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Symmedian Development of the Trimedian and Trisector

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Abstract — *In general, a lot of discussion on the line discusses the symmedian point. This paper discusses the reflection of trimedian and trisector. The discussion is about the side lengths of the symmedian and the area of the symmedian of triangle. The proof is done using a very simple method, namely by using the concept of the height line on the triangle and the trigonometric concepts of the triangle.*

Keywords — Symmedian, area of symmedian triangle, side length of symmedian, trimedian, trisector.

I. INTRODUCTION

In the development of geometry, specifically of the triangle, the reflection of the median line to the bisector line is called the symmedian line [24]. Symmedian lines and their development have been widely discussed in several papers, such as [24] which discusses the concurrency of symmedian lines on triangles that intersect at one point. This point is called a Lemoine point. In [8] the computer is used as a tool to determine and to find various symmedian theorems and Lemoine point. Correspondingly, the Lemoine point which is outside of the triangle called Lemoine-Kiepert has been studied at [9].There is another development to prove the existence of the symmedian; the third circle formed from a symmedian point and two angles on a triangle by identifying the center and radius of the circle [10]. In [1], it discusses the spatial of the second circle Lemoine, while [4] discusses the convex coordinates at symmedian points. Another symmedian development in [16] examines centroid curves, Gorgonne points and symmedian triangles on isotropic planes. Then the orthogonality of the median line and symmedian line has been proven in [21]. Then [12] finds the Tucker circle from a symmedian point and two angles on the triangle.

Many things have been discussed in symmedian in various forms of flat planes, but no one has discussed the development of symmedian lines against the trimedian and trisector. Based on this, the authors are interested in finding the area of the triangle formed from symmedian development against the trimedian and trisector by determining the length of the symmedian line using Steiner's theorem. Then the area of the symmedian triangle formed is compared with the area of the triangle using the trigonometric concepts of the triangle with the use of the sides and angle of the known triangle, which is discussed in [14, 15].

II. LITERATURE REVIEW

A. Trimedian

Before forming the trimedian, the authors develop the Varignon's theorem on a triangle. Trimedian is a line drawn from a vertex that divides the front side of the corner into three equal parts. The development of Varignon's theorem is carried out by dividing three, four and five sides of each triangle [2, 17, 18].

The Varignon's theorem on a triangle states that if each side of any triangle ABC is divided into two equal parts to form a midpoint on each side of the triangle then each of the midpoints is connected to form a triangle in the original triangle, then the area of the triangle formed is $\frac{1}{4}[\Delta ABC]$. Basically, the Varignon's theorem on aquadrilateral is discussed in [2] which is stated as follows.

Teorema 2.1. Given any ΔABC, if sides AB, BC and AC are divided into two same parts then forms points as A_t , B_t, C_t. If three points are connected then

$$[\Delta A\, A_t\, C_t] = [\Delta\, A_t B\, B_t] = [\Delta\, C_t\, B_t C] = [\Delta\, A_t\, B_t C_t] = \frac{1}{4}[\Delta ABC].$$

Proof. See [2]. ■

Furthermore, the Varignon's theorem on the quadraliteral is discussed in several articles, among others [2, 17, 18]. Given any ΔABC, if side $BC = a$, $AC = b$ and $AB = c$ are divided into three same parts then forms points as A_{t_1}, A_{t_2} , B_{t_1}, B_{t_2} and C_{t_1}, C_{t_2} respectively, as shown in the Figure 1.

