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Asymptotic Attractivity Result For Neutral Functional Differential Equation

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Abstract: *In this paper an existence result for local asymptotic attractivity of the solutions is proved for a nonlinear neutral functional differential equation in Banach space under the mixed generalized Lipchitz's and caratheodory conditions which gives the existence as well as asymptotic stability of solutions.*

Keyword and Phrases: *Neutral functional differential equation, fixed point theorem, asymptotic attractive solutions mixed generalized Lipchitz's, caratheodory conditions.*

I. Introduction

Let $\mathbb{R}$ denote the real line and let $\mathrm{I_o} = [-\mathrm{r}, 0]$ and $\mathrm{I} = [0, \mathrm{a}]$ be two closed and bounded intervals in $\mathbb{R}$. Let $\mathrm{J} = I_o \cup \mathrm{I}$, then j is a closed and bounded intervals in $\mathbb{R}$. Let $\mathbb{C}$ denote the Banach space of all continuous real valued functions ϕ on $\mathrm{I_o}$ with the supremum norm $\|.\|_c$ defined by

$$\|\phi\|_c = \sup_{t\in I_o}|\phi(t)|$$,Cleary $\mathbb{C}$ is a Banach algebra with this norm.

Consider the functional differential equation (in short FDE)

$$\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t),x_t)}\right] = g(t,x(t),x_t) \qquad a.e.\ t \in I \qquad (1.1)$$

$$x(t) = \phi(t)\,, \qquad t \in I_o$$

Where $f: I \times \mathbb{R} \times \mathbb{C} \to \mathbb{R} - \{0\}$, $g: I \times \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ and for each $t \in I$, $x_t: I_O \to \mathbb{C}$ is a continuous function defined by $x_t(\theta) = x(t+\theta)$ for all $\theta \in I_{o.}$

By a solution of FDE (1.1), we means a function $x \in (J,\mathbb{R}) \cap BC(J,\mathbb{R}) \cap C(I_o,\mathbb{R})$ that satisfies the equations in (1.1), where $BC(J,\mathbb{R})$ is the space of all bounded and continuous real valued functions on J. The functional differential equations have been the most active area of research since long time. See Hale[8], Henderson[9] and the references therein. But the study of functional differential equations in Banach algebra is very rare in the literature. Very recently the study along this line has been initiated via fixed point theorems. See Dhage and Regan [4] and Dhage[2] and the references therein. The FDE (1.1) is new to the literature and the study of this problem definitely contribute immensely to the area of functional differential equations. See Dhage, Salunkhe and R.Verma [7] and the references therein. In this paper, we prove the uniform local asymptotic attractivity via a classical hybrid fixed point theorem of Dhage[8] which gives the asymptotic stability of the solutions for the FDE (1.1).

II. Auxiliary Results

Let $X = BC(J,\mathbb{R})$ be the space of continuous and bounded real valued functions on I, and let Ω be a subset of X. Let $Q: X \to X$ be an operator and consider the following operator equation in X,

$$x(t) = (Qx)(t) \quad \text{for all} \quad t \in I \qquad (2.1)$$

Below we give different Characterizations of the solutions for the operator equation (2.1) on I.

Definition (2.1): We say that solutions of the equation (2.1) are locally attractive if there exists an $x_0 \in BC(I,\mathbb{R})$, $r > 0$ such that for all solutions $x = x(t)$ and $y = y(t)$ of equation (2.1) belonging to $\overline{\mathcal{B}}_r(x_0) \cap \Omega$, we have that

$$\lim_{t\to\infty}[x(t) - y(t)] = 0 \tag{2.2}$$

when for each $\epsilon > 0, \exists\, T > 0$ such that

$$|x(t) - y(t)| \le \epsilon \tag{2.3}$$

for all $x, y \in \overline{\mathcal{B}}_r(x_0) \cap \Omega$ being solution of (2.1) and for $t \ge T$, we will say that solutions of equation (2.1) are uniformly locally attractive on I.

Remark (2.1): In this paper, however, we will deal with only the local characterization of solutions for equation (2.1) on I.

