^{2/s}} - \frac{4(l+1)y}{((l+1)_s)^{1/s}} + 2\right] y^2$$

*Proof.* We begin with the auxiliary operators ${\mathfrak{P}^*}_{l,s} : C_B[0,\infty) \to C_B[0,\infty)$ as

$$ {\mathfrak{P}^*}_{l,s}(\mathbf{h};y) = \mathfrak{P}_{l,s}(\mathbf{h};y) - \mathbf{h}\left(\frac{(l+1)y}{((l+1)_s)^1/s}\right) + \mathbf{h}(y), \tag{3.1}$$

From Lemma 1, it is clear that the auxiliary operator ${\mathfrak{P}^*}_{l,s}(\mathbf{h};y)$ preserve linear and constant functions. Let $\tilde{h} \in C_B^2[0,\infty), C_B^2[0,\infty) = \{\tilde{h} \in C_B[0,\infty) : \tilde{h}', \tilde{h}'' \in C_B[0,\infty)\}$ and $y, v \in [0,\infty)$.

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By Taylor's expansion we get

$$\tilde{h}(v) = \tilde{h}(y) + (v-y)\tilde{h}'(y) + \int_y^v (v-u)\tilde{h}''(u)du$$

Applying Lemma 2, we get

$$\begin{aligned}
|\mathfrak{P}^*{}_{l,s}(\tilde{h};y) - \tilde{h}(y)| &\le \left|\mathfrak{P}^*{}_{l,s}\left(\int_y^v (v-u)\tilde{h}''(u)du;y\right)\right| \\
&\le \left|\mathfrak{P}_{l,s}\left(\int_y^v (v-u)\tilde{h}''(u)du;y\right)\right| + \left|\int_y^{\frac{(l+1)y}{((l+1)_s)^{1/s}}} \left(\frac{(l+1)y}{((l+1)_s)^{1/s}} - u\right)\tilde{h}''(u)du\right| \\
&\le \nu_2^{\mathfrak{P}_{l,s}}(y)||\tilde{h}''|| + \left|\int_y^{\frac{(l+1)y}{((l+1)_s)^{1/s}}} \left(\frac{(l+1)y}{((l+1)_s)^{1/s}} - u\right)du\right| ||\tilde{h}''(u)||
\end{aligned}$$

$$|\mathfrak{P}^*{}_{l,s}(\tilde{h};y) - \tilde{h}(y)| \le \left[\nu_2^{\mathfrak{P}_{l,s}}(y) + \left(\frac{(l+1)y}{((l+1)_s)^{1/s}} - y\right)^2\right] ||\tilde{h}''(u)|| = \gamma_{l,s}||\tilde{h}''(u)|| \tag{3.2}$$

Also,we find, by Lemma 1

$$|\mathfrak{P}_{l,s}(\mathbf{h};y)| \le \frac{((l+1)_s)^{1/s}}{y^s\Gamma(l+1)} \int_0^\infty \exp(-a_j v)v^n \left|\mathbf{h}\left(\frac{v}{l}\right)\right| dv \le ||\mathbf{h}||. \tag{3.3}$$

Then by 3.1, we find

$$||\mathfrak{P}^*{}_{l,s}(\mathbf{h};y)|| \quad \le ||\mathfrak{P}_{l,s}(\mathbf{h};y)|| + 2||\mathbf{h}|| = 3||\mathbf{h}||, \mathbf{h} \in C_B[0,\infty) \tag{3.4}$$

