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The Exit Time And The Dividend Problem For Compound Poisson Risk Model

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Abstract — *In this paper, we consider the classical risk model with mixed dividend strategy. Integro-differential equations for the Laplace transform of exit times from interval are derived. Unlike previous studies, this paper uses exit time to calculate the expected discounted dividend payments prior to ruin under compound Poisson model. When the claims are exponentially distributed, the analytical solution is presented.*

Keywords — *Classical risk model, Mixed dividend strategy, Dividend payment, Exponential claims, Exit times*

I. INTRODUCTION

The study of dividend strategy in insurance risk models were first proposed by De Finetti(1957), who presented his paper at the 15th International Congress of Actuaries in New York City. But, he found that the optimal strategy must be barrier strategy: that is, any surplus above a certain level would be paid as dividend. As this is not realistic, such a barrier strategy is applied, ultimate ruin of the company is certain. Jeanblanc-Picque and Shiryaev (1995) and Asmussen and Taksar (1997) modified the problem and postulated a dividend rate in the Brownian motion model. They showed that the optimal dividend strategy is now a generalized barrier strategy, that is, threshold strategy. Some down-to-earth calculations for the classical risk model are given in Gerber and Shiu (2004,2006b).

From then on, the study of dividend problems has drawn more and more authors' attention. For example, Lin and Pavlova (2006) and Lin and Sendova (2008) studies the dividend problem under compound Poisson risk model with threshold or multiple threshold strategy. Dividend problem was widely studied in different risk models, such as dual model (see, Avanzi et.al 2007), the spectrally positive Lévy processes(see, Yin et al. 2013, 2014) and Erlang-2 model (see, Avanzi et.al 2018). Recently, the dividend problem is usually studied by combining the other strategies, such as reinsurance, financing and investment(Yao et al. 2010, 2016). The usual method to solve this problem is using optimal control theory. But the dividend problem can also be studied by exit time (see, Fang and Wu 2009; Li et al. 2013).

In this paper, we consider the mixed dividend strategy (see, Li et al. 2014a, 2014b) with $0 < b_1 < b_2$ for the classical risk model. Dividends are paid at a constant rate α whenever the modified surplus is in interval (b_1, b_2). The excess is paid out immediately as the dividends whenever the surplus exceeds the level b_2. Unlike the previous studies, we derive the dividend function by exit time method. We lead to the integro-differential equations for the Laplace transform of exit time. Then, we use this result to calculate the expression of the dividend payment in the case of exponential claim amount distributions.

The rest of the paper is organized as follows. Section 2 gives the model. In Section 3, we defined the exit times of the modified process from interval and obtained the integro-differential equation. Explicit results of the Laplace transform of exit time are given in the case of the exponential claim distribution. For the case of mixture of exponential claim amount distribution, a system of linear equations are obtained. Section 4 presents the application of exit time in dividend problem, and the results in some special case is agree with the known results.

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II. The Model and Mixed Dividend Strategy

We consider the classical model of risk theory with initial surplus $u > 0$. If no dividends were paid the surplus process U_t at time t is given by

$$U_t = u + \mu t - S_t = u + \mu t - \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0,$$

where μ is the premium rate, and $\{S_t\}$ represents the aggregate claims up to time t, $N(t)$ is a Poisson process with intensity λ, and $Y_i, i = 1, \ldots,$ independent of $\{N(t); t > 0\}$, are positive i.i.d. random variables with distribution function

$P(y)$ and density function $p(y)$. Unlike the dividend strategies in Gerber and Shiu (2004) and Gerber and Shiu (2006a), we assume the company will pay dividends to its shareholders according to a mixed dividend strategy with parameters (see, Li et al. 2014a, 2014b). This dividend consists of two parts. The first part of dividends are paid at a constant rate $\alpha \in (0,c)$ whenever the (modified) surplus between the level b_1 and b_2. Additionally, whenever the (modified) surplus reaches the level b_2, the overflow will be paid as the second part of dividends. Under this mixed dividend strategy, the company's (modified) surplus at time t is given by

$$\tilde{U}_t = U_t - D(t), \tag{2.1}$$

where $D(t)$ is the accumulate dividend up to time t.

III. Laplace Transform of exit time

A. Integro-differential equations

We first consider the Laplace Transform of the exit times from an interval for the modified surplus process defined by (2.1). Given the costant $a \in [b_1, b_2]$, we consider the exist time of interval $[0,a]$. Define

$$\begin{aligned} \tau_0^- &= \inf\{t : \tilde{U}_t \le 0\}, \\ \tau_a^+ &= \inf\{t : \tilde{U}_t \ge a\}, \\ \tau_{0,a} &= \tau_0^- \wedge \tau_a^+. \end{aligned} \tag{3.1}$$

Then we can give the following Laplace transforms:

$$\begin{aligned} L_1(u) &= E_u[e^{-\delta\tau_0^-}, \tau_0^- < \tau_a^+], \\ L_2(u) &= E_u[e^{-\delta\tau_a^+}, \tau_a^+ < \tau_0^-], \\ L(u) &= E_u[e^{-\delta\tau_{0,a}}]. \end{aligned}$$

where $\delta > 0$ is a constant.

