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Quasi weak – partial cone b – metric space and some fixed point results

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Abstract - *Let X be a non – empty set. In this paper is given a new space called the quasi weak – partial cone b - metric space X. There are defined right (left) open balls, right (left) closed balls, right topology, left topology, right Cauchy sequences, left Cauchy sequences and right (left) convergent sequences in it. Furthermore, there is proved the existence and uniqueness of a fixed point related to a nonlinear contraction using a comparison function in X. Some results are obtain as corollaries. These results generalize some well – known theorems in quasi weak – partial cone metric space. In addition, as illustrations are given some examples.*

Keywords — *Cauchy sequence, convergent sequence, comparison function, fixed point, quasi weak – partial b – cone metric space*

I. INTRODUCTION

Fixed point theory plays an important role in many fields of Mathematics as Functional Analysis, Economy, Informatics, etc. The study of this theory has developed during the years. Its evolution is focused on two directions, ones the generalization of metric spaces and the other the improvement of contractive conditions.

The authors in [1] expanded the metric space to cone metric space by replacing set of real numbers by a Banach space. The authors had studied fixed points in these spaces which complete Banach contractions and some other contractions. Many authors have worked on these spaces such as [2], [3], [4], [5], [6].

In 1994, Mathew [7] defined a new space which was called partial metric space. Later, in 1999 R.Heckmann [8], in his paper generalized partial metric spaces into weak- partial metric space. Many authors have worked related to fixed point in these spaces. ([9], [10], [11], [12], [13], [14])

Brakat et al [15] have given an interesting new space, generalizing weak partial metric space in weak quasi – partial metric space.

In this paper is defined weak quasi – partial cone b - metric space and are shown some topologic aspects of it. Furthermore, are shown some fixed point theorems and corollaries in this space. As applications of this theory, some results are illustrated by examples. Our obtained results are generalizations of some known results in metric space, cone metric space and partial metric space.

II. PRELIMINARIES

Definition 2.1 [1] Let P be a nonempty subset of E, where E is an ordered Banach space. The set P is called cone if it satisfies the following conditions:

1. $P \neq \{0\}$
2. For every $a, b \in R, ax + by \in P,$ for each $x, y \in P$
3. For every $x \in P$ then $-x \in P$.

The cone P is called normal [1] if for every $x, y \in P$ such that $x \ll y$ then $\|x\| \leq K\|y\|$, where $K > 0$. K is called the normality constant of P.

The authors in [1] have defined a partial ordering relation in cone P as follows:

For each $x, y \in P, x \leq y$ only if $y - x \in P$ and $x < y$ if $, x \leq y$ and $x \neq y$. For every $x, y \in P, x \ll y$ only if $y - x \in intP$.

Definition 2.2. [1] Let P be a cone and X a non – empty set. The map $d: X \times X \to P$ is called a cone metric if it satisfies the following conditions:

1. $d(x, y) = 0$ if and only if $x = y$, for every $x, y \in X$,

2. For every $x, y \in X, d(x,y) = d(y,x)$,
3. For each $x, y, z \in X, d(x,z) \leq d(x,y) + d(y,z)$.

The ordered couple (X, d) is called a cone metric space.

Definition 2.3. [15] Let M be a non – empty set. The map $\rho: M \times M \to R^+$ is called weak partial metric space if it accomplishes the following conditions:

1. $\rho(s,s) = \rho(s,t)$ if and only if $s = t$, for every $s, t \in M$,
2. $\rho(s,s) \leq \rho(s,t)$, for each $(s,t) \in M^2$,
3. $\rho(s,t) = \rho(t,s)$, for each $(s,t) \in M^2$,
4. $\rho(s,t) \leq \rho(s,z) + \rho(z,t)$, for every $s, t, z \in M$.

(M, ρ) is called weak partial metric space.