Using 3.1, 3.2, 3.4 and in light of Peetre's K-functional given as,

$$K_2(\mathbf{h},\eta) = \inf_{\tilde{h}\in C_B^2[0,\infty)} \{||\mathbf{h} - \tilde{h}|| + \eta||\tilde{h}''|| : \tilde{h} \in C_B^2[0,\infty)\}$$

we find

$$\begin{aligned}
|\mathfrak{P}_{l,s}(\mathbf{h};y) - \mathbf{h}(y)| &\le |\mathfrak{P}^*{}_{l,s}(\mathbf{h}-\tilde{h});y) - (\mathbf{h}-\tilde{h})(y)| + |\mathfrak{P}^*{}_{l,s}(\tilde{h};y) - \tilde{h}(y)| \\
&\quad + \left|\mathbf{h}\left(\frac{(l+1)y}{((l+1)_s)^{1/s}}\right) - \mathbf{h}(y)\right| \\
&\le 4||\mathbf{h}-\tilde{h}|| + \gamma_{l,s}||\tilde{h}''|| + \left|\mathbf{h}\left(\frac{(l+1)y}{((l+1)_s)^{1/s}}\right) - \mathbf{h}(y)\right| \\
&\le C\left\{||\mathbf{h}-\tilde{h}|| + \gamma_{l,s}||\tilde{h}''||\right\} + \varpi\left(\mathbf{h}, \left|\frac{(l+1)y}{((l+1)_s)^{1/s}} - y\right|\right) \\
&\le K_2(\mathbf{h},\delta_{l,s}) + \varpi\left(\mathbf{h}, \left|\frac{(l+1)y}{((l+1)_s)^{1/s}} - y\right|\right)
\end{aligned}$$

Now, using the Lorentz-DeVore property [13]

$$K_2(\mathbf{h},\eta) \le C\varpi_2(h,\sqrt{\eta}), \eta > 0$$

we find

$$|\mathfrak{P}_{l,s}(\mathbf{h};y) - \mathbf{h}(y)| = C\varpi_2(\mathbf{h},\sqrt{\gamma}_{l,s}) + \varpi\left(\mathbf{h}, \left|\frac{(l+1)}{((l+1)_s)^{1/s}} - 1\right|y\right)$$

□

## 4. Exponential modulus of continuity

*Next,*

$$||\mathbf{h}||_B = \sup_{y\in[0,\infty)} |\mathbf{h}(y)e^{-By}| < \infty, B > 0,$$

*defines the class of continuous function on $[0,\infty)$ with exponential growth. Then Ditzian [5] mentioned the second order modulus of continuity as*

$$\varpi_2(\mathbf{h},\gamma,B) = \sup_{k\leq\gamma, 0\leq y\leq\infty} |\mathbf{h}(y) - 2\mathbf{h}(y+k) + \mathbf{h}(y+2k)|e^{-By}$$

*The present paper concern with the first order modulus of continuity, which is given by*

$$\varpi_1(\mathbf{h},\gamma,B) = \sup_{k\leq\gamma, 0\leq y\leq\infty} |\mathbf{h}(y) - \mathbf{h}(y+k)|e^{-By}$$

**Theorem 3.** *Let $E \subseteq C[0,\infty)$ such that it contains the polynomials, let $\mathfrak{P}_{l,s}(.;y) : E \to C[0,\infty)$ is sequence of linear positive operators, upholding the test functions $\alpha_s = y^s, s \in \mathbb{N}\cup\{0\}$,and for all constant $B > 0$ and specified $y \in [0,\infty)$, $\mathfrak{P}_{l,s}(.;y)$ holds*

$$\mathfrak{P}_{l,s}\Big((v-y)^2 e^{Bv}; y\Big) \leq C(B,y)\nu_2^{\mathfrak{P}_{l,s}}(y)$$

*Also if $\mathbf{h} \in C^2[0,\infty)\cap E$ and $\mathbf{h}'' \in Lip(\bar{a},B), 0 < \bar{a} \leq 1$ then we find for $y \in [0,\infty)$*

$$\left|\mathfrak{P}_{l,s}(\mathbf{h};y) - \mathbf{h}(y) - \mathbf{h}'(y)\Big(\frac{(l+1)}{((l+1)_s)^{1/s}} - 1\Big)y - \frac{1}{2}\mathbf{h}''(y)\Big(\frac{(l+1)_2}{((l+1)_s)^{2/s}} - \frac{2(l+1)}{((l+1)_s)^{1/s}} + 1\Big)y^2\right|$$
$$\leq \left[e^{2By} + \frac{C(B,y)}{2} + \frac{\sqrt{C(B,y)}}{2}\right]\nu_2^{\mathfrak{P}_{l,s}}(y)\varpi_1\left(\mathbf{h}'', \sqrt{\frac{\nu_4^{\mathfrak{P}_{l,s}}(y)}{\nu_2^{\mathfrak{P}_{l,s}}(y)}}, B\right)$$