To give the Integro-differential equations satisfied by the above Laplace transforms. We introduce the infinitesimal generator G. When $0 < \tilde{U}_t < b_2$, process $\{\tilde{U}_t\}_{t\ge 0}$ is a lévy process with negative jump, and the generator is given by

$$\mathrm{G}g(u) = \begin{cases} \mu g'(u) + \lambda \int_0^{+\infty} [g(u-y) - g(u)] p(y) dy, & 0 \le u < b_1 \\ (\mu - \alpha) g'(u) + \lambda \int_0^{+\infty} [g(u-y) - g(u)] p(y) dy, & b_1 \le u \le b_2 \end{cases}$$

for any continuously differentiable function $g(u)$.

Theorem 3.1. For $u \in (0, b_1)$, $L_1(u)$ satisfies the following integro-differential equation:

$$\mu L_{1'}(u) - (\lambda + \delta) L_1(u) + \lambda \int_0^u L_1(u-y) p(y) dy + \lambda [1 - P(u)] = 0, \tag{3.2}$$

and for $u \in [b_1, a)$, $L_1(u)$ satisfies the the following integro-differential equation:

$$(\mu - \alpha) L_{1'}(u) - (\lambda + \delta) L_1(u) + \lambda \int_0^u L_1(u-y) p(y) dy + \lambda [1 - P(u)] = 0. \tag{3.3}$$

Furthermore, $L_1(0) = 1$ and $L_1(a) = 0$.

Proof. Let $g(u)$ be a continuously differentiable function, and it satisfies the following function

$$\mathrm{G}g(u) - \delta g(u) = 0, u \in (0,a) \tag{3.4}$$

and boundary conditions $g(0) = 1$ and $g(a) = 0$. Now, we apply Dynkin's formula to function $e^{-\delta t} g(\tilde{U}_t)$, then

$$\mathrm{E}_u\left[e^{-\delta t} g(\tilde{U}_t)\right] = g(u) + \mathrm{E}_u\left[\int_0^t e^{-\delta s} (\mathrm{G} - \delta) g(\tilde{U}_s) \mathrm{d}s\right].$$

Note that stopping time τ which is defined by (3.1) is finite, and it follows from optional theorem that

$$\mathrm{E}_u\left[e^{-\delta(\tau_{0,a}\wedge t)}g(\tilde{U}_{(\tau_{0,a}\wedge t)})\right]=g(u)+\mathrm{E}_u\left[\int_0^{\tau_{0,a}\wedge t}e^{-\delta s}(\mathrm{G}-\delta)g(\tilde{U}_s)\mathrm{d}s\right].$$

Now let $t\to+\infty$, it yields that

$$\mathrm{E}_u[e^{-\delta\tau_{0,a}}g(\tilde{U}_{\tau_{0,a}})]=g(u)+\mathrm{E}_u\left(\int_0^{\tau_{0,a}}e^{-\delta s}(\mathrm{G}-\delta)g(\tilde{U}_s)\mathrm{d}s\right). \tag{3.5}$$

By the definition of $\tau_{0,a}$, it follows that

$$\mathrm{E}_u[e^{-\delta\tau_{0,a}}g(\tilde{U}_{\tau_{0,a}})]=g(0)\mathrm{E}_u[e^{-\delta\tau_0^-};\tau_0^-<\tau_a^+]+g(a)\mathrm{E}_u[e^{-\delta\tau_a^+};\tau_a^+<\tau_0^-]. \tag{3.6}$$

Note the boundary condition $g(0)=1$ and $g(a)=0$, we have

$$\mathrm{E}_u[e^{-\delta\tau_{0,a}}g(\tilde{U}_{\tau_{0,a}})]=\mathrm{E}_u[e^{-\delta\tau_0^-};\tau_0^-<\tau_a^+].$$

Substituting (3.4) and above equation into (3.5), we get $g(u)=L_1(u)$, which means $L_1(u)$ is the solution of equation (3.4). Next, we will verify the function (3.5) is equal to the theorem. Note that the integral part

$$\lambda\int_0^{+\infty}[L_1(u-y)-L_1(u)]p(y)dy=\lambda[\int_0^{u}L_1(u-y)p(y)dy+1-P(u)-L_1(u)]$$

where the equality used the definition of $L_1(u-y)=1$ when $y\geq u$. Substituting above equation to (3.4), the theorem is proved.

Theorem 3.2. $L_2(u)$ satisfies integro-differential equation For $u\in(0,b_1)$,

$$\mu L_{2'}(u)-(\lambda+\delta)L_2(u)+\lambda\int_0^u L_2(u-y)p(y)dy=0, \tag{3.7}$$

and for $u\in[b_1,a)$, $L_2(u)$ satisfies integro-differential equation:

$$(\mu-\alpha)L_{2'}(u)-(\lambda+\delta)L_2(u)+\lambda\int_0^u L_2(u-y)p(y)dy=0, \tag{3.8}$$

and the boundary conditions $L_2(0)=0$ and $L_2(a)=1$.