Definition (2.2): A line $y = at + b$ where a, b are real numbers, is called an attractor for the solution $x \in BC(I,\mathbb{R})$ to equation (2.1) if condition in equation (2.2) satisfied. The solution x of equation (2.1) is also called asymptotic to line $y(t) = at + b$ and the line is an asymptote for the solution x on I.

The following definition useful in the sequel.

Definition (2.3): The solutions $x = x(t)$ and $y = y(t)$ of equation (2.1) are said to be locally asymptotically attractive if there exists an $x_0 \in BC(I,\mathbb{R})$, and $r > 0$ such that $x, y \in \overline{\mathcal{B}}_r(x_0) \cap \Omega$ the condition is satisfied and there is a line which is common attractor to them on I. In the case when condition (2.2) is satisfied uniformly with respect to the set $\overline{\mathcal{B}}_r(x_0) \cap \Omega$, i.e. if $\forall\, \epsilon > 0, \exists\, T > 0$ such that the inequality in (2.3) is satisfied for $t \ge T$ and $x, y \in \overline{\mathcal{B}}_r(x_0) \cap \Omega$ being solution of equation (2.1) having a line as a common attractor, then we will say that solutions of equation (2.1) are locally asymptotically attractive on I.

Remark (2.2): Locally asymptotically attractive solutions are asymptotically attractive, but the vice versa may not be true. Similarly, uniformly locally asymptotically attractive solutions are asymptotically attractive, but the vice versa may not be true.

We seek the solution of FDE (1.1) in the space $BC(I,\mathbb{R})$ of continuous and bounded real valued functions defined on I. Defined norm $\|.\|$ and multiplication '$\cdot$' in $BC(I,\mathbb{R})$ by

$$\|x\| = \sup_{t\to I}|x(t)| \text{ , and } (x\cdot y)(t) = x(t)\cdot y(t), \quad t \to I,$$

then clearly $BC(I,\mathbb{R})$ is a Banach algebra with respect to norm $\|.\|$ and multiplication '$\cdot$'. Let $L^1(I,\mathbb{R})$ denote the space of Lebesque integral functions on I, and $\|.\|_{L^1}$ in $L^1(I,\mathbb{R})$ is defined by

$$\|x\|_{L^1} = \int_0^\infty |x(t)|ds .$$

Definition (2.4): An operator $Q: X \to X$ is called Lipschitz if there exists constant $K > 0$ such that $\|Qx - Qy\| \le K\|x - y\|$ for all $x, y \in X$. The constant K is called the Lipschitz constant of Q on$X$.

Definition (2.5): An operator Qon Banach space X in itself is called compact if for any bounded subset S of X, $Q(S)$ is relatively compact subset of X. If Q is continuous and compact, then it is called completely continuous on X.

The following a hybrid fixed point theorem of Dhage [8] is useful for proving the existence result for uniform local asymptotic attractivity of the solutions of FDE (1.1).

Theorem (2.1) Dhage[8]: Let S be a closed convex and bounded subset of the Banach algebra X and let $A, B: S \to X$ be two operators such that

i. A is Lipschitz with the Lipschitz constant K.
ii. B is completely continuous
iii. $AxBx \in S$ for all $x \in S$ and
iv. $MK < 1$, where $M = \|B(S)\| = sup\{\|Bx\|: x \in S\}$.

Then the operator equation

$$AxBx = x \tag{2.4}$$

has a solution and the set of all solutions is compact in S.

***Remark (2.4)*:** If x and y are two solutions of the operator equation (2.4) in S, then we have

$$\|x - y\| \leq \left(\frac{1}{1 - KM}\right) \|Ay\| \|Bx - By\|$$

Form the above inequality, we infer that the uniqueness's is achieved only if B is a constant mapping on S which is not always true.

III. Main Result

We gives an account in words of our hypotheses for prove the main results of this paper.

We consider the following set of hypothesis in the sequel

(A_1) The function $\alpha: I_0 \to I$ is continuous.

(A_2) The function $f: I \times \mathbb{R} \times \mathbb{C} \to \mathbb{R} - \{0\}$ is continuous and there exists a bounded function $\ell: I \to I$ with bound L satisfying

$$|f(t, x, x_t) - f(t, y, y_t)| \leq \ell(t) max\{|x - y|, \|x_t - y_t\|_c\}, \text{ for all, } t \to I \text{ and } x, y \in \mathbb{R}.$$

(A_3) The function $F: I \to \mathbb{R}$ defined by $F(t) = |f(t, 0,0)|$ is bounded with $F_0 = \sup_{t \geq 0} F(t)$.