*Where $\nu_2^{\mathfrak{P}_{l,s}}$ and $\nu_4^{\mathfrak{P}_{l,s}}$ are from Lemma 2.*

*Proof.* For the function $\mathbf{h} \in C^2[0,\infty)$, Taylor's expansion at the point $y \in [0,\infty)$ is given by

$$\mathbf{h}(v) = \mathbf{h}(y) + (v-y)\mathbf{h}'(y) + \frac{(v-y)^2}{2!}\mathbf{h}''(y) + R_2(v,y) \tag{4.1}$$

where

$$R_2(v,y) = \frac{\mathbf{h}''(\xi) - \mathbf{h}''(y)}{2}(v-y)^2$$

and $y < \xi < v$. By operating the operator $\mathfrak{P}_{l,s}(.;y)$ on (4.1) and in light of Lemma 2, we find

$$\left|\mathfrak{P}_{l,s}(\mathbf{h};y) - \mathbf{h}(y) - \mathbf{h}'(y)\nu_1^{\mathfrak{P}_{l,s}}(y) - \frac{1}{2}\mathbf{h}''(y)\nu_2^{\mathfrak{P}_{l,s}}(y)\right| \leq \mathfrak{P}_{l,s}(|R_2(v,y)|;y) \tag{4.2}$$

Next, we required $\mathfrak{P}_{l,s}(|R_2(v,x)|,y)$ to prove the theorem. By simplification, we find

$$|R_2(v,y)| \leq \frac{1}{2}(e^{2By} + e^{Bv})\Big(1 - \frac{|v-y|}{k}\Big)\varpi_1(\mathbf{h}'',k,B)|v-y|^2$$

Consequently, we get

$$\mathfrak{P}_{l,s}(|R_2(v,y)|,y) = \frac{1}{2}\Big[\mathfrak{P}_{l,s}(e^{2By} + e^{Bv})\Big(|v-y|^2 + \frac{|v-y|^3}{k};y\Big)\varpi_1(\mathbf{h}'',k,B) \tag{4.3}$$

With the operator 1.2, we find

$$\mathfrak{P}_{l,s}(v^p e^{Bv};y) = \frac{1}{l!}\Big(\frac{((l+1)_s)^{1/s}}{ly}\Big)^{(l+1)}\int_0^\infty \exp\Big(-\frac{((l+1)_s)^{1/s}}{ly}v\Big)v^l\Big(\frac{v^p\exp(Bv/l)}{l^p}\Big)dv.$$

By simplification

$$\mathfrak{P}_{l,s}(v^p e^{Bv};y) = \frac{1}{l!}\left(\frac{((l+1)_s)^{1/s}}{ly}\right)^{(l+1)}(l+p)!\left[\frac{((l+1_s)^{1/s})}{y} - B\right]^{-(l+p+1)} \tag{4.4}$$

Using 4.4, we find

$$\begin{aligned}\mathfrak{P}_{l,s}\bigg((v-y)^2 e^{Bv};y\bigg) = &\left[\frac{((l+1)_s)^{1/s}}{((l+1)_s)^{1/s} - By}\right]^{(l+3)} \nu_2^{\mathfrak{P}_{l,s}} \\ &\left[1 + \frac{2(l+1)By - 2((l+1)_s)^{1/s}By + B^2y^2}{(l+1)_2 - 2(l+1)((l+1)_s)^{1/s} + ((l+1)_s)^{2/s}}\right]\end{aligned} \tag{4.5}$$