Proof. It follows from the same method of Theorem 3.1, we can get $L_2(u)$ satisfies (3.4). For the the integral part in this equation, we have

$$\int_0^{+\infty}[L_2(u-y)-L_2(u)]p(y)dy=\int_0^u L_2(u-y)p(y)dy$$

which follows that $L_2(u-y)=0$ when $y\geq u$. Substituting above equation to (3.4), the theorem is proved.

Remark 3.1. According to the definition of $L_1(u)$ and $L_2(u)$, we can obtain the process defined by (2.1) started in $(0,a)$ remains in $(0,a)$ for all $t<\tau_{0,a}$. Finally, we can calculate the Laplace transform of the exit times $\tau_{0,a}$ by the following equation

$$E_u[e^{-\delta\tau}]=L_1(u)+L_2(u)$$

From the above equation and the Theorems 3.1 and 3.2, we can get $L(u):=E_u[e^{-\delta\tau}]$ satisfies the integro-differential equation (3.4) with the boundary conditions $L(0)=1$ and $L(a)=1$.

B. Analytical solution under the mixed exponential distribution claim

In this subsection, we calculate $L_1(u)$ and $L_2(u)$ for the case where the individual claim amounts are mixed exponential distribution. The mixture of exponential densities is given by

$$p(y)=\sum_{i=1}^n p_i\beta_i e^{-\beta_i y},\ \ y>0, \tag{3.9}$$

with $0<\beta_1<\beta_2<\cdots<\beta_n$ and $p_i>0$, for $i=1,2,...,n,$ and $\sum_{i=1}^n p_i=1$.

To calculate $L_1(u)$, we apply the operator $\prod_{i=1}^{n}(d/du+\beta_i)$ to equation (3.2), we can get that the function $L_1(u)$ satisfies a homogeneous differential equation
of order $n+1$. Hence, we have

$$L_1(u)=\sum_{k=0}^{n}A_k e^{\upsilon_k u}, \quad 0\le u\le b_1, \tag{3.10}$$

where $\upsilon_0,\upsilon_1,\cdots,\upsilon_n$ are the solutions of the Lundberg's fundamental equation

$$\mu\xi-(\lambda+\delta)+\lambda\sum_{i=1}^{n}p_i\frac{\beta_i}{\beta_i+\xi}(\xi)=0, \tag{3.11}$$

We have

$$-\beta_n<\upsilon_n\cdots<-\beta_2<\upsilon_2<-\beta_1<\upsilon_1<0<\upsilon_0.$$

Substitute (3.10) into (3.2) yields the following equation

$$\sum_{k=0}^{n}A_k[\mu\upsilon_k e^{\upsilon_k u}-(\lambda+\delta)e^{\upsilon_k u}+\lambda\sum_{i=1}^{n}p_i\frac{\beta_i}{\beta_i+\upsilon_k}(e^{\upsilon_k u}-e^{-\beta_i u})]+\lambda\sum_{i=1}^{n}p_i e^{-\beta_i u}=0.$$

Equating the coefficient of $e^{\upsilon_k u}$ with 0, we obtain the Lundberg's fundamental equation (3.11). Equating the coefficient of $e^{-\beta_i u}$ with 0, we obtain the following equation

$$\sum_{k=0}^{n}A_k\frac{\beta_i}{\beta_i+\upsilon_k}=1, \quad i=1,...,n. \tag{3.12}$$

Applying the operator $\prod_{i=1}^{n}(d/du+\beta_i)$ to equation (3.3), we see that $L_1(u)(b_1<u\le a)$ satisfies also a linear differential equation of order $n+1$. Hence, we have

$$L_1(u)=\sum_{k=0}^{n}B_k e^{u_k u}, \quad b_1<u<a. \tag{3.13}$$

We now substitute (3.10) and (3.13) in the integro-differential equation (3.3). We yields the equation

$$(\mu-\alpha)\sum_{k=0}^{n}B_k u_k e^{u_k u}-(\lambda+\delta)\sum_{k=0}^{n}B_k e^{u_k u}+\lambda\sum_{i=1}^{n}p_i\sum_{k=0}^{n}A_k\frac{\beta_i}{\beta_i+\upsilon_k}[e^{\upsilon_k b_1+\beta_i(b_1-u)}-e^{-\beta_i u}]$$
$$+\lambda\sum_{i=1}^{n}p_i\sum_{k=0}^{n}B_k\frac{\beta_i}{\beta_i+u_k}[e^{u_k u}-e^{u_k b_1+\beta_i(b_1-u)}]+\lambda\sum_{i=1}^{n}p_i e^{-\beta_i u}=0. \quad b_1<u<a$$

Now, equating the coefficient of $e^{u_k u}$ with 0, we obtain

$$(\mu-\alpha)u_k-(\lambda+\delta)+\lambda\sum_{i=1}^{n}p_i\frac{\beta_i}{\beta_i+u_k}=0, \quad k=0,1,...,n. \tag{3.14}$$

This means that $u_0,u_1,\cdots,u_n$ are the solutions of equation (3.14). We have

$$-\beta_n<u_n\cdots<-\beta_2<u_2<-\beta_1<u_1<0<u_0.$$

Finally, we equate the coefficient of $e^{-\beta_i u}$ with 0. This leads to the equation

$$\sum_{k=0}^{n}A_k\frac{1}{\beta_i+\upsilon_k}e^{\upsilon_k b_1}=\sum_{k=0}^{n}B_k\frac{1}{\beta_i+u_k}e^{u_k b_1}, \quad i=1,2,...n. \tag{3.15}$$

