

Application of Semi-Analytical Iteration Techniques for the Numerical Solution of Linear and Nonlinear Differential Equations.

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Abstract: *In this research article, we apply four semi-analytical iteration techniques namely: Differential Transform method, variational iteration method, Adomian decomposition method and Temimi and Ansari respectively to both linear and nonlinear differential equations of first, second and third orders. The results obtained reveal, the variational iteration gives a solution in series form which converges to the true solution followed by the Adomian decomposition method and Temimi and Ansari method.*

Keyword: *Temimi and Ansari Method (TAM), Adomian decomposition method (ADM), Differential transform method (DTM), Variational iteration method (VIM)*

I. INTRODUCTION

Most of the problems in engineering and sciences have equations that are nonlinear in form. These equations oftentimes have close form or analytical solutions, whereas, at other times, close form solutions are difficult to come by so approximate solutions are sort to describe the behaviour of the related parameters. Scientists and engineers alike have written extensively on these semi-analytical methods due to the ease with which nonlinear problems are solved by them. The approximate solutions obtained converges to the exact solution under some underlying conditions which satisfies the initial and boundary condition. The differential transform method (DTM) proposed by Zhou [1] is a transformation technique which uses the Taylor series expansion to obtain analytic solution of a differential equation which converges to the exact solution. In this method certain transformation rules are applied to transform the given equation, equation such as algebraic equation in terms of the differential transform of the original transform and the solution of the algebraic equation given the desired solution of the problem. The method has been solved several problems in [2], [3], [4], [5]. Equally elegant and time consuming is Adomian innovative method christened Adomian decomposition method. In this method, the equation under study is split into linear and nonlinear portion. The linear operator representing the linear portion of the equation is inverted and the linear operator is then applied to the equation. Any given conditions are taken into consideration. The nonlinear portion is decomposed into a series of what is called Adomian Polynomials. The method generates a solution in the form of a series whose terms are determined by a recursive relationship using the Adomian Polynomials. ADM has been applied successively by researchers to solve myriad of problems ranging from linear and nonlinear ordinary and partial differential equations. See [6], [7], [8], [9], [10], [11]. Another method worthy of note that came from the Lagrange multiplier method which does not involve small perturbation or linearization is the Ji-Huang He's variational iteration method (VIM). This method gives a convergent solution of both ordinary and partial differential equation without any restrictive assumption that may change the physical nature of the problem under investigation. This novel method solves elegantly, efficiently, easily, and accurately a large class of nonlinear problems which approximate solution converges rapidly to the accurate solution. Extensive applications of this method are found in, [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29]. Most recently, Temimi and Ansari [30] suggested a novel iterative method called TAM. This method is advantageous in that, it does not require the so-called Adomian polynomials, involves less computational work and does not involve restricted assumptions that appear in other iterative methods such as VIM and DTM.



Researchers have used this method to obtain both exact and approximate solutions of many problems. This includes: Multi-point boundary value problems [31], Nonlinear thin flow problems [32], Wave-like equations with variable coefficients [33], Nonlinear problems [34].

In this research article, we employ four semi-analytical methods viz: Differential transform method (DTM), Adomian decomposition method (ADM), Variational Iteration Method (VIM) and Temimi and Ansari method (TAM) to extend the works of Necdet and Konuralp [35] to solve equations frequently occurring in Science and Engineering. To the best of my knowledge, these four methods haven't been compared before.

II. Differential Transform Method (DTM)

Let $u(t)$ be a given analytic function in the given domain D and let $x = x_0$ be an initial point of the function.

Then the k th derivative of $u(t)$ about point $t = t_0$ is defined as follows

$$U(k) = \frac{1}{k!} \left[\frac{d^k u(t)}{dt^k} \right]_{t=t_0} \tag{1}$$

where $u(t)$ is the original function and $U(k)$ is the transformed function

The inverse transforms of $U(k)$ in Eq. (1) is given as

$$u(t) = \sum_{k=0}^{\infty} \frac{(t - t_0)^k}{k!} U(k) \tag{2}$$

Combining Eqs. (1) and (2), the original function, $u(t)$ can be rewritten as a finite series of the form

$$u(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} \left[\frac{d^k u(t)}{dt^k} \right]_{t=t_0} \tag{3}$$