(B_1) The function $\gamma: I \to I$ is measurable and the function $\beta: I \to I$ is continuous.

(B_2) The function $\phi: I \to I$ is continuous and $\lim_{t \to \infty} \phi(t) = 0$.

(B_3) The function $g: I \times \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ is continuous and there exists a continuous function $b: I \times I \to \mathbb{R}$ satisfying $|g(t, s, x_s)| \leq b(t, s)$ for all $t, s \in I$ and $x \in \mathbb{R}$,

where $\lim_{t \to \infty} \int_0^{\beta(t)} b(t, s) ds = 0$

Remark (3.1): Note that if the hypothesis (B_2) and (B_3) hold, then there exist constants $k_1 > 0$ and $k_2 > 0$ such that $k_1 = \sup_{t \geq 0} \phi(t)$ and $k_2 = \sup_{t \geq 0} \gamma(t) = \sup_{t \geq 0} \left[\int_0^{\beta(t)} b(t, s) ds\right]$.

Theorem (3.1): Assume that the hypotheses (A_1) through (A_3) and (B_1) through (B_3) holds, furthermore, if $L(k_1 + k_2) < 1$ where $k_1 = \sup_{t \geq 0} \phi(t) > 0$ and $k_2 = \sup_{t \geq 0} \left[\int_0^{\beta(t)} b(t, s) ds\right]$, $k_1, k_2 > 0$ then the FDE (1.1) has at least one solution in the space $BC(I, \mathbb{R})$. Moreover, solutions of the FDE (1.1) are uniformly locally asymptotically attractive on I.

Proof: **Let** set $X = BC(I, \mathbb{R})$. Consider the closed ball $\overline{\mathcal{B}}_r(0)$ in X center at origin 0 and of radius r, where $r = \frac{F_0(k_1 + K_2)}{1 - L(k_1 + K_2)} > 0$.

Now the FDE (1.1) is equivalent to the functional integral equation (in short FIE

$$x(t) = [f(t, x(t), x_t)] \left(\phi(0) + \int_0^{\beta(t)} g(t, x(s), x_s) ds\right), \text{ if } t \in I \tag{3.1}$$

and $\qquad x(t) = \phi(t)\,,\ \ t \in I_o$

Define the two operators A and B on closed ball $\overline{\mathcal{B}}_r(0)$ in X byType equation here.

$$Ax(t) = f(t, x(t), x_t)$$

and $$Bx(t) = \phi(0) + \int_0^{\beta(t)} g(t, x(s), x_s) ds \tag{3.3}$$

for all $t \in I$, since the hypothesis (A$_2$) holds, the mapping A is well defined and the function Ax is continuous and bounded on I. Again, since the functions ϕ and β are continuous on I. The function Bx is also continuous and bounded in view of hypothesis (B$_3$). Therefore A and B define the operators $A, B: \overline{\mathcal{B}}_r(0) \to X$. We shall show that A and B satisfy all the essential of theorem (2.1).

Firstly, we show that A is a Lipschitz operator on $\overline{\mathcal{B}}_r(0)$. Let $x, y \in \overline{\mathcal{B}}_r(0)$ be arbitrary. Then by hypothesis (A$_2$), we have

$$|Ax(t) - Ay(t)| = |f(t, x(t), x_t) - f(t, y(t), y_t)| \le \ell(t) max\{|x - y|, \|x_t - y_t\|_c\}$$

$$\le L\|x - y\|$$

for all $t \in I$. Taking the supremum over t,

$$\|Ax - Ay\| \le L\|x - y\|\,, \quad \text{for all } x, y \in \overline{\mathcal{B}}_r(0).$$

This shows that A is a Lipschitz on $\overline{\mathcal{B}}_r(0)$ with the Lipschitz constant L.

Now, we show that B is a continuous and compact operator on $\overline{\mathcal{B}}_r(0)$.