Another condition follows from the continuity of $L_1(u)$ at $u=b_1$, it follows that

$$\sum_{k=0}^{n}A_k e^{\upsilon_k b_1}=\sum_{k=0}^{n}B_k e^{u_k b_1}. \tag{3.16}$$

Finally, from the boundary condition $L_1(a)=0$, we have

$$\sum_{k=0}^{n} B_k e^{u_k a} = 0. \tag{3.17}$$

In conclusion, (3.12)and (3.15) ~ (3.17) constitute a system of $2n+2$ linear equations

$$\begin{pmatrix} \frac{\beta_1}{\beta_1+\upsilon_0} & \cdots & \frac{\beta_1}{\beta_1+\upsilon_n} & 0 & \cdots & 0 \\ \vdots & & & & & \\ \frac{\beta_n}{\beta_n+\upsilon_0} & \cdots & \frac{\beta_n}{\beta_n+\upsilon_n} & 0 & \cdots & 0 \\ \frac{e^{\upsilon_0 b_1}}{\beta_1+\upsilon_0} & \cdots & \frac{e^{\upsilon_n b_1}}{\beta_1+\upsilon_n} & -\frac{e^{u_0 b_1}}{\beta_1+u_0} & \cdots & -\frac{e^{u_n b_1}}{\beta_1+u_n} \\ \vdots & & & & & \\ \frac{e^{\upsilon_0 b_1}}{\beta_n+\upsilon_0} & \cdots & \frac{e^{\upsilon_n b_1}}{\beta_n+\upsilon_n} & -\frac{e^{u_0 b_1}}{\beta_n+u_0} & \cdots & -\frac{e^{u_n b_1}}{\beta_n+u_n} \\ e^{\upsilon_0 b_1} & \cdots & e^{\upsilon_n b_1} & -e^{u_0 b_1} & \cdots & -e^{\upsilon_n b_1} \\ 0 & \cdots & 0 & e^{u_0 a} & \cdots & e^{u_n a} \end{pmatrix} \begin{pmatrix} A_0 \\ \\ \vdots \\ \\ A_n \\ B_0 \\ \\ \vdots \\ \\ B_n \end{pmatrix} = \begin{pmatrix} 1 \\ \\ \vdots \\ \\ 1 \\ 0 \\ \\ \vdots \\ \\ 0 \end{pmatrix}$$

from which the coefficients $A_0, A_1, \cdots, A_n, B_0, B_1, \cdots, B_n$ can be determined.

Next, We calculate $L_2(u;b_1,a)$ when the claim density given by (3.9). Applying the operator $\prod_{i=1}^{n}(d/du+\beta_i)$ to equation (3.7), we can get that the function $L_2(u)$ satisfies a homogeneous differential equation of order $n+1$. Hence, we have

$$L_2(u) = \sum_{k=0}^{n} C_k e^{\upsilon_k u}, \quad 0 \le u \le b_1, \tag{3.18}$$

where $\upsilon_0, \upsilon_1, \cdots, \upsilon_n$ are the same as above. Substitute (3.18) into (3.7), we have

$$\sum_{k=0}^{n} C_k [\mu\upsilon_k e^{\upsilon_k u} - (\lambda+\delta)e^{\upsilon_k u} + \lambda \sum_{i=1}^{n} p_i \frac{\beta_i}{\beta_i+\upsilon_k}(e^{\upsilon_k u} - e^{-\beta_i u})] = 0.$$

Equating the coefficient of $e^{-\beta_i u}$ with 0, we obtain the following equations

$$\sum_{k=0}^{n} C_k \frac{1}{\beta_i+\upsilon_k} = 0, \quad i=1,...,n. \tag{3.19}$$

Applying the operator $\prod_{i=1}^{n}(d/du+\beta_i)$ to equation (3.8), we see that $L_2(u)(b_1 < u \le a)$ satisfies also a linear differential equation of order $n+1$. Hence, we have

$$L_2(u) = \sum_{k=0}^{n} D_k e^{u_k u}, \quad b_1 < u < a. \tag{3.20}$$

We now substitute (3.18) and (3.20) into the integro-differential equation (3.7). We yields the equation

$$(\mu-\alpha)\sum_{k=0}^{n} D_k u_k e^{u_k u} - (\lambda+\delta)\sum_{k=0}^{n} D_k e^{u_k u} + \lambda \sum_{i=1}^{n} p_i \sum_{k=0}^{n} C_k \frac{\beta_i}{\beta_i+\upsilon_k}[e^{\upsilon_k b_1 + \beta_i(b_1-u)} - e^{-\beta_i u}]$$

$$+\lambda \sum_{i=1}^{n} p_i \sum_{k=0}^{n} D_k \frac{\beta_i}{\beta_i+u_k}[e^{u_k u} - e^{u_k b_1 + \beta_i(b_1-u)}] = 0. \ b_1 < u < a$$

Inspection the coefficient of the coefficient of $e^{u_k u}$ reveals that $u_0, u_1, ..., u_n$ satisfies (3.14).

