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Comparison of the Area of the Triangle Formed from the Symmedian Line and the Median Line

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Abstract **—** *This paper discusses the area of a triangle formed from the symmedian line and the median line. This discussion includes determining the length of the symmedian line and the comparison of the area of a triangle formed from the symmedian line and the median line. The proof is done by using a very simple method, which is by comparing the bases of the triangles formed and using several theorems such as Stewart's theorem and Steiner's theorem.*

Keywords — *Area comparison of triangle, median, symmedian, symmedian line length.*

I. INTRODUCTION

Within the triangle, there are several special lines that are often discussed, including the perpendicular line, bisector line and median line [2, 12, 14, 15]. The perpendicular line of a triangle is a line that passes through one of the vertices of the triangle and is perpendicular to the side in front of it. A bisector line of a triangle is a line drawn from one corner of the triangle so that it divides the angle into two equal angles. A median line divides the side in front of the corner into two equal parts. In the development of geometry regarding this triangle, a line is found which is a reflection of the median line to the bisector. This line is called the symmedian line [3, 5, 8-10, 13, 15-16]

Previous discussions about symmedians have discussed a lot about symmedian points, which are called Lemoine points, such as a study of the First Lemoine Circle [6, 8, 15]. This theorem proves that if P is the symmedian point of triangle and three parallel lines are drawn to the sides of the triangle where all three lines through point symmedian P then the six points all lie on one circle. Similarly, for Second Lemoine Circle that constructed by drawing three anti parallel lines to the sides of the triangle are through the symmedian point so it will intersect in six points with a side of the triangle. Then, the sixth points will be on one circle [6, 8, 15]. Furthermore, there is Third Lemoine Circle. Let O be a symmedian point on triangle ABC. Then the triangle is partitioned into three triangles, namely ΔBCO, ΔACO and ΔABO. From each triangle, a circle can be made with the center points P, Q, and R. If the points P, Q and R are connected by a line segment, it will form ΔPQR. Additionally, from constructing the three circles of the ΔBCO, ΔACO and ΔABO, will form the intersection of the circle with side and side triangle extension ABC at six points [4, 6, 15]. Futhermore, if M is the centroid point of triangles ABC, and L are the center points of the third Lemoine's circle while K is the simedian point of triangle ABC, then the three points are collinear [15].

This paper discusses the symmedian length and the comparison of the areas of a triangle formed from symmedian and median lines in a very simple way. The length of this symmedian line can be determined by using Stewart's theorem and Steiner's theorem. The further discussion in this paper is about the the comparison of the area of the triangles formed from the symmedian and median lines. In order to determine the area of this triangle, a comparison of the sides of the base is used.

II. LITERATURE REVIEW

Several papers have discussed the previous definition of symmedian [3, 5, 8-10, 13, 15-16].

Definition. 1 (Symmedian Line). Given triangle ABC where *a*, *b* and *c* respectively are the side lengths of BC, AC, and AB. If A_m and A_b are respectively the median line and the bisector line drawn from the angle A, then the reflection of the line A_m against A_b produces the line A_s, which is called a symmedian line.

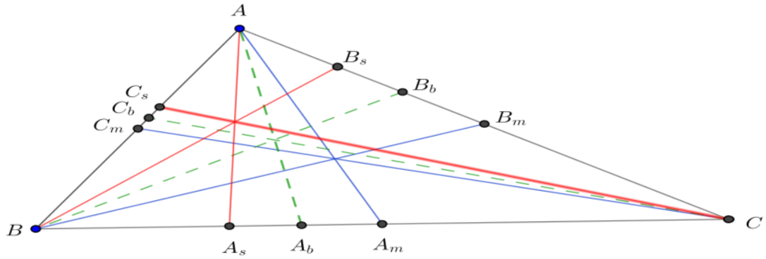

Figure 1. Symmedian line construction on the ABC triangle

In the same way, a symmedian line B_s can be formed from angle B and a symmedian line C_s from angle C, as shown in Figure 1, and that three symmedians meet at a point [7]. Symmedian lines do not always exist in triangles. For example, symmedians are not found in isosceles triangles. This is because the median and bisector lines in the isosceles triangle coincide. Because the symmedian line is a reflection of the median line against the bisector, this line also coincides with the median and bisector lines. [8]. To determine the length of the symmedian line, the Stewart's theorem is used.