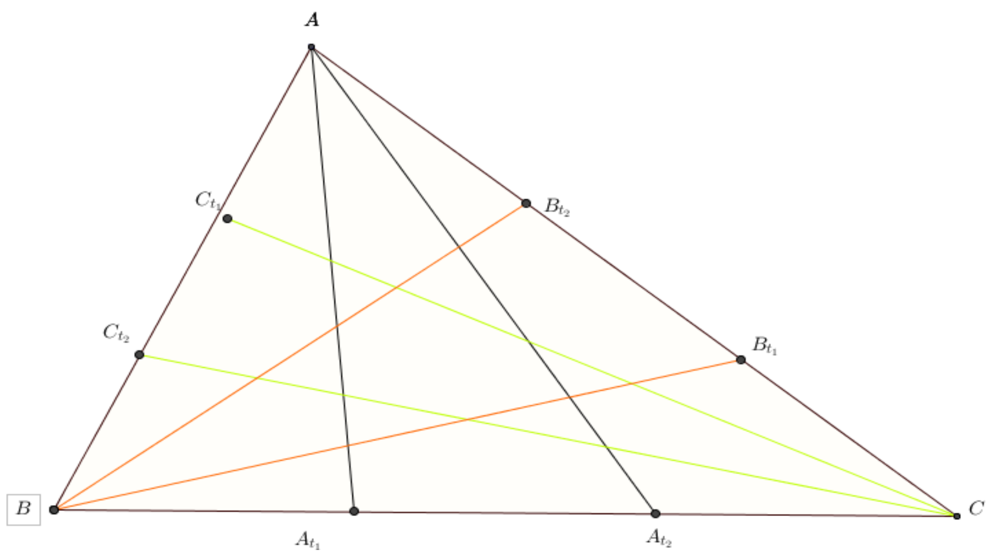

Figure 1: Trimedian of the *ABC* triangle

In Figure 1, AA_{t_1} and AA_{t_2} is trimedian line at line $BC = a$ that divide the line into three equal parts, the side line $BA_{t_1} = A_{t_1}A_{t_2} = A_{t_2}C = \frac{1}{3}a$. BB_{t_1} and BB_{t_2} is trimedian line at line $AC = b$ that divide the line into three equal parts, the side line $CB_{t_1} = B_{t_1}B_{t_2} = B_{t_2}A = \frac{1}{3}b$. CC_{t_1} and CC_{t_2} is trimedian line at line $AB = c$ that divide the line into three equal parts, the side line $AC_{t_1} = C_{t_1}C_{t_2} = C_{t_2}B = \frac{1}{3}c$.

B. Trisector

In addition to dividing the angle into two equal angles in triangle, if two lines are drawn from each vertex to the side in front of it, it can divide the angle into three equal parts [11, 13, 22].

Definition 2.1 Angle trisector has two dividing lines that divide the angle into three equal parts.

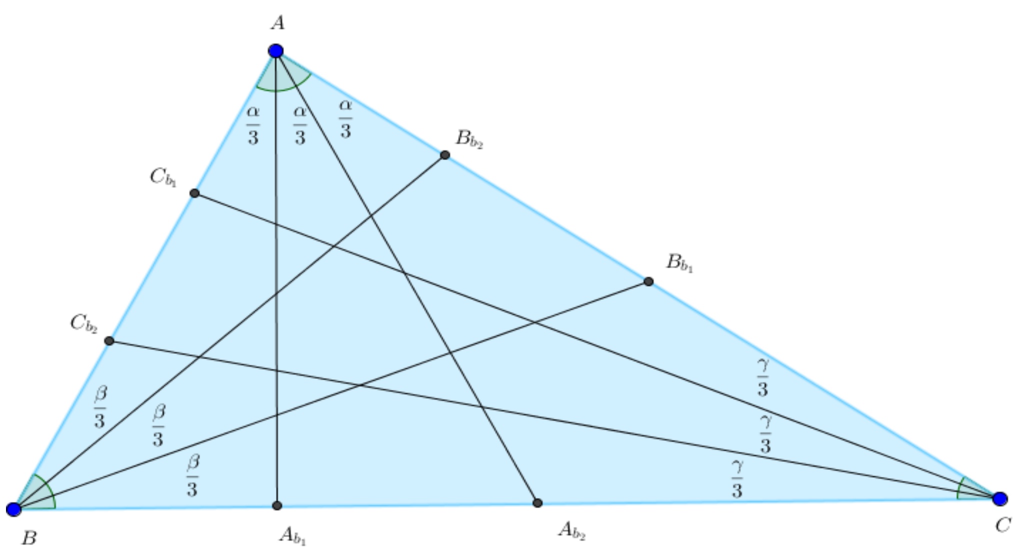

Figure 2: Trisektor on ABC triangle

In Figure 2, AA_{b_1} and AA_{b_2} are angle trisector lines at angle A, that divide an angle into three equal parts. Trisector is often discussed in a theorem, one of them is called the Morley's theorem [5, 6, 11, 13]. Morley's theorem is corner in the form of a trisector that divides each corner into three equal parts. Suppose that the $\angle A$ is given the names a_1 and a_2, the trisector

$\angle B$ is given the names b_1 and b_2 and the trisector $\angle C$ is given the names c_1 and c_2. Let the point D, E and F be the intersection points between the lines a_1 and b_1, a_2 and c_2, b_2 and c_1. If the three points of the intersection are connected then an equilateral triangle, namely ΔDEF, is formed (Figure 3). The figure illustrates the following theorem.