Definition 2.4. Let X be a non – empty set and P a cone. The function $q_w: X \times X \to P$ is called quasi – weak partial cone b - metric if it satisfies the following conditions:

1. $q_w(x,x) = q_w(x,y) = q_w(y,y)$ if and only if $x = y$, for every $x, y \in X$,
2. $q_w(x,x) \leq q_w(x,y)$, for each $(x,y) \in X^2$,
3. $q_w(x,y) \leq s(q_w(x,z) + q_w(z,y))$, for every $x, y, z \in M$ and $s \geq 1$.

The ordered couple (X, q_w) is called quasi weak partial cone b - metric space.

Example 2.5. Let $E = R^2$, $X = R$ and $P = \{(x,y) \in E, x, y \geq 0\}$ be a cone.
Define the map $q_w: X \times X \to P$ such that:

$$q_w(x,y) = \begin{cases} (max\{x,y\}, \frac{1}{y} - \frac{1}{x} + max\{x,y\}), & (x,y) \neq (0,0) \\ (0,0) & (x,y) = (0,0) \end{cases}$$

The map q_w is a quasi – weak partial cone b - metric and (X, q_w) is quasi – weak partial cone metric space with $s \geq 1$.

Below there is defined the topology and there are given some properties of quasi – weak partial cone metric space using the same methods as in Sila E., 2015 [2] for p – quasi cone metric space.

Definition 2.6. Let (X, q_w) be a quasi – weak partial cone b - metric space.
The set

$$B_w^l(x,c) = \{y \in X, q_w(y,x) \ll c + q_w(x,x)\}$$

is called left open ball centered in x with radius $c \gg 0$.
The set

$$B_w^r(x,c) = \{y \in X, q_w(x,y) \ll c + q_w(x,x)\}$$

is called right open ball centered in x with radius $c \gg 0$.

Definition 2.7. Let (X, q_w) be a quasi – weak partial cone b - metric space.
The set

$$B'^l_w(x,c) = \{y \in X, q_w(y,x) \leq c + q_w(x,x)\}$$

is called left closed ball centered in x with radius $c \gg 0$.
The set

$$B'^r_w(x,c) = \{y \in X, q_w(y,x) \leq c + q_w(x,x)\}.$$

is called right closed ball centered in x with radius $c \gg 0$.

Let (X, q_w) be a quasi – weak partial cone b - metric space.
Theorem 2.8 The family $\tau_w^r = \{\phi, X, G \subset X |$ for each $x \in G$, there exists $B_w^r(x,c) \subset G\}$ is a right topology of q_w.
The topology τ_w^r is called right topology obtained by q_w.

Theorem 2.9 The family $\tau_w^l = \{\phi, X, G \subset X |$ for each $x \in G$, there exists $B_w^l(x,c) \subset G\}$ is a left topology of q_w.
The topology τ_w^l is called right topology obtained by the quasi weak partial cone metric q_w.

The following propositions are proved for the right topology τ_w^r in (X, q_w).

Definition 2.10 The set $A \subset X$ is called right open if $A \in \tau_w^r$.

Definition 2.11 The set $V \subset X$ is called open neighborhood of point $a \in X$ if there exists a right open ball $B_w^r(a, c)$ such that $B_w^r(a, c) \subset V$.

Theorem 2.12 The topology τ_w^r in (X, q_w) accomplishes the first axiom of countability.

Definition 2.13. Let (X, q_w) be a quasi – weak partial cone b - metric space and $\{x_n\}_{n\in N}$ a sequence in it.

1. The sequence $\{x_n\}_{n\in N}$ is called right convergent to $x \in X$, if for each $c \gg 0$, there exists $n_0 \in N$, such that for every $n > n_0$ it yields $q_d(x_n, x) \ll c + q(x, x)$. This is denoted $\lim_{n\to\infty} q_d(x_n, x) = q(x, x)$
2. The sequence $\{x_n\}_{n\in N}$ is called left convergent to $x \in X$, if for each $c \gg 0$, there exists $n_0 \in N$, such that for every $n > n_0$ it yields $q_d(x, x_n) \ll c + q(x, x)$. This is denoted $\lim_{n\to\infty} q_d(x, x_n) = q(x, x)$.
3. The sequence $\{x_n\}_{n\in N}$ is called convergent to $x \in X$, if it is right and left convergent to $x \in X$.
4. The sequence $\{x_n\}_{n\in N}$ is called left Cauchy, if for every $n < m$ there exists $\lim_{n,m\to\infty} q_d(x_n, x_m)$ and it is finite.
5. The sequence $\{x_n\}_{n\in N}$ is called right Cauchy, if for every $n < m$ there exists $\lim_{n,m\to\infty} q_d(x_m, x_n)$ and it is finite.
6. The sequence $\{x_n\}_{n\in N}$ is called Cauchy, if it is right and left Cauchy.