Theorem 1. (Stewart's Theorem) Given an ABC triangle, on the side of BC a point D is made with the ratio $BD:DC = r:s$. If the side length of AD is p then

$$a(p^2 + rs) = b^2 r + c^2 s.$$

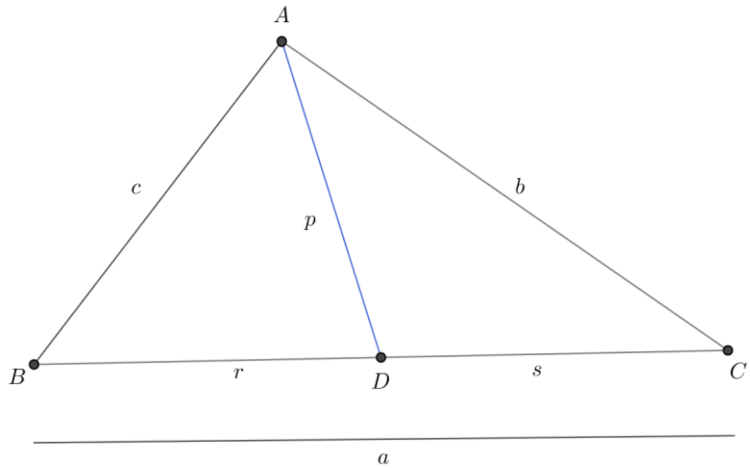

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Figure 2. Stewart's theorem

Proof. The proof can be seen in [1, 2 and 11]. ■

Furthermore, to determine the symmedian length of the angle A, that is A_s by using the Stewart's theorem, it is necessary to determine the length of BA_s and CA_s by using the Steiner's theorem.

Theorem 2. (Steiner's Theorem**)** At any point in ΔABC, if D is a point that lies on the line BC and if the reflection of the line AD on the angle bisector A intersects the line BC at point E, then

$$\frac{BD}{CD} \cdot \frac{BE}{CE} = \frac{(AB)^2}{(AC)^2}.$$

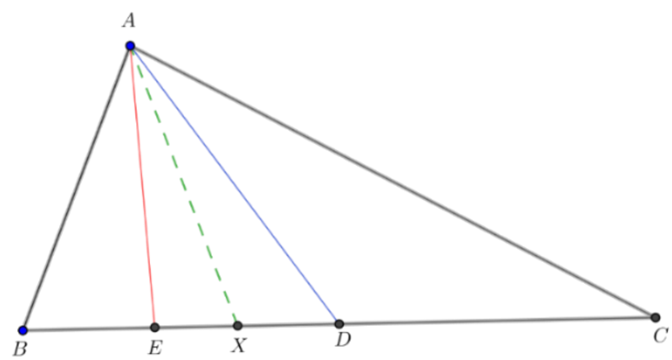

Figure 3. Steiner's theorem

Proof. The proof is discussed in [8]. ■

III. RESULTS

A. Symmedian Line Length

In any ABC triangle, if from point A, a median line AA_m is formed and also bisector line AA_b, then AA_s is a symmedian line which is a reflection of the AA_m line against the AA_b line. The following will determine the length of the symmedian AA_s line using Stewart's Theorem. To determine the length of this AA_s symmedian line, it is necessary to determine the length of the BA_s and CA_s lines first as in the following theorems:

Theorem 3. In any ABC triangle, with $BC = a$, $AC = b$ and $AB = c$, the median line AA_m, the dividing line AA_b, and the symmedian line AA_s are formed from angle A. Then the line lengths CA_s and BA_s are respectively

(i) $CA_s = \frac{ab^2}{b^2 + c^2}$,

(ii) $BA_s = \frac{ac^2}{b^2 + c^2}$.

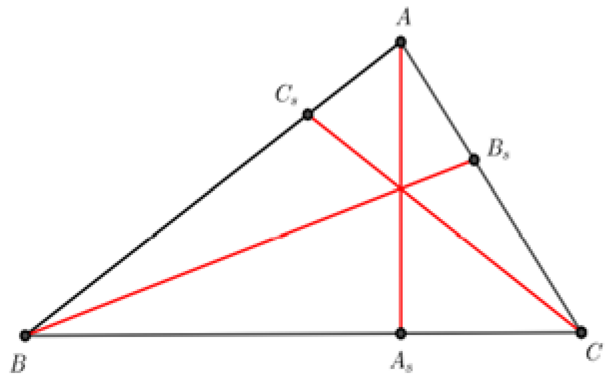

Figure 4. Determine BA_s and CA_s line symmedian on ΔABC

Proof. From Figure 1, if the median and bisector lines are removed from each angle A, B and C, then Figure 4 appears. Figure 4 shows that AA_s, BB_s and CC_s lines are symmedian lines which are formed from angles A, B and C.