We equate the coefficient of $e^{-\beta_i u}$ with 0. This leads to the condition

$$\sum_{k=0}^{n} C_k \frac{1}{\beta_i + \upsilon_k} e^{\upsilon_k b_1} = \sum_{k=0}^{n} D_k \frac{1}{\beta_i + u_k} e^{u_k b_1}, \quad i = 1, 2, \ldots n. \tag{3.21}$$

Another condition follows from the continuity of $L_2(u)$ at $u = b_1$, it follows that

$$\sum_{k=0}^{n} C_k e^{\upsilon_k b_1} = \sum_{k=0}^{n} D_k e^{u_k b_1}. \tag{3.22}$$

Finally, from the boundary condition $L_2(a) = 1$, we have

$$\sum_{k=0}^{n} D_k e^{u_k a} = 1. \tag{3.23}$$

In conclusion, (3.19) and (3.21) ~ (3.23) constitute a system of $2n+2$ linear equations

$$\begin{pmatrix} \frac{1}{\beta_1+\upsilon_0} & \cdots & \frac{1}{\beta_1+\upsilon_n} & 0 & \cdots & 0 \\ \vdots & & & & & \\ \frac{1}{\beta_n+\upsilon_0} & \cdots & \frac{1}{\beta_n+\upsilon_n} & 0 & \cdots & 0 \\ \frac{e^{\upsilon_0 b_1}}{\beta_1+\upsilon_0} & \cdots & \frac{e^{\upsilon_n b_1}}{\beta_1+\upsilon_n} & -\frac{e^{u_0 b_1}}{\beta_1+u_0} & \cdots & -\frac{e^{u_n b_1}}{\beta_1+u_n} \\ \vdots & & & & & \\ \frac{e^{\upsilon_0 b_1}}{\beta_n+\upsilon_0} & \cdots & \frac{e^{\upsilon_n b_1}}{\beta_n+\upsilon_n} & -\frac{e^{u_0 b_1}}{\beta_n+u_0} & \cdots & -\frac{e^{u_n b_1}}{\beta_n+u_n} \\ e^{\upsilon_0 b_1} & \cdots & e^{\upsilon_n b_1} & -e^{u_0 b_1} & \cdots & -e^{\upsilon_n b_1} \\ 0 & \cdots & 0 & e^{u_0 a} & \cdots & e^{u_n a} \end{pmatrix} \begin{pmatrix} C_0 \\ \vdots \\ C_n \\ D_0 \\ \vdots \\ D_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

from which the coefficients $C_0, C_1, \cdots, C_n, D_0, D_1, \cdots, D_n$ can be determined.

To explain the analytical solution, we give the following example.

Example 1. We assume that the individual claim amounts are exponentially distributed with mean $1/\beta$,

$$p(y) = \beta e^{-\beta y}, \quad y > 0.$$

Which is the special case in (3.9) with $n = 1$. In this example, we give the analytical solution of $L_1(u)$ and $L_2(u)$.\\

Applying the operator $(d/du + \beta)$ to equation (3.2), we get that $L_1(u)$ satisfies the differential equation

The solutions of above equation as follows

$$L_1(u) = A_0 e^{ru} + A_1 e^{su}, \quad 0 < u < b_1, \tag{3.24}$$

where the coefficients A_0 and A_1 are to be determined, and $s < r$ being the roots of the characteristic equation $\mu L_{1''}(u) + (\beta\mu - \lambda - \delta) L_{1'}(u) - \beta\delta L_1(u) = 0. \quad 0 < u < b_1.$

$$\mu\xi^2 + (\beta\mu - \lambda - \delta)\xi - \beta\delta = 0.$$

Substituting (3.24) into (3.2) and equating the coefficient of $e^{-\beta u}$ with 0 yields a relation between A_0 and A_1:

$$\frac{\beta}{\beta + r} A_0 + \frac{\beta}{\beta + s} A_1 = 1. \tag{3.25}$$

Applying the operator $(d/du + \beta)$ to equation (3.3), we see that $L_1(u)$ satisfies the differential equation

$$(\mu-\alpha)L_{1''}(u)+(\beta(\mu-\alpha)-\lambda-\delta)L_{1'}(u)-\beta\delta L_1(u)=0.\quad b_1<u<a.$$

Thus,

$$L_1(u)=B_0e^{wu}+B_1e^{vu},\quad b_1<u<a, \tag{3.26}$$

where the coefficients B_0 and B_1 are to be determined, and $w<v$ are the roots of the characteristic equation

$$(\mu-\alpha)\xi^2+[\beta(\mu-\alpha)-\lambda-\delta]\xi-\delta\beta=0.$$

Substitute (3.24) and (3.26) into (3.2), we have

$$A_0\frac{e^{rb_1}}{\beta+r}+A_1\frac{e^{sb_1}}{\beta+s}=B_0\frac{e^{wb_1}}{\beta+w}+B_1\frac{e^{vb_1}}{\beta+v}. \tag{3.27}$$