Theorem 2.2 (Morley's Theorem) Let the adjacent inner trisectors of ΔABC intersect at points, namely point D, E and F. If the intersecting points are connected then an equilateral triangle, namely ΔDEF, is formed.

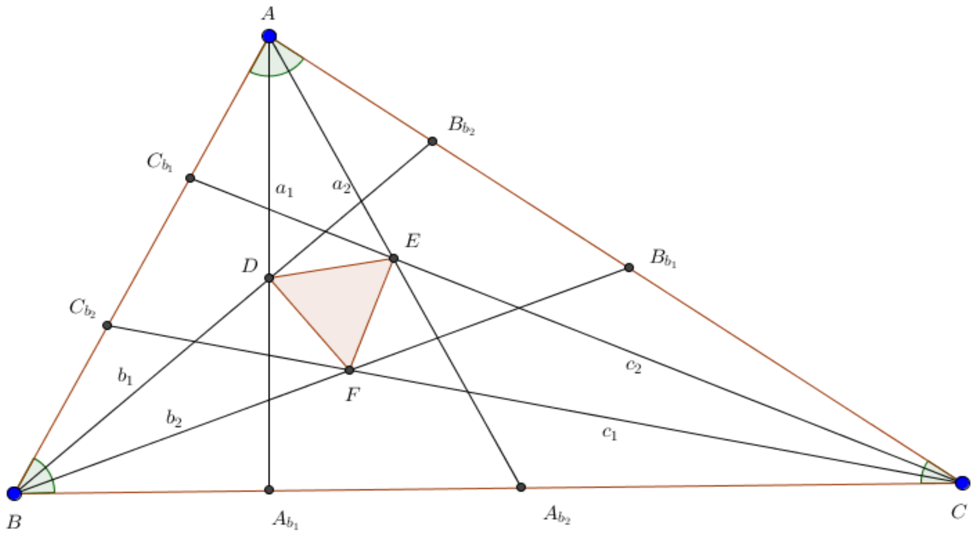

Figure 3: ΔDEF is Morley's triangle of ΔABC

Proof. See [11]. ■

C. Symmedian

Definition of symmedian is discussed in several articles, among others [3, 20, 24].

Definition 2.2 (Symmedian Line). In a triangle ABC, the reflection of the A-median in the A-internal angle bisector is called the A-Symmedian of a triangle ABC. The B-Symmedian and the C-Symmedian of the triangle can be defined similarly.

