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Star-jacobian Matrix of Star with α Coefficient $\bigstar_\alpha$

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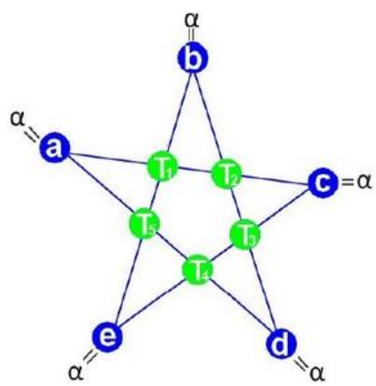
***Abstract** - In this paper, we introduce a new data type of Star-matrices and we define a simple basis Star-Jacobian vector that enables the representation of Star-Jacobian matrices (directly, indirectly) composed of the solution of a Star-System with α coefficient. The resolution of Star-Systems is laid in a basis of the known Gauss' method (method of exception of unknown values) for solution of system of linear equations. When we solve a linear Star-System, we will place the 5 Star-Vectors that are the solution (linearly independent) in the columns of a matrix. The so-called fundamental matrix of the Star-System (5x5). Thereafter, we start with two examples with detailed solutions are presented. This can, in particular, be exploited to obtain arithmetic properties for classes of Star-Matrices. According to a number of different studies, we also note that there is a constant coefficient matrix $C_\alpha^\bigstar$ if we multiply that matrix by $M^{\bigstar+}$ (Star-Matrix directly). Then you will get the matrix $M^{\bigstar-}$ (Star-Matix indirectly). $C_\alpha^\bigstar$* is an orthogonal matrix ($^t C_\alpha^\bigstar = (C_\alpha^\bigstar)^{-1}$). On the other hand, *we study the relationship between two Star-Jacobian matrices of Star with α coefficient, a relationship refers to the correspondence between two Star-Matrices. The results of calculations show that the products between two Star- Jacobian Matices of two Star-System ($\bigstar\alpha_1$, $\bigstar\alpha_2$) with α_1 and α_2 coefficient (directly-indirectly) or (directly-directly) or (indirectly-indirectly) are diagonalizable.*

Keywords - *Coefficient star-matrix, algebra matrix, algebra linear equations matrix, Star-Jacobian matrix, Star.*

I. INTRODUCTION

A star with α coefficient is composed of five numbers outside a, b, c, d, e and five numbers inside T_1, T_2, T_3, T_4, T_5, These last five numbers are written in the form of 5-tuple (T_1, T_2, T_3, T_4, T_5) [1].

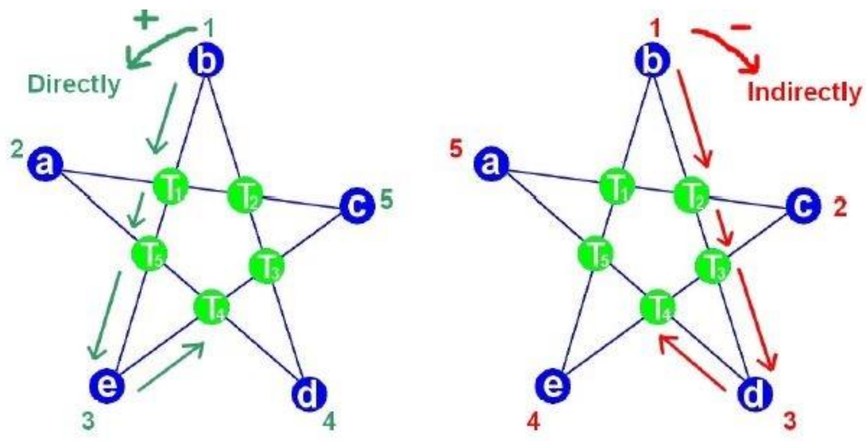


(Fig 1)

In addition to having the sum α in each line.

The scalars α are called the star coefficient if α is a solution of equation: $\alpha = T_1(\alpha)+T_2(\alpha)+T_3(\alpha)+T_4(\alpha)+T_5(\alpha)$ (Noted by $\alpha^\bigstar$), a vector (T_1, T_2, T_3, T_4, T_5) is called a solution vector of this Star-System with coefficient α in five unknowns.

We will use the convention here that the star $\bigstar_\alpha$ has a positive or a negative (Figure 2) orientation besides orientation of a star $\bigstar_\alpha$. Another way to think of a positive orientation is that as we traverse the path following the positive orientation the star $\bigstar_\alpha$ must always be on the left (that is, one may also speak of orientation of a 5×5 matrix, polynomial of degree 5, etc.). It is therefore possible to orient a star with α coefficient $\bigstar_\alpha$ in two different ways, directly different ways, directly and indirectly [2]:



(Fig 2)

The present paper is organized as follows:
In Section 2, we present some preliminary results and notations that will be useful in the sequel. In Section 3, we study the relationship between two star-Jacobian matrix directly and star-Jacobian matrix indirectly of star-system. Finally, in section 4, we study the relationship between two star-matrix (directly, indirectly) of two star-system with coefficient (α_1, α_1).

II. SOME BASIC DEFINITIONS AND NOTATIONS

In this section, we introduce some notations and star-system with α coefficient defined [1, 2].

1. A star-system with α coefficient:

Definition 1. Let a, b, c, d, e and α be real numbers, and let T_1, T_2, T_3, T_4, T_5 be unknowns (also called variables or indeterminates). Then a system of the form

$$\begin{cases} T1 + T2 = \alpha - a - c \\ T2 + T3 = \alpha - b - d \\ T3 + T4 = \alpha - c - e \\ T4 + T5 = \alpha - a - d \\ T5 + T1 = \alpha - b - e \end{cases}$$

is called a star-system with α coefficient in five unknowns. We have also noted ★[a; b; c; d; e; α] = α. The scalars a, b, c, d, e are called the coefficients of the unknowns, and α is called the constant "Chaff" of the star-system in five unknowns.
A vector (T_1, T_2, T_3, T_4, T_5) in R^5 is called a star-solution vector of this star-system if and only if ★[a; b; c; d; e; α] = α.
The solution of a Star-system is the set of values for T_1, T_2, T_3, T_4 and T_5 that satisfies five equations simultaneously.

2. A star-element

A star-element is a term of the five-tuple (T_1, T_2, T_3, T_4, T_5) solution of a star-system ★[a; b; c; d; e; α] = α, wher (T_1, T_2, T_3, T_4, T_5) in R^5. There are many methods to calculate for solution in [9-13].

3. Star-Coefficient or Constant "Chaff"

The star-Coefficient or Constant "Chaff" [1] is also noted by $\alpha^\star$ and is a solution of equation:

$$\alpha = T_1(\alpha) + T_2(\alpha) + T_3(\alpha) + T_4(\alpha) + T_5(\alpha),$$

wher (T_1, T_2, T_3, T_4, T_5) is solution of a star-system: ★[a; b; c; d; e; α] = α.

4. Star-Matrix

The star-system with α coefficient [1] can be written in matrix form $M^\star T = C_\alpha$
Where

$$M^{\star} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

vector $T = (T_1; T_2; T_3; T_4; T_5)$

$$\text{and } C_\alpha = \begin{pmatrix} \alpha - a - c \\ \alpha - b - d \\ \alpha - c - e \\ \alpha - a - d \\ \alpha - b - e \end{pmatrix}.$$

$M^{\star}$ or M_{Staris} called the star-Matrix of the star-system with α coefficient (★[a; b; c; d; e; α] = α).

$M^{\star}$ a matrix is said to be of dimension 5×5. A value called the determinant of $M^{\star}$, that we denote by $|M^{\star}|$ or $|M_{Staris}|$, corresponds to square matrix $M_{\star}$ [3, 4]. Consequently, the determinant of $M^{\star}$ is $|M^{\star}| = 2$.

5. Set-Star

The set-star is constructed from the solution set of linear star-system with α coefficient (★[a; b; c; d; e; α] = α). The Set-star will be noted by $S^{\star}$.

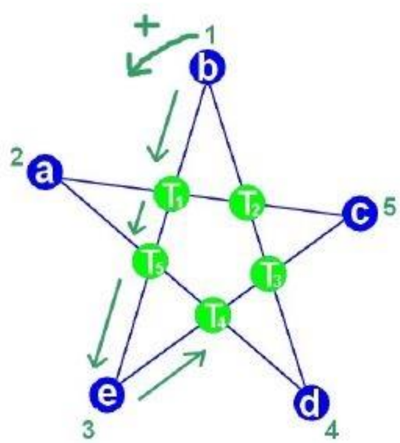
6. Star-System equivalent

Equivalent Star-Systems [1] are those systems having exactly same solution, i.e. Two star-systems are equivalent if solution of on star-system is the solution of other, and vice-versa.