First we show that B is a continuous operator on $\overline{\mathcal{B}}_r(0)$. Let $\epsilon > 0,\ x, y \in \overline{\mathcal{B}}_r(0)$ such that $\|x - y\| \le \epsilon$.

Then

$$|Bx(t) - By(t)| \le \int_0^{\beta(t)} |g(t, x(s), x_s) - g(t, y(s), y_s)| ds$$

$$\le \int_0^{\beta(t)} [|g(t, x(s), x_s)| + |g(t, y(s), y_s)|] ds$$

$$\le \int_0^{\beta(t)} |g(t, x(s), x_s)|\, ds + \int_0^{\beta(t)} |g(t, y(s), y_s)|\, ds$$

$$\le 2\int_0^{\beta(t)} \alpha(t)\, b(t, s) ds = 2\gamma(t) \tag{3.3}$$

Hence by impact of hypothesis (B$_3$), we infer that $\exists\ T > 0$ such that $\gamma(t) \le \epsilon$ for $t \ge T$. Thus for

$t \ge T$ we derive inequality (3.3) that $|Bx(t) - By(t)| \le 2\epsilon$. Further, let us assume that $t \in [0, T]$. Then evaluating similarly to above we obtain,

$$|Bx(t) - By(t)| \leq \int_0^{\beta(t)} |g(t, x(s), x_s) - g(t, y(s), y_s)| ds$$

$$\leq \int_0^{\beta(t)} \mathcal{W}_r^T(g, \epsilon) ds$$

$$\leq \beta_T \mathcal{W}_r^T(g, \epsilon) \tag{3.4}$$

where $\beta_T = sup\{\beta(t): t \in [0, T]\}$ and

$\mathcal{W}_r^T(g, \epsilon) = sup\{|g(t, x(s), x_s) - g(t, y(s), y_s)|: t \in [0, T], s \in [0, \beta_T], x, y \in [-r, 0] \cap [0, r], |x - y| \leq \epsilon\}$.
Obviously, we have in view of the continuity of β that $\beta_T < \infty$. Moreover, from the uniform continuity of the function $g(t, x(s), x_s)$ on the set $[0, T] \times [0, \beta_T] \times [-r, r]$ we derive that $\mathcal{W}_r^T(g, \epsilon) \to 0$ as $\in \to o$. Now from (3.3), (3.4) and above facts, we observed and concluded that the operator B maps on ball $\overline{\mathcal{B}}_r(0)$ continuously into itself.

Next, we show B is compact on $\overline{\mathcal{B}}_r(0)$. And for this, it is enough to show that every sequence $\{Bx_n\}$ in $B(\overline{\mathcal{B}}_r(0))$ has a Cauchy subsequence. Now by (B_2) and (B_3)

$$|Bx_n(t)| \leq |\phi(0)| + \int_0^{\beta(t)} |g(t, x_n(s), (x_n)_s)| ds$$

$$\leq K_1 + \gamma(t)$$

$$\leq K_1 + K_2$$

(3.5)

for all $t \in I$. Taking supremum over t, we obtain $\|Bx_n\| \leq K_1 + K_2$ for all $n \in N$, this shows that sequence $\{Bx_n\}$ is a uniformly bounded sequence in $B(\overline{\mathcal{B}}_r(0))$. Now we show that it is equicontinous.

Let $\epsilon > 0$ be given. Since $\lim_{t\to\infty} \phi(t) = 0$ and $\lim_{t\to\infty} \gamma(t) = 0$, there are constants $T_1 > 0$ and $T_2 > 0$ such that $|\phi(t)| < \frac{\epsilon}{4}$ for all $t \geq T_1$ and $|\gamma(t)| < \frac{\epsilon}{4}$ for all $t \geq T_2$. Let $T = max\{T_1, T_2\}$. Let $t, \tau \in I$ be arbitrary. If $t, \tau \in [0, T]$, then we have

$$|Bx_n(t) - Bx_n(\tau)| \leq |\phi(0) - \phi(0)| + \left|\int_0^{\beta(t)} g(t, x_n(s), x_{n_s}) ds - \int_0^{\beta(\tau)} g(\tau, x_n(s), x_{n_s}) ds\right|$$