Based on the Steiner's theorem,

$$\frac{BA_s}{CA_s} \cdot \frac{BA_m}{CA_m} = \frac{AB^2}{CA^2} = \frac{c^2}{b^2},$$

$$\frac{BA_s}{c^2} = \frac{CA_s}{b^2}. \quad (1)$$

By substituting $BA_s = a - CA_s$ to equation (1) yields

$$\frac{a - CA_s}{c^2} = \frac{CA_s}{b^2}$$

$$b^2(a - CA_s) = c^2 CA_s$$

$$ab^2 = c^2 CA_s + a - b^2 CA_s$$

$$CA_s = \frac{ab^2}{b^2 + c^2}. \quad (2)$$

Furthermore, to determine the length of BA_s, then in the same way, substituting $C = a - BA_s$ gives

$$BA_s = \frac{ac^2}{b^2 + c^2}. \quad (3)$$

After the length of BA_s and CA_s are obtained, then the length of the symmedian line can be determined.

Theorem 4. In any ABC triangle, with $BC = a$, $AC = b$ and $AB = c$, if it is formed median line AA_m, bisector line AA_b and symmedian line AA_s from angle A, then the length of AA_s is

$$AA_s = \frac{bc}{b^2 + c^2}\sqrt{2(b^2 + c^2) - a^2}\,.$$

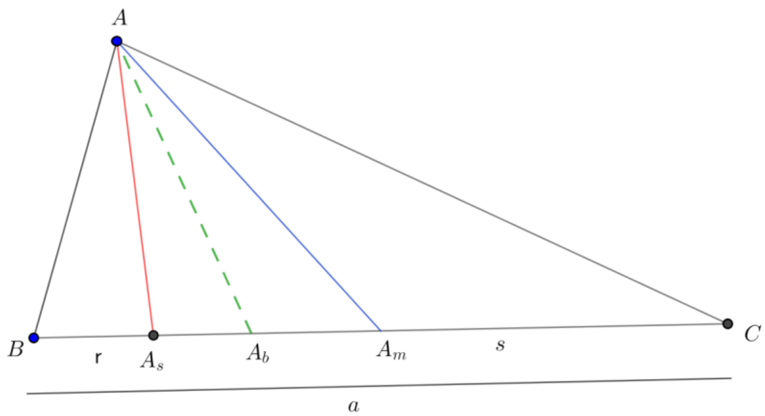

Figure 5. The length of symmedian line on ΔABC

Proof. Figure 5 is resulted from taking specifically the median, bisector and symmedian line of angle A in Figure 1. Based on Stewart's theorem

$$AA_s = \frac{b^2 r + c^2 s - ars}{a}\,. \tag{4}$$

By substituting equations (2) and (3) into (4), it is obtained

$$AA_s = \sqrt{\frac{b^2\left(\frac{ac^2}{b^2 + c^2}\right) + c^2\left(\frac{ab^2}{b^2 + c^2}\right) - a\left(\frac{ac^2}{b^2 + c^2}\right)\left(\frac{ab^2}{b^2 + c^2}\right)}{a}}$$

$$= \sqrt{\frac{2ab^2c^2}{a(b^2 + c^2)} - \frac{a^3b^2c^2}{a(b^2 + c^2)^2}}$$

$$= \sqrt{\frac{b^2c^2(2(b^2 + c^2)) - a^2}{(b^2 + c^2)^2}}$$

$$= \frac{bc}{b^2 + c^2}\sqrt{2(b^2 + c^2) - a^2}$$

B. Comparison of the Area of the Triangle Formed from the Symmedian Line and the Median Line

To determine the ratio of the area of the triangle formed from the symmedian line and the median line on ABC triangle, it is only needed to compare the bases of each of these triangles because all the three are in the same triangle, namely triangle ABC, as shown in Figure 6.