Definition 2.14. The space (X, q_w) is called complete if every Cauchy sequence converges.

Definition 2.15. [16] The map $\varphi: P \to P$, where P is a cone in a Banach space E, is called a comparison function if it satisfies:

1. for all $t \in P$, $\varphi(t) < t$,
2. for all $t_1, t_2 \in P, t_1 < t_2$ it yields $\varphi(t_1) < \varphi(t_2)$
3. $\lim_{n\to\infty} \|\varphi^n(t)\| = 0$, for each $t \in P$.

III. Main results

Theorem 3.1. Let (X, q_w) be a Hausdorff complete quasi – weak partial cone b - metric space with constant of normality $K \geq 1$ and $T: X \to X$ a continuous map which satisfies the following nonlinear contraction:

$$q_w(Tx, Ty) \leq \varphi(\max\{q_w(x, x), q_w(y, y), q_w(x, y), q_w(y, x), q_w(Tx, x), q_w(Ty, y), q_w(x, Ty)\})$$

where $\varphi: X \to P$ is a comparison function. Then the function T has a unique fixed point.

Proof. Let $x_0 \in X$ and $\{x_n\}_{n\in N} \in X$ such that $x_n = Tx_{n-1}$.

If $x_n = x_{n-1} = \cdots = x_1 = x_0$, then the theorem is true.

Suppose that there exists $i \in N$, $x_i \neq x_0$.

$$\begin{aligned} q_w(x_{n+1}, x_n) &= q_w(Tx_n, Tx_{n-1}) \\ &\leq \varphi(\max\{q_w(x_n, x_n), q_w(x_{n-1}, x_{n-1}), q_w(x_n, x_{n-1}), q_w(x_{n-1}, x_n), q_w(x_{n+1}, x_n), q_w(x_n, x_{n-1}), \\ &\quad q_w(x_n, x_n)\} = \varphi(\max\{q_w(x_n, x_n), q_w(x_{n-1}, x_{n-1}), q_w(x_n, x_{n-1}), q_w(x_{n-1}, x_n), q_w(x_{n+1}, x_n)\}) \end{aligned}$$

Case 1.

$$\max\{q_w(x_n, x_n), q_w(x_{n-1}, x_{n-1}), q_w(x_n, x_{n-1}), q_w(x_{n-1}, x_n), q_w(x_{n+1}, x_n)\} = q_w(x_n, x_n)$$

So the following inequality holds

$$q_w(x_{n+1}, x_n) \leq \varphi(q_w(x_n, x_n)) \leq \varphi^2\big(q_w(x_{n-1}, x_{n-1})\big) \leq \cdots \leq \varphi^n(q_w(x_0, x_0)).$$

Case 2.

$$\max\{q_w(x_n, x_n), q_w(x_{n-1}, x_{n-1}), q_w(x_n, x_{n-1}), q_w(x_{n-1}, x_n), q_w(x_{n+1}, x_n)\} = q_w(x_{n-1}, x_{n-1})$$

As a results

$$q_w(x_{n+1}, x_n) \leq \varphi(q_w(x_{n-1}, x_{n-1})) \leq \varphi^2\big(q_w(x_{n-1}, x_{n-1})\big) \leq \cdots \leq \varphi^{n-1}(q_w(x_0, x_0)).$$

Case 3.

$$\max\{q_w(x_n,x_n),q_w(x_{n-1},x_{n-1}),q_w(x_n,x_{n-1}),q_w(x_{n-1},x_n),q_w(x_{n+1},x_n)\}=q_w(x_n,x_{n-1})$$