7. Orientation of a Star with α coefficient

In [2], we choose two directions of travel on this Star with α coefficient can be classified as negatively oriented (clock-wise), positively oriented (counterclockwise).

Where $\star^{+}$ is a star oriented countreclockwise (positively oriented):



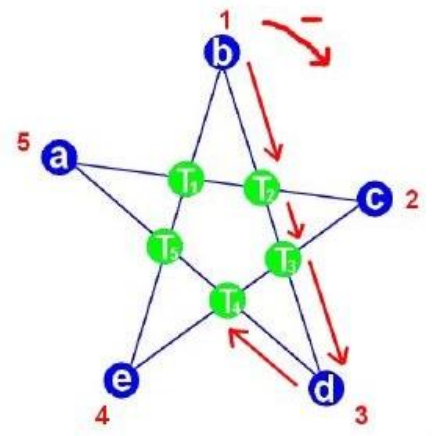
(Fig 3)

In the first case, one obtains a new matrix noted $M^{\star+}$.

Called the star-matrix directly.

$$M^{\bigstar+} = \begin{pmatrix} b & a & e & d & c \\ T1 & T5 & T4 & T3 & T2 \\ T5 & T4 & T3 & T2 & T1 \\ e & d & c & b & a \\ T4 & T3 & T2 & T1 & T5 \end{pmatrix}.$$

We note ★⁻ is a star oriented clockwise (negatively oriented):



(Fig 4)

In the second case, one obtains a new matrix noted $M^{\bigstar-}$. Called the star-matrix indirectly.

$$M^{\bigstar-} = \begin{pmatrix} b & c & d & e & a \\ T2 & T3 & T4 & T5 & T1 \\ T3 & T4 & T5 & T1 & T2 \\ e & d & c & b & a \\ T4 & T5 & T1 & T2 & T3 \end{pmatrix}$$

8. Parametrized Curves

A parametrized differentiable curve is simply a specific subset of R^5 with which certain aspects of differntial calculus can be applied.

Definition 2. A parametrized differentiable curve is a differentiable map α: I →R^5 of an open interval I = (a; b) of the real line R in to R^5.

9. Regular Curves

A parametrized differentiable curve α: I →R^5, We call any point that satisfies α'(t) = 0 a singular point and we will ristrict our study to curves without singular points.

Definition 3. A parametrized differentiable curve is a differentiable α: I →R^5 is said to be regular if α'(t) ≠ 0 for all t ∈ I (see [7,8]).

10. Parametric Arclength: Generalized, a parametric arclength starts with a parametric curve in R^5. This is given by some parametric equations $T_1(t)$; $T_2(t)$; $T_3(t)$; $T_4(t)$; $T_5(t)$, where the parameter t ranges over some given interval. The following formula computes the length of the arc between two points a, b.

Lemma 1: Consider a parametric curve ($T_1(t)$; $T_2(t)$; $T_3(t)$; $T_4(t)$; $T_5(t)$), where t ∈ (a; b). The length of the arc traced by the curve (see [5, 6]), as t ranges overt (a; b) is

$$L=\int_a^b \sqrt{(T'1(t))^2 + (T'2(t))^2 + (T'3(t))^2 + (T'4(t))^2 + (T'5(t))^2}dt$$

10. Orthogonal Matrices:

M 5×5 matrix is orthogonal if $M^t M = I_5$.

Recall the basic property of the transpose (for any M 5×5 matrix): $\forall a,b \in R^5$, $Ma\cdot b = a\cdot M\,{}^t b$.

It implies that requiring M to have the property: $\forall a,b \in R^5$, $Ma\cdot Mv = a\cdot b$,.

Is the same as requiring: (see [15, 16]).

$$\forall a,b \in R^5,\ a\cdot M^t Mb = a\cdot b,$$

Thereafter we start with several examples with detailed solutions are presented.

III. RELATIONSHIP BETWEEN TWO STAR-JACOBIAN MATRIX JM★+ DIRECTLY AND JM★- INDIRECTLY OF STAR WITH α COEFFICIENT

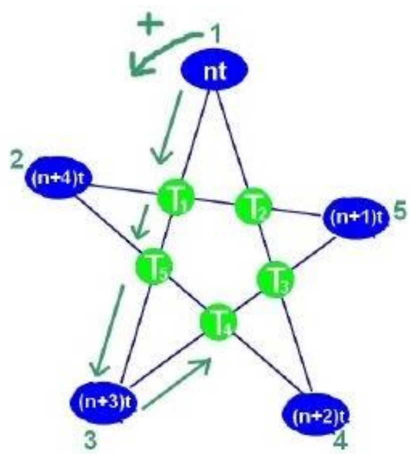
In this section, we study the relationship between two star-Jacobian matrix $JM^{\star+}$ directly and $JM^{\star-}$ indirectly of star-system ★[a; b; c; d; e; α] = α.

1. Star-Jacobian matrix of star-system:

$$\star_1[nt, (n+1)t, (n+2)t, (n+3)t, (n+4)t; \alpha_1] = \alpha_1.$$

We consider two star-matrix $M_1^{\star+}$ directly and $M_1^{\star-}$ indirectly of star-system: $\star_1$[nt, (n+1)t, (n+2)t, (n+3)t, (n+4)t; α_1] = α_1, where the star-Coefficient $\alpha_1^{\star} = \frac{10n+20}{3}t$

$\star_1$[nt, (n+1)t, (n+2)t, (n+3)t, (n+4)t; α_1] = α_1 is a star oriented countre-clockwise (positively oriented):

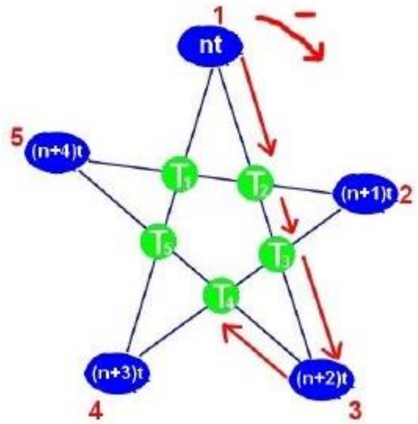


(Fig 5)

In the first case, one obtains a star matrix directly noted $M_1^{\star+}$.

$$M_1^{\star+} = \begin{pmatrix} nt & (n+4)t & (n+3)t & (n+2)t & (n+1)t \\ \frac{1}{3}(2n+4)t & \frac{1}{3}(2n+7)t & \frac{1}{3}(2n-5)t & \frac{1}{3}(2n+13)t & \frac{1}{3}(2n+1)t \\ \frac{1}{3}(2n+7)t & \frac{1}{3}(2n-5)t & \frac{1}{3}(2n+13)t & \frac{1}{3}(2n+1)t & \frac{1}{3}(2n+4)t \\ (n+3)t & (n+2)t & (n+1)t & nt & (n+4)t \\ \frac{1}{3}(2n-5)t & \frac{1}{3}(2n+13)t & \frac{1}{3}(2n+1)t & \frac{1}{3}(2n+4)t & \frac{1}{3}(2n+7)t \end{pmatrix}$$

In the second case, $\star_1$[nt, (n+1)t, (n+2)t, (n+3)t, (n+4)t; α_1] = α_1 is a star oriented clockwise:



(Fig 6)

one obtains a star matrix indirectly noted $M_1^{\star -}$.