Let $y$ be fixed and $n > 2By$, then

$$\begin{aligned}\left[\frac{((l+1)_s)^{1/s}}{((l+1)_s)^{1/s} - By}\right]^{(l+3)} &= \left[1 + \frac{By}{((l+1)_s)^{1/s} - By}\right]^{(l-By)}\left[1 + \frac{By}{((l+1)_s)^{1/s} - By}\right]^{(2+By)} \\ &\leq e^{By}(1+1)^{2+By} = 4(2e)^{By}\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\left[1 + \frac{2(l+1)By - 2((l+1)_s)^{1/s}By + B^2y^2}{(l+1)_2 - 2(l+1)((l+1)_s)^{1/s} + ((l+1)_s)^{2/s}}\right] &\leq \left[1 + \frac{B^2y^2}{(((l+1)_s)^{1/s} + (l+1))^2 + (l+1)}\right] \\ &\leq 1 + B^2y^2\end{aligned} \tag{4.7}$$

Using 4.6 and 4.7in 4.5, we get

$$\mathfrak{P}_{l,s}\bigg((v-y)^2 e^{Bv};y\bigg) \leq 4(2e)^{By}(1+B^2y^2)\nu_2^{\mathfrak{P}_{l,s}}(y) = C(B,y)\nu_2^{\mathfrak{P}_{l,s}}(y) \tag{4.8}$$

Further by Cauchy-Schwarz inequality we get

$$\begin{aligned}\mathfrak{P}_{l,s}\bigg((v-y)^2 e^{Bv};y\bigg) &\leq \sqrt{\mathfrak{P}_{l,s}\bigg((v-y)^2 e^{Bv};y\bigg)}\sqrt{\mathfrak{P}_{l,s}\bigg((v-y)^4 e^{Bv};y\bigg)} \\ &\leq \sqrt{C(B,y)\nu_2^{\mathfrak{P}_{l,s}}(y)}\sqrt{\nu_4^{\mathfrak{P}_{l,s}}(y)}\end{aligned} \tag{4.9}$$

Putting $k = \sqrt{\dfrac{\nu_4^{\mathfrak{P}_{l,s}}(y)}{\nu_2^{\mathfrak{P}_{l,s}}(y)}}$ in 4.2 and using 4.8 and 4.9, we find

$$\begin{aligned}&\left|\mathfrak{P}_{l,s}(\mathbf{h};y) - \mathbf{h}(y) - \mathbf{h}'(y)\left(\frac{(l+1)}{((l+1)_s)^{1/s}} - 1\right)y - \frac{1}{2}\mathbf{h}''(y)\left(\frac{(l+1)_2}{((l+1)_s)^{2/s}} - \frac{2(l+1)}{((l+1)_s)^{1/s}} + 1\right)y^2\right| \\ &\leq \left[e^{2By} + \frac{C(B,y)}{2} + \frac{\sqrt{C(B,y)}}{2}\right]\nu_2^{\mathfrak{P}_{l,s}}(y)\varpi_1\left(\mathbf{h}'', \sqrt{\frac{\nu_4^{\mathfrak{P}_{l,s}}(y)}{\nu_2^{\mathfrak{P}_{l,s}}(y)}}, B\right)\end{aligned}$$

□

**Remark 2.** *The confirmation of convergence of Theorem 3 comes from the condition*

$$\frac{\nu_4^{\mathfrak{P}_{l,s}}(y)}{\nu_2^{\mathfrak{P}_{l,s}}(y)} \to 0, \ l \to 0.$$

## 5. Weighted modulus of continuity

We consider $B_2[0,\infty)$ as the collection of all the function $\mathbf{h}$ fulfill the condition $|\mathbf{h}| \leq \mathcal{M}_h(1+y^2)$, $\mathcal{M}_h$ is some constant not dependent on $y$.
Let $C_2[0,\infty) = C[0,\infty) \cap B_2[0,\infty)$ and by $C_2^p[0,\infty)$, we denote subspace of all continuous functions $\mathbf{h} \in B_2[0,\infty)$ for which $\lim_{y\to\infty} |\mathbf{h}(y)|(1+y^2)^{-1} < \infty$
The weighted modulus of continuity $\Omega(\mathbf{h},\gamma)$, for each $\mathbf{h} \in C_2[0,\infty)$ is defined as

$$\Omega(\mathbf{h},\gamma) = \sup_{|k|<\gamma, y\in\mathbf{R}^+} |\mathbf{h}(y+k) - \mathbf{h}(y)|(1+k^2+y^2+k^2y^2)^{-1}$$