Consequently, it yields
$q_w(x_{n+1},x_n)\le\varphi(q_w(x_n,x_{n-1}))$
In this case there are two options either
$q_w(x_{n+1},x_n)\le\varphi(q_w(x_n,x_{n-1})\le\cdots\le\varphi^n(q_w(x_1,x_0))$ or $q_w(x_{n+1},x_n)\le\varphi(q_w(x_n,x_{n-1})\le\cdots\le\varphi^n(q_w(x_0,x_0))$

Case 4.

$$\max\{q_w(x_n,x_n),q_w(x_{n-1},x_{n-1}),q_w(x_n,x_{n-1}),q_w(x_{n-1},x_n),q_w(x_{n+1},x_n)\}=q_w(x_{n+1},x_n)$$

As a results $q_w(x_{n+1},x_n)\le\varphi(q_w(x_{n+1},x_n))$. This case cannot happen because $\varphi(t)<t$, for each $t\in P$.

Consequently for each case the inequality $q_w(x_{n+1},x_n)\le\varphi^n(c)$, where $c\in P$ and $n\in N$, holds.
Since the cone P is normal with constant of normality K, it yields:
$\|q_w(x_{n+1},x_n)\|\le K\|\varphi^n(c)\|$.
Taking limit in this inequality, it results
$\lim_{n\to+\infty}\|q_w(x_{n+1},x_n)\|\le K\lim_{n\to+\infty}\|\varphi^n(c)\|=0$.
Consequently, $\lim_{n\to+\infty}q_w(x_{n+1},x_n)=0$.
For $n,k\in N$

$$\begin{aligned}q_w(x_{n+k},x_n)&\le s\big(q_w(x_{n+k},x_{n+1})+q_w(x_{n+1},x_n)\big)\le s^2q_w(x_{n+k},x_{n+2})+s^2q_w(x_{n+2},x_{n+1})+sq_w(x_{n+1},x_n)\le\cdots\\&\le s^kq_w(x_{n+k},x_{n+k-1})+s^{k-1}q_w(x_{n+k-1},x_{n+k-2})+\cdots+sq_w(x_{n+1},x_n)\\&\le s^k\varphi^{n+k}(c)+s^{k-1}\varphi^{n+k-1}(c)+\cdots+s\varphi^{n+1}(c)\\&\le[s^{n+k}\varphi^{n+k}(c)+s^{n+k-1}\varphi^{n+k-1}(c)+\cdots+s^{n+1}\varphi^{n+1}(c)\,]\frac{1}{s^n}\le\frac{1}{s^n}\,\frac{s^{n+1}\left(1-\left(s\varphi(c)\right)^k\right)\varphi^n(c)}{1-s\varphi(c)}\\&\le\frac{s\varphi^n(c)}{1-s\varphi(c)}\end{aligned}$$

$$\|q_w(x_{n+k},x_n)\|\le K\left\|\frac{s\varphi^n(c)}{1-s\varphi(c)}\right\|$$

Taking limit of both sides $\lim_{n,k\to+\infty}\|q_d(x_{n+k},x_n)\|\le K\lim_{n\to+\infty}\left\|\frac{s\varphi^n(c)}{1-s\varphi(c)}\right\|=0$

Consequently, the sequence $\{x_n\}_{n\in N}$is left Cauchy.
Using the same technique it can be shown that the sequence $\{x_n\}_{n\in N}$ is right Cauchy. As a result the sequence $\{x_n\}_{n\in N}$ is Cauchy. Since the space (X,q_d) is complete then the sequence $\{x_n\}_{n\in N}$ converges to $u\in X$, so $\lim_{n\to+\infty}Tx_n=u$.
The next step is to show thar u is a fixed point of T, $Tu=u$.
Since $\lim_{n\to+\infty}q_w(Tx_n,u)=q_w(u,u)$ and the map T is continuous $\lim_{n\to+\infty}q_w(T(Tx_{n-1}),u)=q_w(Tu,u)$. Due to the fact that the space is Hausdorff, it yields $q_w(Tu,u)=q_w(u,u)$. As a result $Tu=u$ and u is a fixed point of T.
Below is shown the uniqueness of the fixed point u.
Let $v\in X$, $v\ne u$, another fixed point of map T, so $Tv=v$.