$$M_1^{\star -} = \begin{pmatrix} nt & (n+1)t & (n+2)t & (n+3)t & (n+4)t \\ \frac{1}{3}(2n+1)t & \frac{1}{3}(2n+13)t & \frac{1}{3}(2n-5)t & \frac{1}{3}(2n+7)t & \frac{1}{3}(2n+4)t \\ \frac{1}{3}(2n+13)t & \frac{1}{3}(2n-5)t & \frac{1}{3}(2n+7)t & \frac{1}{3}(2n+4)t & \frac{1}{3}(2n+1)t \\ (n+2)t & (n+3)t & (n+4)t & nt & (n+1)t \\ \frac{1}{3}(2n-5)t & \frac{1}{3}(2n+7)t & \frac{1}{3}(2n+4)t & \frac{1}{3}(2n+1)t & \frac{1}{3}(2n+13)t \end{pmatrix}$$

$$\det(M_1^{\star +}) = \det(M_1^{\star -}) = \frac{n+2}{3} 1250 \times t^5$$

$$M_1^{\star +}(n,t) \times C\alpha_1^{\star}(n,t) = M_1^{\star -}(n,t),$$

we get the Chaff-matrix

$$C\alpha_1^{\star} = \begin{pmatrix} \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{-3}{5} \\ \frac{-3}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-3}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{-3}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{-3}{5} & \frac{2}{5} \end{pmatrix}$$

Theorem 1. For all (n; t) ∈ R×R-{0},

1) There exists a constant matrix « $C\alpha_1^{\star}$» that is wholly independent of (n; t), check for equality: $M_1^{\star +}(n,t) \times C\alpha_1^{\star} = M_1^{\star -}(n,t)$.
2) $\mathrm{Det}(C\alpha_1^{\star}) = |C\alpha_1^{\star}| = 1$.
3) ${}^{t}C\alpha_1^{\star} = (C\alpha_1^{\star})^{-1}$.
4) $M_1^{\star -}(n,t) \times {}^{t}C\alpha_1^{\star} = M_1^{\star +}(n,t)$.

Definition 4. Suppose $f^{\star}$: R → R^5 is a function such that each of its first-order partial derivatives exist on R. This function takes a point t ∈ R as input and produces the vector $f^{\star}(t) \in R^5$ as output. Then the Star-Jacobian vector of $f^{\star}$ is defined to be an 5×1 vector, denoted by $\mathbf{J} f^{\star}$, whose (i, 1)th entry is $\mathbf{J} f^{\star} = (\frac{\partial f1}{\partial t}, \frac{\partial f1}{\partial t}, \frac{\partial f1}{\partial t}, \frac{\partial f1}{\partial t}, \frac{\partial f1}{\partial t})$. (see [17])

On the other hand

$$f_1^{\bigstar}(t) = \left(\tfrac{1}{3}(2n+4)t, \tfrac{1}{3}(2n+1)t, \tfrac{1}{3}(2n+13)t, \tfrac{1}{3}(2n-5)t, \tfrac{1}{3}(2n+7)t\right)$$

$f_1^{\bigstar}(t)$ is solution of a star-system $\bigstar_1$[nt, (n+1)t, (n+2)t, (n+3)t, (n+4)t; α_1] = α_1.
We obtain the Star-Jacobian vector of $f_1^{\bigstar}(t)$:

$$Jf_1^{\bigstar}(t) = \left(\tfrac{1}{3}(2n+4), \tfrac{1}{3}(2n+1), \tfrac{1}{3}(2n+13), \tfrac{1}{3}(2n-5), \tfrac{1}{3}(2n+7)\right)$$

Definition 5. Suppose $M^{\bigstar}$: R → $R^5 \times R^5$ is a Matrix function such that each of its first-order partial derivatives exist on R. This Matrix function takes a point t ∈ R as input and produces the Matrix $M^{\bigstar}(t) \in R^5 \times R^5$ as output.
Then the Star-Jacobian Matrix of $M^{\bigstar}$ is defined to be an 5×5 Matrix, denoted by $JM^{\bigstar}$, whose (i; j)th entry is

$$JM^{\bigstar} = \begin{pmatrix} \frac{\partial M^{\bigstar}11}{\partial t} & \frac{\partial M^{\bigstar}12}{\partial t} & \frac{\partial M^{\bigstar}13}{\partial t} & \frac{\partial M^{\bigstar}14}{\partial t} & \frac{\partial M^{\bigstar}15}{\partial t} \\ \frac{\partial M^{\bigstar}21}{\partial t} & \frac{\partial M^{\bigstar}22}{\partial t} & \frac{\partial M^{\bigstar}23}{\partial t} & \frac{\partial M^{\bigstar}24}{\partial t} & \frac{\partial M^{\bigstar}25}{\partial t} \\ \frac{\partial M^{\bigstar}31}{\partial t} & \frac{\partial M^{\bigstar}32}{\partial t} & \frac{\partial M^{\bigstar}33}{\partial t} & \frac{\partial M^{\bigstar}34}{\partial t} & \frac{\partial M^{\bigstar}35}{\partial t} \\ \frac{\partial M^{\bigstar}41}{\partial t} & \frac{\partial M^{\bigstar}42}{\partial t} & \frac{\partial M^{\bigstar}43}{\partial t} & \frac{\partial M^{\bigstar}44}{\partial t} & \frac{\partial M^{\bigstar}45}{\partial t} \\ \frac{\partial M^{\bigstar}51}{\partial t} & \frac{\partial M^{\bigstar}52}{\partial t} & \frac{\partial M^{\bigstar}53}{\partial t} & \frac{\partial M^{\bigstar}54}{\partial t} & \frac{\partial M^{\bigstar}55}{\partial t} \end{pmatrix}$$

Here is an example of star-matrix:

$$M_1^{\bigstar+} = \begin{pmatrix} nt & (n+4)t & (n+3)t & (n+2)t & (n+1)t \\ \frac{1}{3}(2n+4)t & \frac{1}{3}(2n+7)t & \frac{1}{3}(2n-5)t & \frac{1}{3}(2n+13)t & \frac{1}{3}(2n+1)t \\ \frac{1}{3}(2n+7)t & \frac{1}{3}(2n-5)t & \frac{1}{3}(2n+13)t & \frac{1}{3}(2n+1)t & \frac{1}{3}(2n+4)t \\ (n+3)t & (n+2)t & (n+1)t & nt & (n+4)t \\ \frac{1}{3}(2n-5)t & \frac{1}{3}(2n+13)t & \frac{1}{3}(2n+1)t & \frac{1}{3}(2n+4)t & \frac{1}{3}(2n+7)t \end{pmatrix}$$

And

$$M_1^{\bigstar-} = \begin{pmatrix} nt & (n+1)t & (n+2)t & (n+3)t & (n+4)t \\ \frac{1}{3}(2n+1)t & \frac{1}{3}(2n+13)t & \frac{1}{3}(2n-5)t & \frac{1}{3}(2n+7)t & \frac{1}{3}(2n+4)t \\ \frac{1}{3}(2n+13)t & \frac{1}{3}(2n-5)t & \frac{1}{3}(2n+7)t & \frac{1}{3}(2n+4)t & \frac{1}{3}(2n+1)t \\ (n+2)t & (n+3)t & (n+4)t & nt & (n+1)t \\ \frac{1}{3}(2n-5)t & \frac{1}{3}(2n+7)t & \frac{1}{3}(2n+4)t & \frac{1}{3}(2n+1)t & \frac{1}{3}(2n+13)t \end{pmatrix}$$

So, the Star-Jacobian Matrix of $M_1^{\bigstar+}$ *is* given by :

$$JM_1^{\bigstar+} = \begin{pmatrix} n & (n+4) & (n+3) & (n+2) & (n+1) \\ \frac{1}{3}(2n+4) & \frac{1}{3}(2n+7) & \frac{1}{3}(2n-5) & \frac{1}{3}(2n+13) & \frac{1}{3}(2n+1) \\ \frac{1}{3}(2n+7) & \frac{1}{3}(2n-5) & \frac{1}{3}(2n+13) & \frac{1}{3}(2n+1) & \frac{1}{3}(2n+4) \\ (n+3) & (n+2) & (n+1) & n & (n+4) \\ \frac{1}{3}(2n-5) & \frac{1}{3}(2n+13) & \frac{1}{3}(2n+1) & \frac{1}{3}(2n+4) & \frac{1}{3}(2n+7) \end{pmatrix}$$

and the Star-Jacobian Matrix of $M_1^{\bigstar-}$ is given by :

$$JM_1^{\bigstar -} = \begin{pmatrix} n & (n+1) & (n+2) & (n+3) & (n+4) \\ \frac{1}{3}(2n+1) & \frac{1}{3}(2n+13) & \frac{1}{3}(2n-5) & \frac{1}{3}(2n+7) & \frac{1}{3}(2n+4) \\ \frac{1}{3}(2n+13) & \frac{1}{3}(2n-5) & \frac{1}{3}(2n+7) & \frac{1}{3}(2n+4) & \frac{1}{3}(2n+1) \\ (n+2) & (n+3) & (n+4) & n & (n+1) \\ \frac{1}{3}(2n-5) & \frac{1}{3}(2n+7) & \frac{1}{3}(2n+4) & \frac{1}{3}(2n+1) & \frac{1}{3}(2n+13) \end{pmatrix}$$

By means of elementary calculation, it is easy to deduce the following results.
For all n ∈ R we have,

1) $\det(JM_1^{\bigstar +}) = \det(JM_1^{\bigstar -})$
2) There exists a constant matrix « $C\alpha_1^{\bigstar}$» that is wholly independent of (n; t), check for equality :

$$JM_1^{\bigstar +}(n,t) \times C\alpha_1^{\bigstar} = JM_1^{\bigstar -}(n,t).$$

3) $\det(C\alpha_1^{\bigstar}) = | C\alpha_1^{\bigstar}| = 1$.
4) ${}^t C\alpha_1^{\bigstar} = (C\alpha_1^{\bigstar})^{-1}$.
5) $JM_1^{\bigstar -}(n,t) \times {}^t C\alpha_1^{\bigstar} = JM_1^{\bigstar +}(n,t)$.