The boundary condition $L_1(a)=0$ implies

$$B_0e^{wa}+B_1e^{va}=0. \tag{3.28}$$

From the continuity of the function $L_1(u)$ at $u=b_1$

$$A_0e^{rb_1}+A_1e^{sb_1}=B_0e^{wb_1}+B_1e^{vb_1}. \tag{3.29}$$

From (3.27) and (3.29), we have

$$A_0=\frac{\beta+r}{r-s}(e^{wb_1}\frac{w-s}{\beta+w}B_0+e^{vb_1}\frac{v-s}{\beta+v}B_1)e^{-rb_1}$$

$$A_1=\frac{\beta+r}{s-r}(e^{wb_1}\frac{w+r}{\beta+w}B_0+e^{vb_1}\frac{v-r}{\beta+v}B_1)e^{-sb_1},$$

and from (3.28), we get

$$B_0=-e^{(v-w)a}B_1.$$

Substituting above equations into (3.27), the constants B_0 and B_1 can be solved as

$$B_0=\frac{s-r}{\beta}\frac{(\beta+u)(\beta+v)e^{(v-w)a}}{I},$$

$$B_1=\frac{r-s}{\beta}\frac{(\beta+u)(\beta+v)}{I},$$

where

$$I=(\beta+w)[(v-s)e^{-rb_1}-(v-r)e^{-sb_1}]e^{vb_1}-(\beta+v)[(w-s)e^{-rb_1}-(w-r)e^{-sb_1}]e^{wb_1+(v-w)a}.$$

Then, the constants A_0 and A_1 can be given by

$$A_0=\frac{\beta+r}{\beta}\frac{-(\beta+v)(w-s)e^{wb_1+(v-w)a}+(\beta+w)(v-s)e^{vb_1}}{I}e^{-rb_1},$$

$$A_1=\frac{\beta+s}{\beta}\frac{(\beta+v)(w-r)e^{wb_1+(v-w)a}-(\beta+w)(v-r)e^{vb_1}}{I}e^{-sb_1}.$$

From above discussion, the expression of $L_1(u)$ is given by

$$\begin{aligned}L_1(u)=&\frac{\beta+r}{\beta I}[-(\beta+v)(w-s)e^{wb_1+(v-w)a}+(\beta+w)(v-s)e^{vb_1}]e^{r(u-b_1)}\\&+\frac{\beta+s}{\beta I}[(\beta+v)(w-r)e^{wb_1+(v-w)a}-(\beta+w)(v-r)e^{vb_1}]e^{s(u-b_1)},0\le u\le b_1,\end{aligned} \tag{3.30}$$

$$L_1(u)=\frac{s-r}{\beta I}(\beta+w)(\beta+v)e^{wu+(v-w)a}+\frac{r-s}{\beta I}(\beta+w)(\beta+v)e^{vu},b_1<u<a. \tag{3.31}$$

Now, we give explicit expressions for $L_2(u)$ when the claim sizes are exponentially distributed.

Similarly, we applying the operator $(d/du+\beta)$ to equation (3.7) and (3.8), we can get that $L_2(u)$ satisfies the same differential equation as above.

Then, we have

$$L_2(u)=C_0e^{ru}+C_1e^{su}, \quad 0<u\le b_1, \tag{3.32}$$

$$L_2(u)=D_0e^{wu}+D_1e^{vu}, \quad b_1<u<a, \tag{3.33}$$

where the coefficients C_0, C_1, D_0 and D_1 are to be determined, and r, s, w and v are the same as above.

Substituting (3.32) into (3.7) equating the coefficient of $e^{-\beta u}$ with 0 yields the condition

$$\lambda\beta(\frac{C_0}{r+\beta}+\frac{C_1}{s+\beta})=0,$$

Then, we can rewrite

$$L_2(u)=\zeta[(r+\beta)e^{ru}-(s+\beta)e^{su}], \quad 0\le u\le b_1 \tag{3.34}$$

where ζ dose not depend on u. Substituting (\ref{th20}) and (\ref{th21}) into (\ref{theo4}) equating the coefficient of $e^{-\beta u}$ with 0 yields the condition

$$\zeta(e^{rb_1}-e^{sb_1})=D_0\frac{e^{wb_1}}{\beta+w}+D_1\frac{e^{vb_1}}{\beta+v}. \tag{3.35}$$

Note that $L_2(a)=1$, which implies

$$D_0e^{ua}+D_1e^{va}=1. \tag{3.36}$$

The continuity of the function $L_2(u;b_1,a)$ at $u=b_1$ yields the condition

$$\zeta[(r+\beta)e^{rb_1}+(s+\beta)e^{sb_1}]=D_0e^{wb_1}+D_1e^{vb_1}. \tag{3.37}$$

From above discussion, we have

$$D_0=\frac{\beta+w}{v-w}e^{-wb_1}[(v-r)e^{rb_1}-(v-s)e^{sb_1}]\zeta$$

$$D_1=\frac{\beta+v}{w-v}e^{-vb_1}[(w-r)e^{rb_1}-(w-s)e^{sb_1}]\zeta.$$

Substituting above equations into (3.37), the coefficient ζ can be given by

$$\zeta=\frac{(w-v)e^{w(b_1-a)+vb_1}e^{-(r+s)b_1}}{I},$$

where I is same as before.