$$\begin{aligned}q_w(u,v)=q_w(Tu,Tv)&\le\varphi(\max\{q_w(u,u),q_w(v,v),q_w(u,v),q_w(v,u),q_w(u,u),q_w(v,v),q_w(v,u)\})\\&=\varphi(\max\{q_w(u,u),q_w(v,v),q_w(u,v),q_w(v,u)\})\end{aligned}$$

Case 1. $\max\{q_w(u,u),q_w(v,v),q_w(u,v),q_w(v,u)\}=q_w(u,u)$

$$q_w(u,v)\le\varphi\big(q_w(u,u)\big)<q_w(u,u)$$

This case cannot happen and $u=v$.

Case 2. $\max\{q_w(u,u),q_w(v,v),q_w(u,v),q_w(v,u)\}=q_w(v,v)$

$$q_w(u,v)\le\varphi\big(q_w(v,v)\big)<q_w(v,v)$$

This case cannot happen and $u=v$.

Case 3. $\max\{q_w(u,u),q_w(v,v),q_w(u,v),q_w(v,u)\}=q_w(u,v)$

$$q_w(u,v)\le\varphi\big(q_w(u,v)\big)<q_w(u,v)$$

Case 4. $\max\{q_w(u,u), q_w(v,v), q_w(u,v), q_w(v,u)\} = q_w(v,u)$
It is true that $q_w(u,v) < q_w(v,u)$.
Furthermore $q_w(v,u) < q_w(u,v)$. Consequently $q_w(v,u) = q_w(u,v)$ and it happens only when $u = v$.
As a result, u is a unique fixed point of T.

The following example illustrates Theorem 3.1.

Example 3.2. Let $P = \{(x,y) \in R^2, x,y > 0\}$ be a cone with $K = 1$ and $X = [0,1]$.
Define the map $q_w: X \times X \longrightarrow P$ such that:

$$q_w(x,y) = \begin{cases} (\max\{x,y\}, \min\left\{\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{x}\right\} + \max\{x,y\}), & (x,y) \neq (0,0) \\ (0,0), & x = y = 0 \end{cases}$$

$q_w(x,y)$ is a quasi – weak partial cone metric and (X, q_w) is quasi – weak partial cone metric space.
Let $T: X \longrightarrow X$, $Tx = \frac{x}{6}$ be a continuous map and $\varphi: X \longrightarrow X \times X$ a comparison function such that $\varphi(x) = (\frac{x}{3}, \frac{x}{3})$.
Below is shown that the map T completes the nonlinear contraction condition of Theorem 3.1.
For this are taken the following cases:
Case 1. $x < y$

$$q_w(Tx,Ty) = q_w\left(\frac{x}{6}, \frac{y}{6}\right) = \left(\max\left\{\frac{x}{6}, \frac{y}{6}\right\}, \min\left\{\frac{6}{x} - \frac{6}{y}, \frac{6}{y} - \frac{6}{x}\right\} + \max\left\{\frac{x}{6}, \frac{y}{6}\right\}\right) = \left(\frac{y}{6}, \frac{6(x-y)}{xy} + \frac{y}{6}\right) < \left(\frac{y}{6}, \frac{y}{6}\right)$$

$$\varphi(\max\{q_w(x,x), q_w(y,y), q_w(x,y), q_w(y,x), q_w(Tx,x), q_w(Ty,y), q_w(x,Ty)\})$$

$$= \varphi\left(\max\left\{(x,x), (y,y), \left(y, \frac{(x-y)}{xy} + y\right), \left(y, \frac{(x-y)}{xy} + y\right), \left(x, \frac{-5}{x} + x\right), \left(y, \frac{-5}{y} + y\right), \left(y, \frac{1}{x} - \frac{6}{y} + y\right) \text{or } (y, \frac{6}{y} - \frac{1}{x} + y)\right\}\right)$$

$$= (\frac{y}{3}, \frac{y}{3})$$

Consequenlty,

$$q_d(Tx,Ty) \leq \varphi(\max\{q_w(x,x), q_w(y,y), q_w(x,y), q_w(y,x), q_w(Tx,x), q_w(Ty,y), q_w(x,Ty)\})$$