$$\leq \left|\int_0^{\beta(t)} g(t, x_n(s), x_{n_s}) ds - \int_0^{\beta(t)} g(\tau, x_n(s), x_{n_s}) ds\right| +$$

$$\left|\int_0^{\beta(t)} g(\tau, x_n(s), x_{n_s}) ds - \int_0^{\beta(\tau)} g(\tau, x_n(s), x_{n_s}) ds\right|$$

$$\leq \int_0^{\beta(t)} |g(t, x_n(s), (x_n)_s) - g(\tau, x_n(s), (x_n)_s)| ds + \left|\int_{\beta(\tau)}^{\beta(t)} |g(\tau, x_n(s), (x_n)_s)| ds\right|$$

$$\leq \int_0^{\beta_T} |g(t, x_n(s), (x_n)_s) - g(\tau, x_n(s), (x_n)_s)| ds + |\gamma(t) - \gamma(\tau)|$$

By the uniform continuity of the function ϕ, γ on $[0, T]$ and the function g in $[0, T] \times [0, \beta_T] \times [-r, r]$, we obtain $|Bx_n(t) - Bx_n(\tau)| \to 0$ as $t \to \tau$.

If $t, \tau \geq T$,then we have

$$|Bx_n(t) - Bx_n(\tau)| \leq |\phi(0) - \phi(0)| + \left| \int_0^{\beta(t)} g\big(t, x_n(s), x_{n_s}\big) ds - \int_0^{\beta(\tau)} g\big(\tau, x_n(s), x_{n_s}\big) ds \right|$$

$$\leq |\phi(0)| + |\phi(0)| + \gamma(t) + \gamma(\tau)$$

$$< \epsilon \text{ as } t \to \tau.$$

Similarly, if $t, \tau \in$ I with $t < T < \tau$, then we have

$$|Bx_n(t) - Bx_n(\tau)| \leq |Bx_n(t) - Bx_n(T)| + |Bx_n(T) - Bx_n(\tau)|$$

If $t \to \tau$ then $t \to T$ and $T \to \tau$, therefore it follows that $|Bx_n(t) - Bx_n(T)| \to 0$ and

$|Bx_n(T) - Bx_n(\tau)| \to 0$ as$t \to \tau$. A result, $|Bx_n(t) - Bx_n(\tau)| \to 0$ as $t \to \tau$. Hence sequence $\{Bx_n\}$ is an equicontinous sequence of functions in X. By Arzela-Ascoli theorem yields that $\{Bx_n\}$ has a uniformly convergent subsequence on the compact subset $[0, T]$ of $\mathbb{R}$. Without of generality, call the subsequence the sequence itself. We show that sequence $\{Bx_n\}$ is Cauchy in X.

Now $|Bx_n(t) - Bx_n(t)| \to 0$ as$n \to \infty$ for all $t \in [0, T]$. Then for given $\epsilon > 0$ there exists an$n_0 \in N$ such that

$$\sup_{0 \leq P \leq T} \left[\int_0^{\beta(p)} |g(t, x_m(s), (x_m)_s) - g(t, x_n(s), (x_n)_s)| \, ds < \frac{\epsilon}{2} \right]$$

for all $m, n \geq n_{0.}$ Therefore if $m, n \geq n_{0,}$ then we have

$$\|Bx_m - Bx_n\| = \sup_{o \leq t < \infty} \left| \int_0^{\beta(t)} |g(t, x_m(s), (x_m)_s) - g(t, x_n(s), (x_n)_s)| \, ds \right|$$

$$\leq \sup_{o \leq p \leq T} \left| \int_0^{\beta(t)} |g(t, x_m(s), (x_m)_s) - g(t, x_n(s), (x_n)_s)| \, ds \right|$$

$$+ \sup_{p \geq T} \left\{ \int_0^{\beta(p)} [|g(t, x_m(s), (x_m)_s)| + |g(t, x_n(s), (x_n)_s)|] ds \right\}$$

$$\leq \epsilon$$

This shows sequence $\{Bx_n\} \subset B\big(\overline{\mathcal{B}}_r(0)\big) \subset X$ is Cauchy. Since X is complete, $\{Bx_n\}$ is convergence to a point in $B\big(\overline{\mathcal{B}}_r(0)\big)$ is relatively compact and consequently B is a continuous and compact operator on $\overline{\mathcal{B}}_r(0)$.