Then, the constants D_0 and D_1 as follows

$$D_0=\frac{(\beta+w)[(v-s)e^{-rb_1}-(v-r)e^{-sb_1}]e^{vb_1-wa}}{I},$$

$$D_1=\frac{(\beta+v)[(w-s)e^{-rb_1}-(w-r)e^{-sb_1}]e^{w(b_1-a)}}{I}.$$

Thus,

$$L_2(u)=\frac{(w-v)e^{w(b_1-a)+vb_1}e^{-(r+s)b_1}}{I}[(r+\beta)e^{ru}-(s+\beta)e^{su}], \quad 0<u\le b_1. \tag{3.38}$$

Finally,

$$L_2(u)=\frac{1}{I}\{(\beta+w)[(v-s)e^{-rb_1}-(v-r)e^{-sb_1}]e^{vb_1-wa+wu}+(\beta+v)[(w-s)e^{-rb_1}-(w-r)e^{-sb_1}]e^{w(b_1-a)+vu}\},\ b_1<u<a. \quad (3.39)$$

From (3.30), (3.31), (3.38), and (3.39), we can get the Laplace transform of the exit time

$$E_u[e^{-\delta\tau}]=\frac{\beta+r}{\beta I}\{[-(\beta+v)(w-s)e^{wb_1+(v-w)a}+(\beta+w)(v-s)e^{vb_1}]e^{-rb_1}+\beta(w-v)e^{w(b_1-a)+vb_1-(r+s)b_1}\}e^{ru}$$
$$+\frac{\beta+s}{\beta I}\{[(\beta+v)(w-r)e^{wb_1+(v-w)a}-(\beta+w)(v-r)e^{vb_1}]e^{-sb_1}$$
$$+\beta(w-v)e^{w(b_1-a)+vb_1-(r+s)b_1}\}e^{su},\ if\ 0<u\le b_1,$$
$$E_u[e^{-\delta\tau}]=\frac{\beta+w}{\beta I}\{(s-r)(\beta+v)e^{va}+\beta[(v-s)e^{-rb_1}-(v-r)e^{-sb_1}]e^{b_1v}\}e^{(u-a)w}$$
$$+\frac{\beta+v}{\beta I}\{(\beta+w)(s-r)+\beta[(w-s)e^{-rb_1}-(w-r)e^{-sb_1}]e^{w(b_1-a)}\}e^{vu},if\ b_1<u<a.$$

Remark 3.2. Let us compare our results with known result.

Let $b_2\to\infty$, where the mixed dividends becomes a threshold strategy. When $a=b_2$, Laplace transform $L_1(u)=L(u)$ is same as (10.1) of Gerber and Shiu (2006a).

From the above results, we get

$$\lim_{b_2\to\infty}A_0=\frac{\beta+r}{\beta}\frac{(w-s)e^{sb_1}}{(w-s)e^{sb_1}-(w-r)e^{rb_1}},$$
$$\lim_{b_2\to\infty}A_1=\frac{\beta+s}{\beta}\frac{(r-w)e^{rb_1}}{(w-s)e^{sb_1}-(w-r)e^{rb_1}}.$$

Thus,

$$L_1(u)=\frac{1}{\beta}\frac{(\beta+r)(w-s)e^{sb_1+ru}+(\beta+s)(r-w)e^{rb_1+su}}{(w-s)e^{sb_1}-(w-r)e^{rb_1}},\quad 0<u\le b_1, \quad (3.40)$$

which is same as (10.17) in Gerber and Shiu (2006a).

Similarly,

$$\lim_{a\to\infty}B_0=\frac{r-s}{\beta}\frac{\beta+w}{(w-s)e^{-rb_1}-(w-r)e^{-sb_1}}e^{-wb_1},$$
$$\lim_{a\to\infty}B_1=0.$$

Thus,

$$L_1(u)=\frac{r-s}{\beta}\frac{\beta+w}{(w-s)e^{-rb_1}-(w-r)e^{-sb_1}}e^{w(u-b_1)},\quad u>b_1.$$

This result is same as (10.19) in Gerber and Shiu (2006a).

Remark 3.3. For some special case, above functions can be used to make a decision for insurer. For example, let $\delta=0$ and $a=b_1$, function $L_1(u)=P(T_0<T_{b_1})$ is the probability of no dividend, and $L_2(u)=P(T_{b_1}<T_0)$ is the probability of dividend will occur before ruin. Shareholders can compare two probability to make a decision for investment.

IV. Applications to dividend value function

A. Barrier strategy

In the mixed dividend problem, let $b_1=b_2=a$, which leads to the barrier dividend strategy (see Gerber and Shiu, 2004). Whenever the surplus is above the level b_2, the excess will be paid as dividends, when the surplus is below b nothing is paid out. We define the aggregate dividends paid in the time interval [0,t] by $D_b(t)$. In this special case, we have $\tilde{U}_t=U_t-D_b(t)$.