2. Star-Jacobian matrix of star-system:

$$\bigstar_2[t, t^2, t^3, t^4, t^5; \alpha_2] = \alpha_2.$$

Next, we study the relationship between two star-Jacobian matrix $JM_2^{\bigstar +}$ directly and $JM_2^{\bigstar -}$ indirectly of star-system $\bigstar_2[t, t^2, t^3, t^4, t^5; \alpha_2] = \alpha_2$.
In this case $\alpha_2 = \frac{2}{3}(t + t^2 + t^3 + t^4 + t^5)$ (see[1])
For all t ∈ R, the star-system :

$$\bigstar_2[t, t^2, t^3, t^4, t^5; \tfrac{2}{3}(t+t^2+t^3+t^4+t^5)] = \tfrac{2}{3}(t + t^2 + t^3 + t^4 + t^5)$$

has a unique solution $f_2^{\bigstar}$

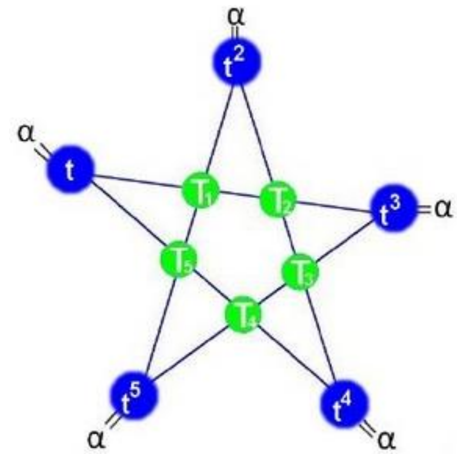
The star-function $f_2^{\bigstar}$: R → R^5 defined by :

$$f_2^{\bigstar}(t) = \left(\tfrac{1}{3}t+\tfrac{1}{3}t^2-\tfrac{2}{3}t^3+\tfrac{4}{3}t^4-\tfrac{2}{3}t^5, -\tfrac{2}{3}t+\tfrac{1}{3}t^2+\tfrac{1}{3}t^3-\tfrac{2}{3}t^4+\tfrac{4}{3}t^5, \tfrac{4}{3}t-\tfrac{2}{3}t^2+\tfrac{1}{3}t^3+\tfrac{1}{3}t^4-\tfrac{2}{3}t^5, -\tfrac{2}{3}t+\tfrac{4}{3}t^2-\tfrac{2}{3}t^3+\tfrac{1}{3}t^4+\tfrac{1}{3}t^5, \tfrac{1}{3}t-\tfrac{2}{3}t^2+\tfrac{4}{3}t^3-\tfrac{2}{3}t^4+\tfrac{1}{3}t^5\right).$$

We obtain the Star-Jacobian vector of $f_2^{\bigstar}(t)$.

$$J f_2^{\bigstar}(t) = \begin{pmatrix} (\frac{1}{3}+\frac{2}{3}t-2t^2+\frac{16}{3}t^3-\frac{10}{3}t^4 \\ -\frac{2}{3}+\frac{2}{3}t+t^2-\frac{8}{3}t^3+\frac{20}{3}t^4 \\ \frac{4}{3}-\frac{4}{3}t+t^2+\frac{4}{3}t^3-\frac{10}{3}t^4 \\ -\frac{2}{3}+\frac{8}{3}t-t^2+\frac{4}{3}t^3+\frac{5}{3}t^4 \\ \frac{1}{3}-\frac{4}{3}t+4t^2-\frac{8}{3}t^3+\frac{5}{3}t^4 \end{pmatrix}.$$

Here is an example two of star-matrix:

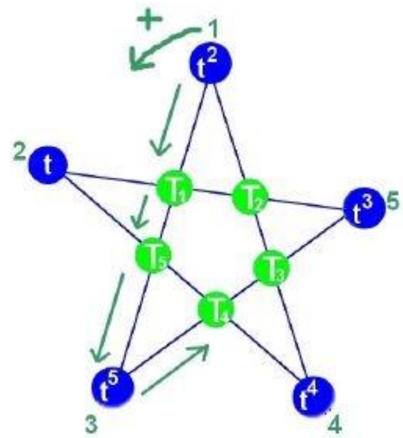


(Fig 7)

Also known as second Star-system,

$$\bigstar_2^{+}[t, t^2, t^3, t^4, t^5; \alpha_2] = \alpha_2 = \frac{2}{3}(t + t^2 + t^3 + t^4 + t^5).$$

This is a star oriented countreclockwise (positively oriented):



(Fig 8)

We obtains a star matrix directly noted $M_2^{\bigstar+}$

$$M_2^{\bigstar+} = \begin{pmatrix} t^2 & t & t^5 & t^4 & t^3 \\ T_1 & T_5 & T_4 & T_3 & T_2 \\ T_5 & T_4 & T_3 & T_2 & T_1 \\ t^5 & t^4 & t^3 & t^2 & t \\ T_4 & T_3 & T_2 & T_1 & T_5 \end{pmatrix}$$

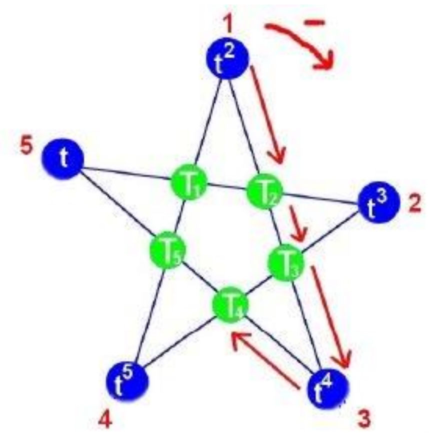
Where

$$\begin{cases} \frac{1}{3}t + \frac{1}{3}t^2 - \frac{2}{3}t^3 + \frac{4}{3}t^4 - \frac{2}{3}t^5 \\ -\frac{2}{3}t + \frac{1}{3}t^2 + \frac{1}{3}t^3 - \frac{2}{3}t^4 + \frac{4}{3}t^5 \\ \frac{4}{3}t - \frac{2}{3}t^2 + \frac{1}{3}t^3 + \frac{1}{3}t^4 - \frac{2}{3}t^5 \\ -\frac{2}{3}t + \frac{4}{3}t^2 - \frac{2}{3}t^3 + \frac{1}{3}t^4 + \frac{1}{3}t^5 \\ \frac{1}{3}t - \frac{2}{3}t^2 + \frac{4}{3}t^3 - \frac{2}{3}t^4 + \frac{1}{3}t^5 \end{cases}$$

In the other case, also known as second Star-system,

$$\bigstar_2^{+}[t, t^2, t^3, t^4, t^5; \alpha_2] = \alpha_2 = \frac{2}{3}(t + t^2 + t^3 + t^4 + t^5)$$

is a star oriented clockwise:



(Fig 9)

We get a star matrix indirectly noted $M_2^{\bigstar -}$

$$M_2^{\bigstar -} = \begin{pmatrix} t^2 & t^3 & t^4 & t^5 & t \\ T_2 & T_3 & T_4 & T_5 & T_1 \\ T_3 & T_4 & T_5 & T_1 & T_2 \\ t^4 & t^5 & t & t^2 & t^3 \\ T_4 & T_5 & T_1 & T_2 & T_3 \end{pmatrix}$$