2.1 Fundamental Operations of the Differential Transform Method

| Original functions | Transformed function |
|-------------------------------------|--|
| $y(t) = \alpha v(t) \pm \beta w(t)$ | $Y(k) = \alpha V(k) \pm \beta W(k)$ |
| $y(t) = \alpha v(t)$ | $Y(k) = \alpha V(k)$ |
| $y(t) = \frac{dv(t)}{dt}$ | $Y(k) = (k + 1)V(k + 1)$ |
| $y(t) = \frac{d^2v(t)}{dt^2}$ | $Y(k) = (k + 1)(k + 2)V(k + 2)$ |
| $y(t) = \frac{d^r(t)}{dt^r}$, | $Y(k) = \frac{(k+r)!V(k+r)}{k!}$ |
| $y(t) = v(t)w(t)$ | $Y(k) = \sum_{n=0}^k V(n)W(k - n)$ |
| $y(t) = t^r$ | $Y(k) = \delta(k - r) = \begin{cases} 1, & \text{if } k = r \\ 0, & \text{if } k \neq r \end{cases}$ |

| | |
|--|--|
| $y(t) = v^3$ | $Y(k) = \sum_{k_1}^k \sum_r^{k_1} v(r)v(k_1 - r)v(k - r)$ |
| $y(t) = \left[\frac{dv(t)}{dt^2}\right]^2$ | $Y(k) = \sum_r^k (k + 1)(k - r + 1)(r + 1)V(k - r + 1)$ |
| $y(t) = e^{\lambda t}$ | $Y(k) = \frac{\lambda^k}{k!}$ |
| $y(t) = (1 + t)^r$ | $Y(k) = \frac{r(r-1)\dots(r-k+1)}{k!}$ |
| $y(t) = \sin(nt + \alpha)$ | $Y(k) = \frac{n^k}{k!} \sin\left(\frac{nk}{2} + \alpha\right)$ |
| $y(t) = \cos(nt + \alpha)$ | $Y(k) = \frac{n^k}{k!} \cos\left(\frac{nk}{2} + \alpha\right)$ |

III. Adomian Decomposition Method (ADM)

. A brief outline of the method is as follows.

Consider a general nonlinear differential equation of the form

$$D(y) = f(x) \tag{4}$$

where D is a nonlinear differential operator comprising both the linear and nonlinear terms, while $f(x)$ is any differentiable function of x

Decomposing the linear term in Eq. (4) into the form $L + R$, where L is the highest order derivative that is invertible and R is the remainder of the linear term.

Rewriting Eq. (4) in operator form, we have

$$L[y] + R[y] + N[y] = f(x)$$

$$L[y] = f(x) - Ry(x) - Ny(x) \tag{5}$$

While $N(y)$ is a nonlinear term and g is the source term.

Applying the inverse operator L^{-1} of both sides of Eq. (5), we obtain

$$L^{-1}(Ly(x)) = L^{-1}(f(x)) - L^{-1}(Ry(x)) - L^{-1}(Ny(x))$$

Where $L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot) dx dx$

$$y(x) = \varphi_0(x) + g(x) - L^{-1}R(y(x)) - L^{-1}N(y(x)) \tag{6}$$

Where $g(x)$ is the term obtained from integrating the source term. That is, $[L^{-1}(g)]$ and φ_0 from the given conditions

Now rewriting the solution and nonlinear terms as decomposition series of the form

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \text{ and } N(y(x)) = \sum_{n=0}^{\infty} A_n(x), \tag{7}$$

where the $A_n^{(s)}$ are the Adomian polynomials obtained using the formula

$$A_k = \frac{1}{k!} \frac{\partial^k}{\partial \lambda^k} \left[N \left(\sum_{n=0}^{\infty} y_n \lambda^n \right) \right]_{\lambda=0}, k = 0, 1, 2 \dots$$

Putting Eq. (7) into Eq. (6), we obtain the solution in the form a decomposition series

$$\sum_{n=0}^{\infty} y_n(x) = y(x) = \varphi_0(x) + g(x) - L^{-1}R(\sum_{n=0}^{\infty} y_n(x)) - L^{-1}N(\sum_{n=0}^{\infty} A_n(x)) \tag{8}$$