Let T_b be the ruin time of this process, then we can define the present value of all dividends until ruin T_b by $D_b = \int_0^{T_b} e^{-\delta t} dD_b(t)$, here, δ can be interpreted as the interest force. The expectation of present value of all dividends is given by

$$V_b(u,b_2) = \mathrm{E}_u[D_b].$$

The above dividend function when the process is modeled by the Brownian motion process is given by Gerber and Shiu (2004). We proved that this function can be derived by the exit time in Li et al. (2013). The following lemma follows from this study.

Lemma 4.1. For $0 \le u \le b_2$, one has $V_b(u,b_2) = \dfrac{L_2(u)}{L_{2'}(b_2)}$.

The proof of this lemma is similar to Theorem 9 in Li et al. (2013).

Example 2. This example assumes that the process is compound Poisson risk model and the individual claim amounts are exponentially distributed. From above lemma and the previous discussion, we can get the expectation of present value of dividend function $V_b(u,b_2)$. The analytical solution of exit time is given by Example 1, and we have

$$V_b(u,b_2) = \frac{(r+\beta)e^{ru} - (s+\beta)e^{su}}{r(r+\beta)e^{rb_2} - s(s+\beta)e^{sb_2}}, 0 < u \le b_2$$

where all the parameters are given in Section 2.

B. Threshold strategy

In this case, let $b_2 \to +\infty$, which leads to the threshold dividend strategy. When the surplus is above b_1, dividends are paid at a constant rate α, and no dividends are paid whenever the surplus is below b_1. In this special case, we define the aggregate dividends paid in the time interval [0,t] by $D_d(t)$, and the surplus is $\tilde{U}_t = U_t - D_d(t)$. Similarly, let T_d defines the ruin time of this process. The expectation of present value of dividends is given by

$$V_d(u,b_1) = \mathrm{E}[D_d] = \mathrm{E}\left[\alpha \int_0^{T_d} e^{-\delta s} I(\tilde{U}_t > b_1) ds\right],$$

where $I(\cdot)$ is the indicator function. To calculate this function by exit time, we need to give the following exit time

$$\tau_a^- = \inf\{t : \tilde{U}_t \le a\},$$

$$L_3(u) = E_u[e^{-\delta \tau_a^-}].$$

Note that we assume $b_2 \to \infty$, and the integro-differential equation satisfied by $L_3(u)$ can be given by the following Lemma, which the proof is similar with Theorem 3.1.

Lemma 4.2. $L_3(u), u > b_1$ satisfies the the following integro-differential equation:

$$(\mu - \alpha)L_{3'}(u) - (\lambda + \delta)L_3(u) + \lambda \int_0^u L_3(u-y)p(y)dy + \lambda[1 - P(u)] = 0,$$

with $L_3(b_1) = 1$ and $L_3(+\infty) = 0$.

Note that the threshold dividend function $V_d(u,b_1)$ under Poisson process is studied by Gerber and Shiu (2006a), and Li et al.(2013) proved that this function can be calculated by exit time when the risk model are one-diffusion processes. This result can be extended to the Poisson risk model.

Lemma 4.3. In this threshold dividend problem, when $u \in [0,b_1]$, it has

$$V_d(u;b_1) = \frac{\alpha}{\delta} \frac{L_{3'}(b_1)L_2(u)}{L_{3'}(b_1) - L_2'(b_1)},$$

and when $u > b_1$, it has

$$V_d(u;b_1)=\frac{\alpha}{\delta}+\frac{\alpha}{\delta}\frac{L_{2'}(b_1)L_3(u)}{L_{3'}(b_1)-L_2'(b_1)},$$

where the $L_2(u)$ is calculated under $a=b_1$ and $b_2\to+\infty$.

Example 3. This example gives the threshold dividend function when the process is compound Poisson risk model and the individual claim amounts are exponentially distributed. We first calculate the expression of $L_3(u)$, and it follows from the similar program as Example 1, we have

$L_3(u)=e^{w(u-b_1)}, u\geq b_1.$

When $a=b_1$, (3.38) gives the following expression:

$$L_2(u)=\frac{e^{-(r+s)b_1}[(r+\beta)e^{ru}-(s+\beta)e^{su}]}{(r+\beta)e^{-sb_1}-(s+\beta)e^{-rb_1}},\quad 0<u\leq b_1.$$

Consider the boundary condition $L_1(0)=1$ of the Remark 3.2 in equation (3.40), which gives the following equation:

$$(\beta+r)(w-s)e^{sb_1}+(\beta+s)(r-w)e^{rb_1}=\beta((w-s)e^{sb_1}-(w-r)e^{rb_1}).$$

According to the results of Lemma 4.3, we substitute $L_2(u)$ and $L_3(u)$ into $V_d(u,b_1)$ and use above equation arrange it, and it has

$$V_d(u;b_1)=\frac{-\alpha w}{\delta\beta}\frac{(r+\beta)e^{ru}-(s+\beta)e^{su}}{(r-w)e^{rb_1}-(s-w)e^{sb_1}},\ 0<u\leq b_1,$$

and

$$V_d(u;b_1)=\frac{\alpha}{\delta}(1-e^{w(u-b)})+V_d(b_1;b_1)e^{w(u-b)},\ x>b_1,$$

which is same with the results (6.14) and (6.15) in Gerber and Shiu (2006a).

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