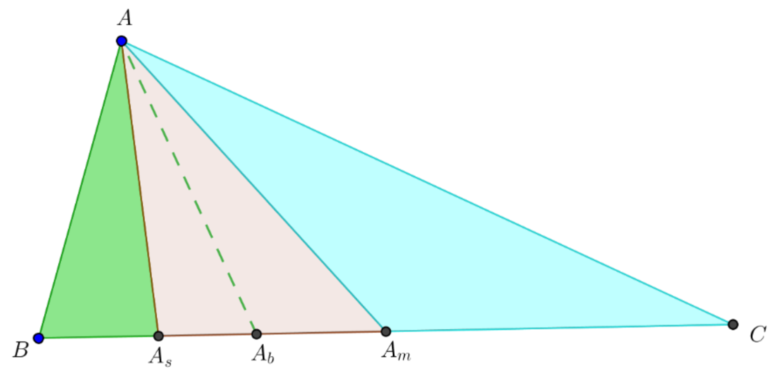

Figure 6. The areas of triangles between symmedian line and median line on ΔABC

In Figure 6, the symmedian line AA_s and the median line AA_m form three triangles, namely ΔBAA_s, ΔA_sAA_m dan ΔCAA_m. Since the areas of these triangles are going to be compared by comparing their bases, first the length of the sides of the triangle must be determined. Previously the length of ΔBA_s is obtained as in equation (3). Next, the length of A_sA_m and CA_m will be determined.

Theorem 5. In any ABC triangle, with $BC = a$, $AC = b$ and $AB = c$, each A_m and A_s are the median points and the intersection of the symmedian line AA_s with BC, respectively. If A_sA_m is a line formed from point A_m and point A_s, then the length of the line A_sA_m is

$$A_sA_m = \frac{a(b^2 - c^2)}{2(b^2 + c^2)}.$$

Proof. From Figure 6, it is known that the length of the A_sA_m line is the difference between the BA_m and BA_s line lengths, namely

$$A_sA_m = BA_m - BA_s = \frac{1}{2}a - \frac{ac^2}{b^2 + c^2}$$

$$= \frac{a(b^2 + c^2) - 2ac^2}{2(b^2 + c^2)}$$

$$= \frac{ab^2 + ac^2 - 2ac^2}{2(b^2 + c^2)}$$

$$A_sA_m = \frac{a(b^2 - c^2)}{2(b^2 + c^2)}. \quad (5)$$

Since the length of $CA_m = 1/2BC$ then

$$CA_m = 1/2a. \quad (6)$$

Theorem 6. In any ABC triangle, if from angle A is drawn a symmedian line AA_s and median line AA_m, then ΔBAA_s, $\Delta A_sAA_m$and ΔCAA_m are formed with the ratio of areas :

$$L\Delta BAA_s : L\Delta A_sAA_m : L\Delta CAA_m = \frac{c^2}{b^2 + c^2} : \frac{b^2 - c^2}{2(b^2 + c^2)} : \frac{1}{2}.$$

Proof. Consider ΔBAA_s, $\Delta A_sAA_m$and ΔCAA_m. The three triangles have the same height because they are in ΔABC. Thus, the ratio of the area is obtained by comparing the bases line of each of the triangles, namely

$$L\Delta BAA_s : L\Delta A_sAA_m : L\Delta CAA_m = BA_s : A_sA_m : CA_m. \quad (7)$$

Substituting equations (3), (5) and (6) into (7) yields

$$L\Delta BAA_s : L\Delta A_sAA_m : L\Delta CAA_m = \frac{ac^2}{b^2+c^2} : \frac{a(b^2-c^2)}{2(b^2+c^2)} : \frac{1}{2}a\,,$$

$$L\Delta BAA_s : L\Delta A_sAA_m : L\Delta CAA_m = \frac{c^2}{b^2+c^2} : \frac{(b^2-c^2)}{2(b^2+c^2)} : \frac{1}{2}. \quad (8)$$

To simplify the comparation form of the three triangles, two triangles will be compared, namely ΔBAA_s and ΔA_sAA_m and also ΔA_sAA_m and ΔCAA_m. From equation (8),

$$L\Delta BAA_s : L\Delta A_sAA_m = \frac{c^2}{b^2+c^2} : \frac{b^2-c^2}{2(b^2+c^2)}$$
$$= c^2 : \frac{b^2-c^2}{2},$$

and also

$$L\Delta A_sAA_m : L\Delta CAA_m = \frac{(b^2-c^2)}{(b^2+c^2)} : 1.$$

VI. CONCLUSIONS

Symmedian lines are formed from the reflection of the median line against the bisector. To determine the length of this line, Stewart's theorem and Steiner's theorem can be used. Comparing the area of the triangle formed from the symmedian line and the median line can be determined by using the comparison of the bases of the triangle.

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