Where $y_0(x) = \varphi_0(x) + g(x)$ is the zeroth component of $y_n(x)$

The subsequent members of the series are obtained recursively using

$$y_{k+1} = -L^{-1}R(y_k(x)) - L^{-1}(A_k(x)), \quad k \geq 0 \tag{9}$$

Then exact solution of the problem is the limit of the recursive relation

$$y(x) = \lim_{n \rightarrow \infty} \sum_{k=0}^n y_k(x) \tag{10}$$

IV. He's Variational Iteration method (VIM)

. The basic idea of the VIM is as follows

Consider the ordinary differential equation of the form

$$Ly + N(y) = f(x), \quad x \in I \tag{11}$$

Where L and N are linear and nonlinear operators respectively, and $f(x)$ is any given inhomogeneous terms defined for $x \in I$

We defined a correctional functional for Eq. (11) as follows

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau) \left(Ly_n(\tau) + N(\tilde{y}_n(\tau)) - f(\tau) \right) d\tau \tag{12}$$

Where $\lambda(\tau)$ is a Lagrange multiplier obtained through variational theory, $y_n(x)$ is the n th approximation of $y(x)$ and $\tilde{y}_n(x)$ is a restricted variation meaning $\delta\tilde{y}_n(x) = 0$

By imposing the variation of both sides of Eq. (12) and taking the restricted variation we obtained

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \left(\int_0^x \lambda(\tau) Ly_n(\tau) d\tau \right) \tag{13}$$

$$\delta y_{n+1}(x) = \delta y_n(x) + \left[\lambda(\tau) \left(\int_0^\tau Ly_n(\xi) d\xi \right) \right]_{\tau=0}^{\tau=x} - \int_0^x \lambda^1(\tau) \left(\int_0^\tau L\delta y_n(\xi) d\xi \right) d\tau \tag{14}$$

Now by applying the stationary condition, the value of the Lagrange multiplier, $\lambda(\tau)$ can be found. Then the successive approximations, $y_n(x), n = 0, 1, 2, 3 \dots$ Can be found in the form

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau) \left(Ly_n(\tau) + N(y_n(\tau)) - f(\tau) \right) d\tau \tag{15}$$

The exact solution is then obtained as the limit of the successive approximations from Eq. (15)

$$y(x) = \lim_{n \rightarrow \infty} y_n(x)$$

V. Temimi and Ansari Method (TAM)

Consider the general differential equation in operator form as follows

$$L(y(x)) + N(y(x)) + f(x) = 0, B \left(y, \frac{dy}{dx} \right) = 0 \tag{16}$$

Where x is the independent variable, $y(x)$ is an unknown function, $f(x)$ is a given known function, L is a linear operator, N is a nonlinear operator and B is a boundary operator.

To implement TAM, we first assume an initial guess of the form, $y_0(x)$ that satisfy the equation as follows

$$L(y_0(x)) + f(x) = 0, B \left(y_0, \frac{dy_0}{dx} \right) = 0 \tag{17}$$

We consider the next iteration as follows

$$L(y_1(x)) + N(y_0(x)) + f(x) = 0, B \left(y_1, \frac{dy_1}{dx} \right) = 0 \tag{18}$$

Continuing the same way to obtain the subsequent terms, the general equation of the method becomes

$$L(y_{n+1}(x)) + N(y_n(x)) + f(x) = 0, B \left(y_{n+1}, \frac{dy_{n+1}}{dx} \right) = 0 \tag{19}$$

From Eq. (19), each $y(x)$ is considered alone as a solution for Eq. (16). This method easy to implement, straightforward and direct. The method gives better approximate solution which converges to the exact solution with only few members.

VI. EXPERIMENTAL EVALUATION

In this subsection, some linear and nonlinear problems are solved using the Adomian decomposition, Differential Transform method, Variational iteration method and the Temimi and Ansari method. This is to determine whether the approximate solution converges to the exact solution or there is marked error between them.

Example 6.1

Solve the differential equation, $\frac{dy}{dx} - y = 0$, subject to the condition, $y(0) = 1$

Solution by ADM.

Writing the given equation in operator form

$$Ly = y \tag{20}$$

Taking the inverse transform of both sides

$$L^{-1}(Ly) = L^{-1}(y)$$

$$y(x) - y(0) = L^{-1}(y)$$

Using the initial condition, we obtain

$$y(x) = 1 + L^{-1}(y)$$

Let $y(x) = \sum_{n=0}^{\infty} y_n(x)$

The decomposition of the problems becomes

$$\sum_{n=0}^{\infty} y_n(x) = 1 + L^{-1}(\sum_{n=0}^{\infty} y_n(x)) \tag{21}$$

Comparing both sides, we get the first five iterative solutions as

$$y_0(x) = 1$$

$$y_1(x) = x$$

$$y_2(x) = \frac{x^2}{2!}$$

$$y_3(x) = \frac{x^3}{3!}$$

$$y_4(x) = \frac{x^4}{4!}$$

Th recursive relation of the problem becomes

$$y_{n+1}(x) = \sum_{n=0}^{\infty} y_n(x), n \geq 0 \tag{22}$$

$$y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$y(x) = e^x$ (converges to the exact solution)