Where

$$\begin{cases} \frac{1}{3}t + \frac{1}{3}t^2 - \frac{2}{3}t^3 + \frac{4}{3}t^4 - \frac{2}{3}t^5 \\ -\frac{2}{3}t + \frac{1}{3}t^2 + \frac{1}{3}t^3 - \frac{2}{3}t^4 + \frac{4}{3}t^5 \\ \frac{4}{3}t - \frac{2}{3}t^2 + \frac{1}{3}t^3 + \frac{1}{3}t^4 - \frac{2}{3}t^5 \\ -\frac{2}{3}t + \frac{4}{3}t^2 - \frac{2}{3}t^3 + \frac{1}{3}t^4 + \frac{1}{3}t^5 \\ \frac{1}{3}t - \frac{2}{3}t^2 + \frac{4}{3}t^3 - \frac{2}{3}t^4 + \frac{1}{3}t^5 \end{cases}$$

We have

$$\det(M_2^{\star+}(t))=\det(M_2^{\star-}(t))=\frac{1}{3}(2t^{25} - 8t^{20} + 12t^{15} - 8t^{10} + t^5).$$

$$M_2^{\star+}(t)\times C\alpha_2^{\star} = M_2^{\star-}(t).$$

we obtains the Chaff-matrix

$$C\alpha_2^{\star}(t)= \frac{1}{t^4+t^3+t^2+t+1}\begin{pmatrix} t^3+t^2 & t^4+t^3 & -t^3-t^2-t & t+1 & t^2+t \\ t^2+t & t^3+t^2 & t^4+t^3 & -t^3-t^2-t & t+1 \\ t+1 & t^2+t & t^3+t^2 & t^4+t^3 & -t^3-t^2-t \\ -t^3-t^2-t & t+1 & t^2+t & t^3+t^2 & t^4+t^3 \\ t^4+t^3 & -t^3-t^2-t & t+1 & t^2+t & t^3+t^2 \end{pmatrix}$$

where

$$det\begin{pmatrix} t^3+t^2 & t^4+t^3 & -t^3-t^2-t & t+1 & t^2+t \\ t^2+t & t^3+t^2 & t^4+t^3 & -t^3-t^2-t & t+1 \\ t+1 & t^2+t & t^3+t^2 & t^4+t^3 & -t^3-t^2-t \\ -t^3-t^2-t & t+1 & t^2+t & t^3+t^2 & t^4+t^3 \\ t^4+t^3 & -t^3-t^2-t & t+1 & t^2+t & t^3+t^2 \end{pmatrix}$$

$$=t^4+t^3+t^2+t+1$$

For all $t \in R - \{0\}$ we have the following results,

1) $M_2^{\star+}(t) \times C\alpha_2^{\star}(t) = M_2^{\star-}(t)$.
2) $\det(C\alpha_2^{\star}(t)) = | C\alpha_2^{\star}(t)| = 1$.
3) ${}^t(C\alpha_2^{\star}(t))= (C\alpha_2^{\star}(t))^{-1}$.
4) $M_2^{\star-}(t) \times {}^tC\alpha_2^{\star}(t) = M_2^{\star+}(t)$.

So, the Star-Jacobian Matrix of $M_2^{\star+}(t)$ *is* give by:

$$JM_2^{\star+}(t) = \begin{pmatrix} 2t & 1 & 5t^4 & 4t^3 & 3t^2 \\ \frac{\partial T_1}{\partial t} & \frac{\partial T_5}{\partial t} & \frac{\partial T_4}{\partial t} & \frac{\partial T_3}{\partial t} & \frac{\partial T_2}{\partial t} \\ \frac{\partial T_5}{\partial t} & \frac{\partial T_4}{\partial t} & \frac{\partial T_3}{\partial t} & \frac{\partial T_2}{\partial t} & \frac{\partial T_1}{\partial t} \\ 5t^4 & 4t^3 & 3t^2 & 2t & 1 \\ \frac{\partial T_4}{\partial t} & \frac{\partial T_3}{\partial t} & \frac{\partial T_2}{\partial t} & \frac{\partial T_1}{\partial t} & \frac{\partial T_5}{\partial t} \end{pmatrix}$$

Where

$$\begin{cases} \frac{\partial T_1}{\partial t} = \frac{1}{3}+\frac{2}{3}t-2t^2+\frac{8}{3}t^3-\frac{10}{3}t^4 \\ \frac{\partial T_2}{\partial t} = -\frac{2}{3}+\frac{2}{3}t+t^2-\frac{8}{3}t^3+\frac{20}{3}t^4 \\ \frac{\partial T_3}{\partial t} = \frac{4}{3}-\frac{4}{3}t+t^2+\frac{4}{3}t^3-\frac{10}{3}t^4 \\ \frac{\partial T_4}{\partial t} = -\frac{2}{3}+\frac{8}{3}t-2t^2+\frac{4}{3}t^3+\frac{5}{3}t^4 \\ \frac{\partial T_5}{\partial t} = \frac{1}{3}-\frac{4}{3}t+4t^2-\frac{8}{3}t^3+\frac{5}{3}t^4 \end{cases}$$

and the Star-Jacobian Matrix of $M_2^{\bigstar -}(t)$ *is* give by :

$$JM_2^{\bigstar +}(t) = \begin{pmatrix} 2t & 3t^2 & 4t^3 & 5t^4 & 1 \\ \frac{\partial T_2}{\partial t} & \frac{\partial T_3}{\partial t} & \frac{\partial T_4}{\partial t} & \frac{\partial T_5}{\partial t} & \frac{\partial T_1}{\partial t} \\ \frac{\partial T_3}{\partial t} & \frac{\partial T_4}{\partial t} & \frac{\partial T_5}{\partial t} & \frac{\partial T_1}{\partial t} & \frac{\partial T_2}{\partial t} \\ 4t^3 & 5t^4 & 1 & 2t & 3t^2 \\ \frac{\partial T_4}{\partial t} & \frac{\partial T_5}{\partial t} & \frac{\partial T_1}{\partial t} & \frac{\partial T_2}{\partial t} & \frac{\partial T_3}{\partial t} \end{pmatrix}$$

We get det($JM_2^{\bigstar +}(t)$) = det($JM_2^{\bigstar -}(t)$).

IV. RELATIONSHIP BETWEEN TWO STAR-JACOBIAN MATRIX $JM^{\bigstar +}$ DIRECTLY AND $JM^{\bigstar -}$ INDIRECTLY OF TWO STAR-SYSTEM WITH α_1 AND α_2 COEFFICIENT.

In this section, we study the relationship between two star-matrix $M^{\bigstar +}$ directly and $M^{\bigstar -}$ indirectly of two star-system $\bigstar_1[a; b; c; d; e; \alpha_1] = \alpha_1$ and $\bigstar_2[a; b; c; d; e; \alpha_2] = \alpha_2$. For a 5x5 star-matrix $M^{\bigstar}$ the inverse is $(M^{\bigstar})^{-1}$ [14]. By means of elementary calculation, it is easy to deduce the following results.

Theorem 2. For all $(n; t) \in \mathbb{R}\times\mathbb{R}\backslash\{0\}$, we have

1) $(M_2^{\bigstar -}(t))^{-1} \times M_1^{\bigstar -}(n,t) = {}^t((M_2^{\bigstar +}(t))^{-1} \times M_1^{\bigstar +}(n,t))$.
2) Matrix $(M_2^{\bigstar +}(t))^{-1} \times M_1^{\bigstar +}(n,t)$ is diagonalizable.
3) $(M_2^{\bigstar -}(t))^{-1} \times M_1^{\bigstar +}(n,t) = {}^t((M_2^{\bigstar +}(t))^{-1} \times M_1^{\bigstar -}(n,t))$.
4) Matrix $(M_2^{\bigstar +}(t))^{-1} \times M_1^{\bigstar -}(n,t)$ is diagonalizable.
5) $(JM_2^{\bigstar -}(t))^{-1}\times JM_1^{\bigstar -}(n,t)={}^t((JM_2^{\bigstar +}(t))^{-1}\times JM_1^{\bigstar +}(n,t))$
6) Matrix $(JM_2^{\bigstar +}(t))^{-1}\times JM_1^{\bigstar +}(n,t)$ is diagonalizable.
7) $(JM_2^{\bigstar -}(t))^{-1}\times JM_1^{\bigstar +}(n,t)={}^t((JM_2^{\bigstar +}(t))^{-1}\times JM_1^{\bigstar -}(n,t))$
8) Matrix $(JM_2^{\bigstar +}(t))^{-1}\times JM_1^{\bigstar -}(n,t)$ is diagonalizable.