Next, we show that $AxBx \in \overline{\mathcal{B}}_r(0), \ \forall\, x \in \overline{\mathcal{B}}_r(0)$. Let $x \in \overline{\mathcal{B}}_r(0)$ be arbitrary, then

$$Ax(t)Bx(t) \leq |Ax(t)||Bx(t)|$$

$$\leq |f(t, x(t), x_t)| \left(|\phi(0)| + \int_0^{\beta(t)} |g(t, x(s), x_s)| \, ds \right)$$

$$\leq [|f(t,x(t),x_t)-f(t,0,0)|+|f(t,o,o)|]\left(|\phi(0)|+\int_0^{\beta(t)} b(t,s)ds\right)$$

$$\leq \ell(t)max\{|x|,\|x_t\|_c\}+F(t)\big(|\phi(0)|+\gamma(t)\big)$$

$$\leq [L+F_0](K_1+K_2)$$

$$\leq L(K_1+K_2)+F_0(K_1+K_2)$$

$$=\frac{F_0(K_1+K_2)}{1-L(K_1+K_2)}$$

$$=r$$

for all $t\in I$. Taking the supremum over t, we obtain $\|AxBx\|\leq r$ for all $x\in\overline{\mathcal{B}}_r(0)$. Hence hypothesis (iii) of the theorem (2.1) holds. Here one has

$$M=\left\|B\big(\overline{\mathcal{B}}_r(0)\big)\right\|=sup\{\|Bx\|:x\in\overline{\mathcal{B}}_r(0)\}$$

$$=sup\left\{\sup_{t\geq 0}\left[|\phi(0)|+\int_0^{\beta(t)}|g(t,x(s),x_s)|\,ds\right]:x\in\overline{\mathcal{B}}_r(0)\right\}$$

$$\leq \sup_{t\geq 0}|\phi(o)|+\sup_{t\geq o}\gamma(t)$$

$$\leq K_1+K_2$$

and therefore $MK=L(K_1+K_2)<1$. Now we apply theorem (2.1) to conclude that the FDE (1.1) has a solution on I.

Finally, we show the uniform locally asymptotic attractivity of the solutions for FDE (1.1). Let x and y be any two solutions of the FDE (1.1) in $\overline{\mathcal{B}}_r(0)$ defined on I. Then we have

$$|x(t)-y(t)|\leq\left|f(t,x(t),x_t)\left(\phi(0)+\int_0^{\beta(t)}g(t,x(s),x_s)ds\right)\right|$$

$$+\left|f(t,y(t),y_t)\left(\phi(0)+\int_0^{\beta(t)}g(t,y(s),y_s)ds\right)\right|$$

$$\leq|f(t,x(t),x_t)|\left(|\phi\epsilon(0)|+\int_0^{\beta(t)}|g(t,x(s),x_s)|\,ds\right)+|f(t,x(t),x_t)|\left(|\phi(0)|+\int_0^{\beta(t)}|g(t,x(s),x_s)|\,ds\right)$$

$$\leq 2(Lr+F_0)\big(|\phi(0)|+\gamma(t)\big)\quad\text{for all } t\in I.$$

Since $\lim_{t\to\infty}\phi(t)=0$ and $\lim_{t\to\infty}\gamma(t)=0$, for $\epsilon>o$, there are real numbers $T'>0$ and $T''>0$ such that $|\phi(t)|<\frac{\epsilon}{2(Lr+F_0)}$ for all $t\geq T'$ and $\gamma(t)<\frac{\epsilon}{2(Lr+F_0)}$ for all $t\geq T''$. If we choose $T^*=Max\{T',T''\}$ then from the above inequality it follows that $|x(t)-y(t)|\leq\epsilon$ for all $t\geq T^*$. Since $\lim_{t\to\infty}x(t)=0$, it follows that the zero function $x(t)=0,t\geq 0$, i.e. the axis of t, is a local attractor for all solutions of the FDE (1.1) on I. Consequently the FDE (1.1) has a solution and all solutions are uniformly locally asymptotic attractive on I. The proof is complete.

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