Case 2. $y < x$

$$q_w(Tx,Ty) = q_w\left(\frac{x}{6}, \frac{y}{6}\right) = \left(\max\left\{\frac{x}{6}, \frac{y}{6}\right\}, \min\left\{\frac{6}{x} - \frac{6}{y}, \frac{6}{y} - \frac{6}{x}\right\} + \max\left\{\frac{x}{6}, \frac{y}{6}\right\}\right) = \left(\frac{x}{6}, \frac{6(y-x)}{xy} + \frac{x}{6}\right) < \left(\frac{x}{6}, \frac{x}{6}\right)$$

$$\varphi(\max\{q_w(x,x), q_w(y,y), q_w(x,y), q_w(y,x), q_w(Tx,x), q_w(Ty,y), q_w(x,Ty)\})$$

$$= \varphi\left(\max\left\{(x,x), (y,y), \left(x, \frac{(y-x)}{xy} + x\right), \left(x, \frac{(y-x)}{xy} + x\right), \left(x, \frac{-5}{x} + x\right), \left(y, \frac{-5}{y} + y\right), \left(x, \frac{1}{x} - \frac{6}{y} + x\right) or (x, \frac{6}{y} - \frac{1}{x} + x)\right\}\right)$$

$$= (\frac{x}{3}, \frac{x}{3})$$

As a result,

$$q_w(Tx,Ty) \leq \varphi(\max\{q_w(x,x), q_w(y,y), q_w(x,y), q_w(y,x), q_w(Tx,x), q_w(Ty,y), q_w(x,Ty)\})$$

Case 3. $x = y \neq 0$

$$q_w(Tx,Ty) = q_w\left(\frac{x}{6}, \frac{x}{6}\right) = \left(\frac{x}{6}, \frac{x}{6}\right) < \left(\frac{x}{3}, \frac{x}{3}\right) = \varphi(\max\{q_w(x,x), q_w(y,y), q_w(x,y), q_w(y,x), q_w(Tx,x), q_w(Ty,y), q_w(x,Ty)\})$$

Case 4. $x = y = 0$
This is a trivial case.
Since the map T accomplishes the condition of Theorem 3.1, it has a unique fixed point $x = 0$.

Corollary 3.3 Let (X, q_w) be a Hausdorff complete quasi – weak partial cone b - metric space with constant of normality $K \geq 1$ and $T: X \to X$ a continuous map which satisfies the following nonlinear contraction:

$$q_w(T(x), T(y)) \leq \varphi(q_w(x,y))$$

where $\varphi: P \to P$ is a comparison function, then the map T has a unique fixed point.
Proof.

$$q_w\big(T(x), T(y)\big) \leq \varphi\big(q_w(x,y)\big) \leq \varphi(\max\{q_w(x,x), q_w(y,y), q_w(x,y), q_w(y,x), q_w(Tx,x), q_w(Ty,y), q_w(x,Ty)\})$$

This accomplishes the conditions of Theorem 3.1, consequently the map T has a unique fixed point.

Corollary 3.4. Let (X, q_w) be a Hausdorff complete quasi – weak partial cone b - metric space with constant of normality $K \geq 1$ and $T: X \to X$ a continuous map which satisfies the following nonlinear contraction:
$q_w(Tx, Ty) \leq h \max\{q_w(x, x), q_w(y, y), q_w(x, y), q_w(y, x), q_w(Tx, x), q_w(Ty, y), q_w(x, Ty)\}$
where $x, y \in X$ and $0 < h < 1$, then the map T has a unique fixed point.