**Theorem 4.** *Let $\mathbf{h}'' \in C_2^p[0,\infty)$, and $y > 0$. Then, we find*

$$\left|\mathfrak{P}_{l,s}(\mathbf{h};y) - \mathbf{h}(y) - \mathbf{h}'(y)\left(\frac{(l+1)}{((l+1)_s)^{1/s}} - 1\right)y - \frac{1}{2}\mathbf{h}''(y)\left(\frac{(l+1)_2}{((l+1)_s)^{2/s}} - \frac{2(l+1)}{((l+1)_s)^{1/s}} + 1\right)y^2\right| \leq \frac{8(1+y^2)}{l}\Omega(\mathbf{h}'', l^{-1/2})$$

*Proof.* By Taylor's expansion, we find

$$\begin{aligned}\mathfrak{P}_{l,s}(\mathbf{h};y) - \mathbf{h}(y) &= \mathfrak{P}_{l,s}(\mathbf{h}(v) - \mathbf{h}(y)); y) \\ &= \mathfrak{P}_{l,s}\left((v-y)\mathbf{h}'(y) + \frac{(v-y)^2}{2}\mathbf{h}''(y) + R_2(v,y)(v-y)^2; y\right)\end{aligned}$$

where $R_2(v,y) = [\mathbf{h}''(\eta) - \mathbf{h}''(y)]/2$ and continuous function $R_2$ goes to vanish at 0 and $v < \eta < y$. Using Lemma 2

$$\left|\mathfrak{P}_{l,s}(\mathbf{h};y) - \mathbf{h}(y) - \mathbf{h}'(y)\left(\frac{(l+1)}{((l+1)_s)^{1/s}} - 1\right)y - \frac{1}{2}\mathbf{h}''(y)\left(\frac{(l+1)_2}{((l+1)_s)^{2/s}} - \frac{2(l+1)}{((l+1)_s)^{1/s}} + 1\right)y^2\right| \leq \mathfrak{P}_{l,s}(|R_2(v,y)|(v-y)^2; y)$$

Applying the inequality $|\eta - y| \leq |v - y|$ and computing we find

$$|R_2(v,y)| \leq 8(1+y^2)\left(1 + \frac{(v-y)^4}{\gamma}\right)\Omega(\mathbf{h}'',\gamma)$$

Then we find by Lemma2 that

$$\begin{aligned}\mathfrak{P}_{l,s}(|R_2(v,y)|(v-y)^2; y) &= 8(1+y^2)\Omega(\mathbf{h}'',\gamma)\left\{\nu_2^{\mathfrak{P}_{l,s}}(y) + \frac{1}{\gamma^4}\nu_6^{\mathfrak{P}_{l,s}}(y)\right\} \\ &= 8(1+y^2)\Omega(\mathbf{h}'',\gamma)\Bigg[\left(\frac{(l+1)_2}{((l+1)_s)^{2/s}} - \frac{2(l+1)}{((l+1)_s)^{1/s}} + 1\right)y^2 + \frac{1}{\gamma^4} \\ &\quad \Bigg[\Bigg(\frac{(l+1)_6}{((l+1)_s)^{6/s}} - \frac{6(l+1)_5}{((l+1)_s)^{5/s}} + \frac{15(l+1)_4}{((l+1)_s)^{4/s}} - \frac{20(l+1)_3}{((l+1)_s)^{3/s}} \\ &\quad + \frac{15(l+1)_2}{((l+1)_s)^{2/s}} - \frac{6(l+1)}{((l+1)_s)^{1/s}} + 1\Bigg)y^6\Bigg]\Bigg]\end{aligned}$$

By taking $\gamma = \frac{1}{\sqrt{(l+1)}}$, then theorem complete. □

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