Solution by VIM

The correctional for Eq. (20), we have

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau)(Ly_n(\tau) - y_n(\tau))d\tau, n \geq 0 \tag{23}$$

The first five iterations using eq. (23) gives

$$y(0) = y_0(x) = 1$$

$$y_1(x) = 1 + x$$

$$y_2(x) = 1 + x + \frac{x^2}{2!}$$

$$y_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$y_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

The solution of the problem becomes

$$y(x) = \lim_{n \rightarrow \infty} y_n(x)$$

$$y(x) = e^x$$

Solution by DTM

Taking the DTM of both sides of Eq. (20)

$$(k + 1)! Y(k + 1) = Y(k)$$

Rearranging gives

$$Y(k + 1) = \frac{Y(k)}{(k+1)!} \tag{24}$$

So that for $k = 0,1,2, \dots$, we have the following

$$Y(1) = 1, Y(2) = \frac{1}{2!}, Y(3) = \frac{1}{3!}, Y(4) = \frac{1}{4!}, Y(5) = \frac{1}{5!}, \dots Y(n) = \frac{1}{n!}$$

Now the series expansion becomes

$$y(x) = \sum_{k=0}^n x^n Y(k)$$

$$y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!}$$

$y(x) = e^x$ (converges to the real solution)

Solution by TAM

Writing Eq. (20) in the form gives

$$Ly - y = 0 \tag{25}$$

Where $L(y) = y^1$, $N(y) = -y$, $f(x) = 0$

The first iterate become

$$L(y_0(x)) = 0$$

Taking the inverse operator, we obtain

$$\int_0^x y_0^1(\tau) d\tau = \int_0^x 0 d\tau + c$$

$$y_0(x) = 1$$

The second iterate of the problem now become

$$L(y_1(x)) + N(y_0(x)) + f(x) = 0, \quad y_1(0) = 1 \tag{26}$$

Solving the above gives

$$y_1(x) = 1 + x$$

The problem for the third iterate now become

$$L(y_2(x)) + N(y_1(x)) + f(x) = 0, \quad y_2(0) = 1 \tag{27}$$

Solving Eq. (27) gives

$$y_2(x) = 1 + x + \frac{x^2}{2!}$$

Following the same procedures, we get the fourth and fifth iterates as

$$y_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$y_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

Continuing in the same way, we get

$$y(x) = \lim_{n \rightarrow \infty} y_n(x)$$

$y(x) = e^x$ (converges to the real or exact solution)

Example 6.2 Solve the nonlinear differential equation, $y' + y^2 = 1$, subject to $y(0) = 0$

Solution by ADM

Writing the given equation in operator form

$$Ly + y^2 = 1, \tag{28}$$

Where the linear operator L is the highest order derivative

Taking the inverse operator of both sides, we get

$$L^{-1}(Ly) + L^{-1}(y^2) = L^{-1}(1)$$

$$y(x) - y(0) = x - L^{-1}(y^2)$$

Let $y(x) = \sum_{n=0}^{\infty} y_n(x)$

$$\sum_{n=0}^{\infty} y_n(x) = x - L^{-1}(\sum_{n=0}^{\infty} y_n(x)^2)$$

$$y_0(x) + y_1(x) + y_2(x) + \dots = x - L^{-1}(y_0(x) + y_1(x) + y_2(x) + \dots)^2 \tag{29}$$

Equating both sides, we obtain

$$y_0(x) = x$$

The recursive relation for the problem become

$$y_{n+1}(x) = -L^{-1}(A_n^2), n \geq 0$$

$$y_1(x) = -L^{-1}(A_0) = -L^{-1}(x^2) = -\frac{x^3}{3}$$

$$y_2(x) = -L^{-1}(A_1) = -L^{-1}(2y_0y_1) = -\frac{2x^5}{15}$$

$$y_3(x) = -L^{-1}(A_2) = -L^{-1}(2y_0y_2 + y_1^2) = -\frac{17x^7}{315}$$

The solution in series is of the form

$$y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \dots$$

$$y(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \dots$$

Which converges to the exact solution

$$y(x) = \frac{e^{2x} - 1}{e^{2x} + 1} = \tanh(x)$$

Solution by VIM

The correctional functional for Eq. (28) is given by

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau)(Ly_n(\tau) + y_n^2(\tau) - 1)d\tau, \quad n \geq 0$$