Proof. Taking $\varphi(t) = ht$, in Theorem 3.1, it yields that the map T has a unique fixed point.

Remark 3.5. Corollary 3.4 generalizes the results of Ciric Lj. B. (1974)[17] in quasi – weak partial cone b - metric space.

Corollary 3.6. Let (X, q_w) be a Hausdorff complete quasi – weak partial cone b - metric space with constant of normality $K \geq 1$ and $T: X \to X$ a continuous map which satisfies the following nonlinear contraction:

$$q_w(Tx, Ty) \leq \lambda_1 \, q_w(x, x) + \lambda_2 \, q_w(y, y) + \lambda_3 \, q_w(x, y) + \lambda_4 \, q_w(y, x) + \lambda_5 \, q_w(Tx, x) + \lambda_6 \, q_w(Ty, y) + \lambda_7 \, q_w(x, Ty)$$

where $x, y \in X$ and $0 < \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 < \frac{1}{7}$, then the map T has a unique fixed point.

Proof. From the contraction condition, it yields

$$\begin{aligned} q_w(Tx, Ty) &\leq \lambda_1 \, q_w(x, x) + \lambda_2 \, q_w(y, y) + \lambda_3 \, q_w(x, y) + \lambda_4 \, q_w(y, x) + \lambda_5 \, q_w(Tx, x) + \lambda_6 \, q_w(Ty, y) + \lambda_7 \, q_w(x, Ty) \\ &\leq (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7)(q_w(x, x) + q_w(y, y) + q_w(x, y) + q_w(y, x) + q_w(Tx, x) \\ &\quad + q_w(Ty, y) + q_w(x, Ty)) \\ &\leq 7(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7)\max\{q_w(x, x), q_w(y, y), \\ &\quad q_w(x, y), q_w(y, x), q_w(Tx, x), q_w(Ty, y), q_w(x, Ty)\} \end{aligned}$$

Denoting $7(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7) = h, 0 < h < 1$, it completes the contraction condition of Corollary 3.4. As a result, the map T has a unique fixed point.

Remark 3.7. Corollary 3.6 is a generalization of theorem Hardy-Rogers (1973) [18] in quasi – weak partial cone b - metric space.

Corollary 3.8. Let (X, q_w) be a Hausdorff complete quasi – weak partial cone b - metric space with constant of normality $K \geq 1$ and $T: X \to X$ a continuous map which satisfies the following nonlinear contraction:
$q_w(Tx, Ty) \leq h \max\{ q_w(Tx, x), q_w(Ty, y)\}$
where $x, y \in X$ and $0 < h < 1$, then the map T has a unique fixed point.

Remark 3.9. Corollary 3.8 generalizes theorem of Bianchini R. M. T. (1972) [19] quasi – weak partial cone b - metric space.

Corollary 3.10. Let (X, q_w) be a Hausdorff complete quasi – weak partial cone b - metric space with constant of normality $K \geq 1$ and $T: X \to X$ a continuous map which satisfies the following nonlinear contraction:

$$q_w(Tx, Ty) \leq h \, q_w(x, y)$$

where $x, y \in X$ and $0 < h < 1$, then the map T has a fixed point.
Proof.
$q_w(Tx, Ty) \leq h \, q_w(x, y) \leq h \max\{q_w(x, x), q_w(y, y), q_w(x, y), q_w(y, x), q_w(Tx, x), q_w(Ty, y), q_w(x, Ty)\}$.
It completes the contraction of Corollary 3.4 so the map T has a unique fixed point.

Remark 3.11. Corollary 3.10 generalizes Theorem of Banach (Huang L.G., Zhang X. 2007) [1] in quasi – weak partial cone b - metric space.

IV. CONCLUSIONS

This paper is a contribution in Fixed Point Theory. It gives a new space called quasi – weak partial cone b - metric space which is a generalization of cone metric space. There are defined right and left topologies and Cauchy convergences in this space. Furthermore, there is proved an important result which emphasizes the existance and uniqueness of a fixed point for a nonlinear contraction functions. In addition, there are obtained some corollaries which generalize some well – known results.

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