Proof

We have

$$\left(M_2^{\bigstar +}(t)\right)^{-1} = \begin{pmatrix} \frac{-2}{t^6-t} & 0 & \frac{1}{t^6-t} & \frac{1}{t^5+t^4+t^3+t^2+t} & \frac{3}{t^6-t} \\ \frac{2t-1}{t^6-t} & 0 & \frac{-1}{t^5-1} & \frac{1}{t^5-1} & \frac{-2}{t^5-1} \\ \frac{2t+1}{2t^6-2t} & \frac{-1}{2t^6-2t} & \frac{-3}{2t^6-2t} & \frac{1}{2t^6-2t} & \frac{-1}{t^6-t} \\ \frac{-t+2}{2t^6-2t} & \frac{t-2}{2t^6-2t} & \frac{3t-2}{2t^6-2t} & \frac{-1}{2t^5-2} & \frac{1}{t^5+t^4+t^3+t^2+t+1} \\ \frac{-1}{t^5-1} & \frac{1}{t^5-1} & \frac{1}{t^5-1} & \frac{-1}{t^6-t} & \frac{1}{t^5-1} \end{pmatrix}$$

$$(M_2^{\bigstar +}(t))^{-1} \times M_1^{\bigstar +}(n,t) = C^{++}(n,t)$$

Where

$$C^{++}(n,t) = \begin{pmatrix} \frac{-n+nt+3t-4}{t^5-1} & \frac{-n+nt+2t-3}{t^5-1} & \frac{-n+nt+t-2}{t^5-1} & \frac{-n+nt-1}{t^5-1} & \frac{-n+nt+4t}{t^5-1} \\ \frac{-n+nt+4t}{t^5-1} & \frac{-n+nt+3t-4}{t^5-1} & \frac{-n+nt+2t-3}{t^5-1} & \frac{-n+nt+t-2}{t^5-1} & \frac{-n+nt-1}{t^5-1} \\ \frac{-n+nt+t-1}{t^5-1} & \frac{-n+nt+4t}{t^5-1} & \frac{-n+nt+3t-4}{t^5-1} & \frac{-n+nt+2t-3}{t^5-1} & \frac{-n+nt+t-2}{t^5-1} \\ \frac{-n+nt+t-1}{t^5-1} & \frac{-n+nt-1}{t^5-1} & \frac{-n+nt+4t}{t^5-1} & \frac{-n+nt+3t-4}{t^5-1} & \frac{-n+nt+2t-3}{t^5-1} \\ \frac{-n+nt+2t-3}{t^5-1} & \frac{-n+nt+t-2}{t^5-1} & \frac{-n+nt-1}{t^5-1} & \frac{-n+nt+4t}{t^5-1} & \frac{-n+nt+3t-4}{t^5-1} \end{pmatrix}$$

And

$$\left(M_2^{\star-}(t)\right)^{-1} = \begin{pmatrix} \frac{-2t+1}{2t^6-2t} & \frac{2t-1}{2t^6-2t} & \frac{2t-3}{2t^6-2t} & \frac{1}{2t^6-2t} & \frac{1}{t^5+t^4+t^3+t^2+t} \\ \frac{1}{t^6-t} & \frac{-1}{t^6-t} & \frac{-1}{t^6-t} & \frac{1}{t^5-1} & \frac{-1}{t^6-t} \\ \frac{2}{t^5-1} & 0 & \frac{-1}{t^5-1} & \frac{1}{t^5+t^4+t^3+t^2+t} & \frac{-2}{t^6-t} \\ \frac{t-2}{t^6-t} & 0 & \frac{1}{t^6-t} & \frac{-1}{t^6-t} & \frac{2}{t^6-t} \\ \frac{-t-2}{2t^6-2t} & \frac{1}{2t^5-2} & \frac{3}{2t^5-2} & \frac{-1}{2t^5-2} & \frac{1}{t^5-1} \end{pmatrix}$$

We obtain $(M_2^{\star-}(t))^{-1} \times M_1^{\star-}(n,t) = C^{--}(n,t) = {}^t(C^{++}(n,t))$

And $(M_2^{\star+}(t))^{-1} \times M_1^{\star-}(n,t) = C^{+-}(n,t)$

Where

$$C^{+-}(n,t) = \begin{pmatrix} \frac{-n+nt+2t-1}{t^5-1} & \frac{-n+nt+3t-2}{t^5-1} & \frac{-n+nt+4t-3}{t^5-1} & \frac{-n+nt-4}{t^5-1} & \frac{-n+nt+t}{t^5-1} \\ \frac{-n+nt+t}{t^5-1} & \frac{-n+nt+2t-1}{t^5-1} & \frac{-n+nt+3t-2}{t^5-1} & \frac{-n+nt+4t-3}{t^5-1} & \frac{-n+nt-4}{t^5-1} \\ \frac{-n+nt-4}{t^5-1} & \frac{-n+nt+t}{t^5-1} & \frac{-n+nt+2t-1}{t^5-1} & \frac{-n+nt+3t-2}{t^5-1} & \frac{-n+nt+4t-3}{t^5-1} \\ \frac{-n+nt+4t-3}{t^5-1} & \frac{-n+nt-4}{t^5-1} & \frac{-n+nt+t}{t^5-1} & \frac{-n+nt+2t-1}{t^5-1} & \frac{-n+nt+3t-2}{t^5-1} \\ \frac{-n+nt+3t-2}{t^5-1} & \frac{-n+nt+4t-3}{t^5-1} & \frac{-n+nt-4}{t^5-1} & \frac{-n+nt+t}{t^5-1} & \frac{-n+nt+2t-1}{t^5-1} \end{pmatrix}$$

We get $(M_2^{\star-}(t))^{-1} \times M_1^{\star+}(n,t) = C^{-+}(n,t) = {}^t(C^{+-}(n,t))$

The results obtained show:

1) $(M_2^{\star+}(t))^{-1} \times M_1^{\star+}(n,t) = C^{++}(n,t)$.

2) $(M_2^{\star-}(t))^{-1} \times M_1^{\star-}(n,t) = C^{--}(n,t) = {}^t(C^{++}(n,t))$.

3) $(M_2^{\star+}(t))^{-1} \times M_1^{\star-}(n,t) = C^{+-}(n,t)$.

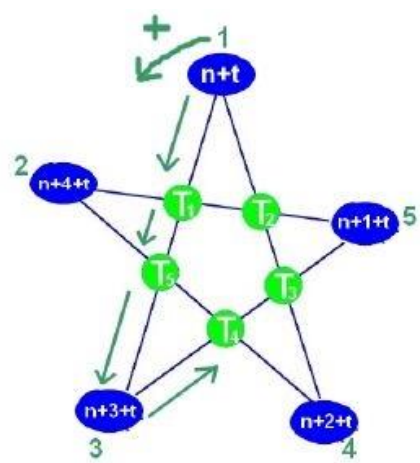
4) $(M_2^{\star-}(t))^{-1} \times M_1^{\star+}(n,t) = C^{-+}(n,t) = {}^t(C^{+-}(n,t))$.