Where the Lagrange multiplier, $\lambda(\tau)$ is obtained optimally via the variational theory

$$y_{n+1}(x) = y_n(x) - \int_0^x (Ly_n(\tau) + y_n^2(\tau) - 1)d\tau$$

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(\frac{\partial}{\partial \tau} (y_n(\tau)) + y_n^2(\tau) - 1 \right) d\tau$$

Let $y(0) = y_0(x) = 0$

The first four iterates of the problem becomes

$$y_1(x) = x$$

$$y_2(x) = x - \frac{x^3}{3}$$

$$y_3(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \dots$$

Therefore, the exact solution of the problem become

$$y(x) = \tanh(x)$$

Solution by TAM

Recall Eq. (28) in operator form

$$Ly + y^2 - 1 = 0 \tag{30}$$

Where $Ly(x) = y^1, N(y) = y^2, f(x) = -1$

The first iterate of the problem gives

$$L(y_0(x)) = 1, y_0(0) = 0$$

Taking the inverse operator of both sides and substituting the initial condition, we obtain

$$y_0(x) = x$$

The second iterate of the problem become

$$L(y_1(x)) + N(y_0(x)) + f(x) = 0, y_1(0) = 0 \tag{31}$$

Taking the inverse transform operator of both sides, we get

$$\int_0^x y_1'(\tau)d\tau = \int_0^x (1 - \tau^2)d\tau$$

Applying the initial condition, we obtain

$$y_1(x) = x - \frac{x^3}{3}$$

The third iterate of the become

$$L(y_2(x)) + N(y_1(x)) + f(x) = 0, y_2(0) = 0$$

Taking the inverse operator of both sides, we get

$$\int_0^x y_2'(\tau) d\tau = \int_0^x \left[1 - \left(\tau - \frac{\tau^3}{3} \right)^2 \right] d\tau$$

Applying the initial condition, we obtain

$$y_2(x) = x - \frac{x^3}{3} + \frac{2x^5}{15}$$

Continuing in the same way, we get the fourth iterate as

$$y_2(x) = x - \frac{x^3}{3} + \frac{2x^5}{15}$$

$$y_3(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315}$$

$$y(x) = \lim_{n \rightarrow \infty} y_n(x)$$

The exact solution becomes

$$y(x) = \tanh(x)$$

Solution by DTM

Rewritten Eq. (28) in standard form and taking the Differential transform of both sides, we get

$$(k + 1)Y(k + 1) = \delta(k) - \sum_{r=0}^k Y(r)Y(k - r) \tag{32}$$

$$\Rightarrow Y(k + 1) = \frac{1}{k+1} [\delta(k) - \sum_{r=0}^k Y(r)Y(k - r)] \tag{33}$$

Therefore, for $k \geq 0$, then the values $Y(1), Y(2), Y(3) \dots$ are obtained as follows

$$Y(1) = 1, Y(3) = \frac{1}{3}, Y(5) = \frac{1}{5}, Y(7) = \frac{1}{7}$$

Similarly, for $k \geq 1$, the odd conditions vanished

Hence, $Y(x) = \sum_{k=0}^{\infty} t^k Y(k)$ become

$$Y(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$$

VII. DISCUSSION OF RESULTS AND CONCLUDING REMARKS

In this paper, we employ four semi-analytical iterative methods to solve linear and nonlinear differential equations. Selected problems were successfully solved with all four of the methods. The results obtained showed, the differential transformation method gives a rapidly convergent solution which requires less computational work provided the order of the equation is not high, Variational iteration method elegantly gives a convergent solution in a series of steps, Adomian decomposition method produce a solution which is in the form of a decomposing infinite series and Temimi and Ansari solves both linear and nonlinear problem in much easier iterative steps which converges to the exact solution without the application of any assumption. Thus, in comparison, the differential transform method is mathematically involving if the order is high, variational iteration method is efficient and powerful, Adomian decomposition method is easier to apply and Temimi & Ansari method gives accurate and reliable result identical to the exact solution. In all, they are found to be efficient, reliable, less time consuming and elegant depending on the problem been examined.

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