Example. We consider two star-matrix $M_3^{\star+}(n,t)$ directly and $M_3^{\star-}(n,t)$ indirectly of star-system:

$$\star_3[n+t, (n+1)+t, (n+2)+t, (n+3)+t, (n+4)+t; \alpha_3] = \alpha_3.$$

where the star-Coefficient $\alpha_3^{\star} = \frac{10}{3}(t+n+2)$

and $\star_3^{+}[n+t, (n+1)+t, (n+2)+t, (n+3)+t, (n+4)+t; \alpha_3] = \alpha_3$ is a star oriented countreclockwise (positively oriented):

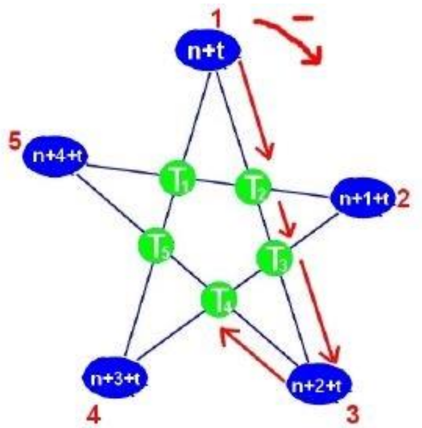


(Fig 10)

In the first case, we obtain a star matrix directly noted $M_3^{\star+}(n,t)$:

$$M_3^{\star+} = \begin{pmatrix} n+t & (n+4)+t & (n+3)+t & (n+2)+t & (n+1)+t \\ \frac{2}{3}(t+n+2) & \frac{1}{3}(2t+2n+7) & \frac{1}{3}(2t+2n-5) & \frac{1}{3}(2t+2n+13) & \frac{1}{3}(2t+2n+1) \\ \frac{1}{3}(2t+2n+7) & \frac{1}{3}(2t+2n-5) & \frac{1}{3}(2t+2n+13) & \frac{1}{3}(2t+2n+1) & \frac{2}{3}(t+n+2) \\ (n+3)+t & (n+2)+t & (n+1)+t & n+t & (n+4)+t \\ \frac{1}{3}(2t+2n-5) & \frac{1}{3}(2t+2n+13) & \frac{1}{3}(2t+2n+1) & \frac{2}{3}(t+n+2) & \frac{1}{3}(2t+2n+7) \end{pmatrix}$$

In the second case, $\bigstar_3^-$[n+t, (n+1)+t, (n+2)+t, (n+3)+t, (n+4)+t; α_3] = α_3 is a star oriented clockwise:



(Fig 11)

We get a star matrix indirectly noted $M_3^{\star-}(n,t)$:

$$M_3^{\star-} = \begin{pmatrix} n+t & (n+1)+t & (n+2)+t & (n+3)+t & (n+4)+t \\ \frac{1}{3}(2t+2n+1) & \frac{1}{3}(2t+2n+13) & \frac{1}{3}(2t+2n-5) & \frac{1}{3}(2t+2n+7) & \frac{2}{3}(t+n+2) \\ \frac{1}{3}(2t+2n+13) & \frac{1}{3}(2t+2n-5) & \frac{1}{3}(2t+2n+7) & \frac{2}{3}(t+n+2) & \frac{2}{3}(2t+2n+1) \\ (n+2)+t & (n+3)+t & (n+4)+t & n+t & (n+1)+t \\ \frac{1}{3}(2t+2n-5) & \frac{1}{3}(2t+2n+7) & \frac{2}{3}(t+n+2) & \frac{1}{3}(2t+2n+1) & \frac{1}{3}(2t+2n+13) \end{pmatrix}$$

$\det(M_3^{\star+}(n,t)) = \det(M_3^{\star-}(n,t)) = \frac{n+t+2}{3} \times 1250$.

Consequently:

$$(M_3^{\star+}(n,t)) \times (M_1^{\star+}(n,t))^{-1} = \begin{pmatrix} \frac{10n+3t+20}{10nt+20t} & \frac{3}{10n+20} & \frac{3}{10n+20} & \frac{3}{10n+20} & 0 \\ \frac{1}{5n+10} & \frac{5n+t+10}{5nt+10t} & \frac{1}{5n+10} & \frac{1}{5n+10} & 0 \\ \frac{1}{5n+10} & \frac{1}{5n+10} & \frac{5n+t+10}{5nt+10t} & \frac{1}{5n+10} & 0 \\ \frac{3}{10n+20} & \frac{3}{10n+20} & \frac{3}{10n+20} & \frac{10n+3t+20}{10nt+20t} & 0 \\ \frac{1}{5n+10} & \frac{1}{5n+10} & \frac{1}{5n+10} & \frac{1}{5n+10} & \frac{1}{t} \end{pmatrix}$$

The characteristic polynomial of a square matrix $(M_3^{\star+}(n,t)) \times (M_1^{\star-}(n,t))^{-1}$.

For all $(n; t) \in R \times R \backslash \{0\}$ and $\lambda \in R$:

$$|(M_3^{\star+}(n,t))\times(M_1^{\star-}(n,t))^{-1} - \lambda I| = \det\begin{pmatrix} \frac{10n+3t+20}{10nt+20t}-\lambda & \frac{3}{10n+20} & \frac{3}{10n+20} & \frac{3}{10n+20} & 0 \\ \frac{1}{5n+10} & \frac{5n+t+10}{5nt+10t}-\lambda & \frac{1}{5n+10} & \frac{1}{5n+10} & 0 \\ \frac{1}{5n+10} & \frac{1}{5n+10} & \frac{5n+t+10}{5nt+10t}-\lambda & \frac{1}{5n+10} & 0 \\ \frac{3}{10n+20} & \frac{3}{10n+20} & \frac{3}{10n+20} & \frac{10n+3t+20}{10nt+20t}-\lambda & 0 \\ \frac{1}{5n+10} & \frac{1}{5n+10} & \frac{1}{5n+10} & \frac{1}{5n+10} & \frac{1}{t}-\lambda \end{pmatrix}$$

$$= -\lambda^5 + \frac{5n+t+10}{nt+2t}\lambda^4 - \frac{10n+4t+20}{nt^2+2t^2}\lambda^3 + \frac{10n+6t+20}{nt^3+2t^3}\lambda^2 - \frac{5n+4t+10}{nt^4+2t^4}\lambda + \frac{n+t+2}{nt^5+2t^5}$$

$$= \frac{nt^2+2t^2}{nt^5+2t^5}(\lambda - \frac{1}{t})^4\times(\lambda - \frac{n+t+2}{nt+2t}).$$

Eigenvalues of the characteristic polynomial: $\lambda = \frac{1}{t}$ and $\lambda = \frac{n+t+2}{nt+2t}$.

We find the eigenvectors associated with each of the eigenvalues.

Case 1: $\lambda = \frac{1}{t}$.

$$V_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, V_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, V_4 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

Case 2: $\lambda = \frac{n+t+2}{nt+2t}$, $V_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

So, we have the passage matrix P^{++}

$$P^{++} = \begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$(P^{++})^{-1} = \begin{pmatrix} \frac{-1}{5} & \frac{4}{5} & \frac{-1}{5} & \frac{-1}{5} & \frac{-1}{5} \\ \frac{-1}{5} & \frac{-1}{5} & \frac{4}{5} & \frac{-1}{5} & \frac{-1}{5} \\ \frac{-1}{5} & \frac{-1}{5} & \frac{-1}{5} & \frac{4}{5} & \frac{-1}{5} \\ \frac{-1}{5} & \frac{-1}{5} & \frac{-1}{5} & \frac{-1}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{-1}{5} & \frac{-1}{5} & \frac{-1}{5} & \frac{-1}{5} \end{pmatrix}$$

and

$$D(n,t) = \begin{pmatrix} \frac{1}{t} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{t} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{t} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{t} & 0 \\ 0 & 0 & 0 & 0 & \frac{n+t+2}{nt+2t} \end{pmatrix} = \frac{1}{t}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1+\frac{t}{n+2} \end{pmatrix}$$

$$(M_3^{\star+}(n,t))\times(M_1^{\star-}(n,t))^{-1} = P^{++} \times D(n,t) \times (P^{++})^{-1}.$$

Theorem 3. For all (n; t) ∈ R×R\{0}, we have

1) $(M_3^{\bigstar -}(n,t))^{-1} \times M_1^{\bigstar -}(n,t) = {}^t((M_3^{\bigstar +}(n,t))^{-1} \times M_1^{\bigstar +}(n,t))$
2) Matrix $(M_3^{\bigstar +}(n,t))^{-1} \times M_1^{\bigstar +}(n,t)$ is a symmetric matrix.
3) $(M_3^{\bigstar +}(n,t))^{-1} \times M_1^{\bigstar -}(n,t) = {}^t((M_3^{\bigstar -}(n,t))^{-1} \times M_1^{\bigstar +}(n,t))$.
4) Matrix $(M_3^{\bigstar +}(n,t))^{-1} \times M_1^{\bigstar -}(n,t)$ is diagonalizable.
5) $\det(JM_1^{\bigstar -}(n,t)) = \det(JM_1^{\bigstar +}(n,t)) = \det(M_1^{\bigstar -}(n,1)) = \det(M_1^{\bigstar +}(n,1))$
6) $(JM_1^{\bigstar -}(n,t))^{-1} \times M_3^{\bigstar -}(n,t) = (JM_1^{\bigstar +}(t))^{-1} \times M_3^{\bigstar +}(n,t)$
7) Matrix $(JM_1^{\bigstar -}(t))^{-1} \times M_3^{\bigstar -}(n,t)$ is a symmetric matrix.
8) $(JM_1^{\bigstar -}(t))^{-1} \times M_3^{\bigstar +}(n,t) = {}^t((JM_1^{\bigstar +}(t))^{-1} \times M_3^{\bigstar -}(n,t))$
9) Matrix $(JM_1^{\bigstar -}(t))^{-1} \times M_3^{\bigstar +}(n,t)$ is diagonalizable.

Proof

Let (n; t) ∈ R×R\{0}, we have

1) $(M_3^{\bigstar -}(t))^{-1} \times M_1^{\bigstar -}(n,t) = {}^t((M_3^{\bigstar +}(t))^{-1} \times M_1^{\bigstar +}(n,t)) =$

$$\begin{pmatrix} \frac{4t^2+5nt+10t}{5n+5t+10} & \frac{-t^2}{5n+5t+10} & \frac{-t^2}{5n+5t+10} & \frac{-t^2}{5n+5t+10} & \frac{-t^2}{5n+5t+10} \\ \frac{-t^2}{5n+5t+10} & \frac{4t^2+5nt+10t}{5n+5t+10} & \frac{-t^2}{5n+5t+10} & \frac{-t^2}{5n+5t+10} & \frac{-t^2}{5n+5t+10} \\ \frac{-t^2}{5n+5t+10} & \frac{-t^2}{5n+5t+10} & \frac{4t^2+5nt+10t}{5n+5t+10} & \frac{-t^2}{5n+5t+10} & \frac{-t^2}{5n+5t+10} \\ \frac{-t^2}{5n+5t+10} & \frac{-t^2}{5n+5t+10} & \frac{-t^2}{5n+5t+10} & \frac{4t^2+5nt+10t}{5n+5t+10} & \frac{-t^2}{5n+5t+10} \\ \frac{-t^2}{5n+5t+10} & \frac{-t^2}{5n+5t+10} & \frac{-t^2}{5n+5t+10} & \frac{-t^2}{5n+5t+10} & \frac{4t^2+5nt+10t}{5n+5t+10} \end{pmatrix}$$

$$= (M_3^{\bigstar +}(n,t))^{-1} \times (M_1^{\bigstar +}(n,t))$$

2) $(M_3^{\bigstar +}(n,t))^{-1} \times (M_1^{\bigstar +}(n,t))$ is a symmetric matrix.
3) $(M_3^{\bigstar +}(n,t))^{-1} \times M_1^{\bigstar -}(n,t) = {}^t((M_3^{\bigstar -}(n,t))^{-1} \times M_1^{\bigstar +}(n,t))$
4)

$$\begin{pmatrix} \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{-4t^2-3nt-6t}{5n+5t+10} \\ \frac{-4t^2-3nt-6t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} \\ \frac{t^2+2nt+4t}{5n+5t+10} & \frac{-4t^2-3nt-6t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} \\ \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{-4t^2-3nt-6t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} \\ \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{-4t^2-3nt-6t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} \end{pmatrix}$$

$((M_3^{\star -}(n,t))^{-1} \times M_1^{\star +}(n,t)) =$

$$\begin{pmatrix} \frac{t^2+2nt+4t}{5n+5t+10} & \frac{-4t^2-3nt-6t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} \\ \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{-4t^2-3nt-6t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} \\ \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{-4t^2-3nt-6t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} \\ \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{-4t^2-3nt-6t}{5n+5t+10} \\ \frac{-4t^2-3nt-6t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} & \frac{t^2+2nt+4t}{5n+5t+10} \end{pmatrix}$$

$(M_3^{\star +}(n,t))^{-1} \times M_1^{\star -}(n,t) = {}^t((M_3^{\star -}(n,t))^{-1} \times M_1^{\star +}(n,t))$.

5) $(M_3^{\star -}(n,t))^{-1} \times M_1^{\star +}(n,t)$ is diagonalizable.

6) $\det(JM_1^{\star -}(n,t)) = \det(JM_1^{\star +}(n,t)) = \det(M_1^{\star -}(n,1)) = \det(M_1^{\star +}(n,1)) = \frac{1250+2500n}{3}$.

7) We have

$$(JM_1^{\star -}(n,t))^{-1} \times M_3^{\star -}(n,t) = \begin{pmatrix} \frac{5n+t+10}{5n+10} & \frac{t}{5n+10} & \frac{t}{5n+10} & \frac{t}{5n+10} & \frac{t}{5n+10} \\ \frac{t}{5n+10} & \frac{5n+t+10}{5n+10} & \frac{t}{5n+10} & \frac{t}{5n+10} & \frac{t}{5n+10} \\ \frac{t}{5n+10} & \frac{t}{5n+10} & \frac{5n+t+10}{5n+10} & \frac{t}{5n+10} & \frac{t}{5n+10} \\ \frac{t}{5n+10} & \frac{t}{5n+10} & \frac{t}{5n+10} & \frac{5n+t+10}{5n+10} & \frac{t}{5n+10} \\ \frac{t}{5n+10} & \frac{t}{5n+10} & \frac{t}{5n+10} & \frac{t}{5n+10} & \frac{5n+t+10}{5n+10} \end{pmatrix}$$

8) Matrix $(JM_1^{\star -}(t))^{-1} \times M_3^{\star -}(n,t)$ is a symmetric matrix.

9) $(JM_1^{\star -}(t))^{-1} \times M_3^{\star +}(n,t) = {}^t((JM_1^{\star +}(t))^{-1} \times M_3^{\star -}(n,t)) =$

$$\begin{pmatrix} \frac{2n+t+4}{5n+10} & \frac{-3n+t-6}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} \\ \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{-3n+t-6}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} \\ \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{-3n+t-6}{5n+10} & \frac{2n+t+4}{5n+10} \\ \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{-3n+t-6}{5n+10} \\ \frac{-3n+t-6}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} \end{pmatrix}$$

$JM_1^{\star +}(t))^{-1} \times M_3^{\star -}(n,t) =$

$$\begin{pmatrix} \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{-3n+t-6}{5n+10} \\ \frac{-3n+t-6}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} \\ \frac{2n+t+4}{5n+10} & \frac{-3n+t-6}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} \\ \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{-3n+t-6}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} \\ \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{2n+t+4}{5n+10} & \frac{-3n+t-6}{5n+10} & \frac{2n+t+4}{5n+10} \end{pmatrix}$$

V. CONCLUSIONS

Now it is easy to identify the Star-Jacobian matrix directly $JM^{\bigstar+}$ (choice of positive direction) per definition (orientation of a Star with α coefficient) by solving the Star system with α coefficient $\bigstar[T_1, T_2, T_3, T_4, T_5; \alpha] = \alpha$, similarly the Star-Jacobian matrix indirectly $JM^{\bigstar-}$ (choice of negative direction) of which can be obtained to using the same technique. Based on the above analysis it is concluded some properties of Star-Jacobian matrix (dirctly, indirectly) multiplication. In particular, our analysis confirms that :

i) Determinant of square (5x5) Star-matrix directly ($M^{\bigstar+}$) is equal to determinant of Star-matrix indirectly ($M^{\bigstar-}$).

ii) It's the same as what we found of the Star-Matrix : Determinant of square (5x5) Star- Jacobian matrix directly ($JM^{\bigstar+}$) is equal to determinant of Star- Jacobian matrix indirectly ($JM^{\bigstar-}$).

iii) There is a constant coefficient matrix $C\alpha^{\bigstar}$ such that if we multiply that matrix by $M^{\bigstar+}$ (Star-Matrix directly), Then you will get the matrix $M^{\bigstar-}$ (Star-Matix indirectly). $C\alpha^{\bigstar}$ is an orthogonal matrix (${}^{t}C\alpha^{\bigstar} = (C\alpha^{\bigstar})^{-1}$).

iv) The products between two Star- Jacobian Matices of two Star-System ($\bigstar\alpha_1$, $\bigstar\alpha_2$) with α_1 and α_2 coefficient directly-indirectly ($JM^{\bigstar+}$,$JM^{\bigstar-}$) or directly-directly ($JM^{\bigstar+}$,$JM^{\bigstar+}$) or indirectly-indirectly ($JM^{\bigstar-}$,$JM^{\bigstar-}$) are diagonalizable.

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