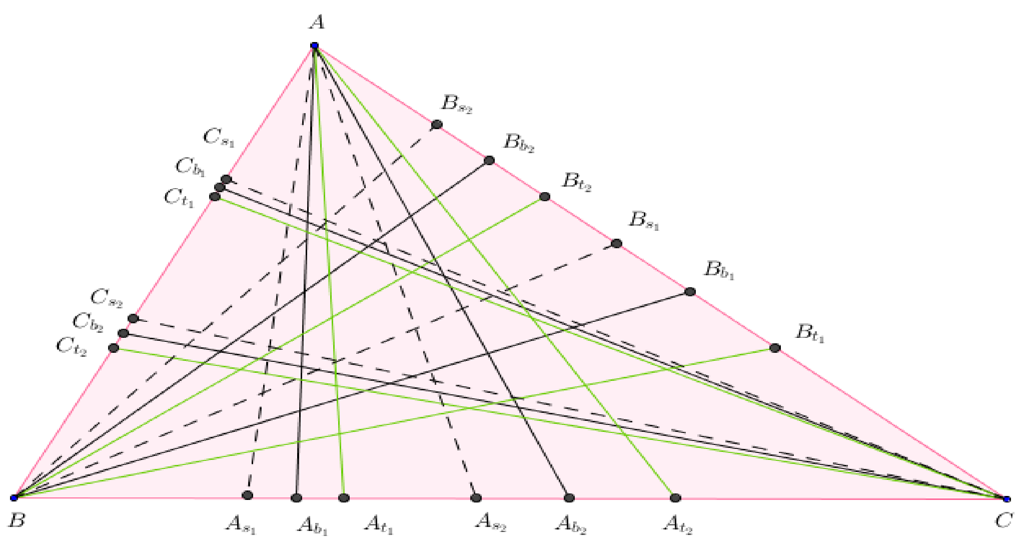

Figure 4. Symmedian line construction of the *ABC* triangle

In Figure 4, given triangle ΔABC where a, b and c are the side lengths. If A_{t_1} and A_{t_2} are the trimedian line and A_{b_1}, A_{b_2} are trisector line drawn from the angle A, then the reflection of the line A_{t_1} against A_{b_1} produces the line A_{s_1} , the reflection of the line A_{t_2} against A_{b_2} produces the line A_{s_2} which is called a symmedian line [24]. In the same way a symmedian line B_{s_1} and B_{s_2} can be formed from angle B and a symmedian line C_{s_1} and C_{s_2} from angle C.

III. SYMMEDIAN SIDE LENGTH OF TRIANGLE

The length of BA_{s_1}, BA_{s_2}, CA_{s_1} and CA_{s_2} must be determined using Steiner's theorem. Further discussion in this paper is about the area of the triangles formed from the symmedian.

Teorema 3.2 Given any ΔABC, then the symmedian are formed on the triangle namely $A_{s_1}, BC = a,\ AC = b$ and $AB = c$. The lengths of the sides CA_{s_1} and BA_{s_1} respectively are

$$BA_{s_1} = \frac{2ac^2}{2c^2 + b^2},$$

$$CA_{s_1} = \frac{b^2 a}{2c^2 + b^2}.$$

Proof. Based on the Steiner's theorem, it is obtained that

$$\frac{BA_{s_1}}{CA_{s_1}} \frac{BA_{t_1}}{CA_{t_1}} = \frac{c^2}{b^2}, \tag{1}$$

$$\frac{BA_{s_2}}{CA_{s_2}} \frac{BA_{t_2}}{CA_{t_2}} = \frac{c^2}{b^2}. \tag{2}$$

Because A_{t_1} is trimedian, then $BA_{t_1} = \frac{1}{3}a$ and $CA_{t_1} = \frac{2}{3}a$. Then it is obtained that

$$\frac{BA_{t_1}}{CA_{t_1}} = \frac{1}{2}.$$

Substituting equation (1) into equation (4) yields

$$\frac{BA_{s_1}}{c^2} = \frac{2\,CA_{s_1}}{b^2}. \tag{3}$$

Because $BA_{s_1} = a - CA_{s_1}$ and $CA_{s_1} = a - BA_{s_1}$, equation (3) becomes

$$\frac{a - CA_{s_1}}{c^2} = \frac{2\,CA_{s_1}}{b^2},$$

$$CA_{s_1} = \frac{b^2 a}{2c^2 + b^2}. \tag{4}$$

Substituting equation (4) into equation (3) gives

$$BA_{s_1} = \frac{2ac^2}{2c^2 + b^2}. \tag{5}$$

In the same way by using Steiner's theorem, it is obtained that

$$BC_{s_1} = \frac{2ca^2}{2a^2 + b^2}, \tag{6}$$

$$AC_{s_1} = \frac{cb^2}{2a^2 + b^2}, \tag{7}$$

$$AB_{s_1} = \frac{2ba^2}{2a^2 + c^2}, \tag{8}$$

$$CB_{s_1} = \frac{bc^2}{2a^2 + c^2}. \tag{9}$$

IV. THE AREA OF A SYMMEDIAN TRIANGLE $A_{s_1}B_{s_1}C_{s_1}$

Reflection of each trimedian on the trisector will produce symmedian. Then if symmedian points are connected, symmedian triangle $A_{s_1}B_{s_1}C_{s_1}$ is obtained. To determine the area of the symmedian $A_{s_1}B_{s_1}C_{s_1}$ of triangle uses the trigonometric concepts of the triangle, as the following theorem states.

Teorema 4.1 Given any ΔABC, $\angle A = \alpha$, $\angle B = \beta$, $\angle C = \gamma$ and $A_{s_1}B_{s_1}C_{s_1}$ is symmedian triangle which is formed from points A_{s_1}, B_{s_1} and C_{s_1}, then

$$[\Delta A_{s_1}B_{s_1}C_{s_1}] = \frac{4a^4b^2 + 2b^2c^4}{(2a^2+b^2)(2c^2+b^2)(2a^2+c^2)}[\Delta ABC].$$

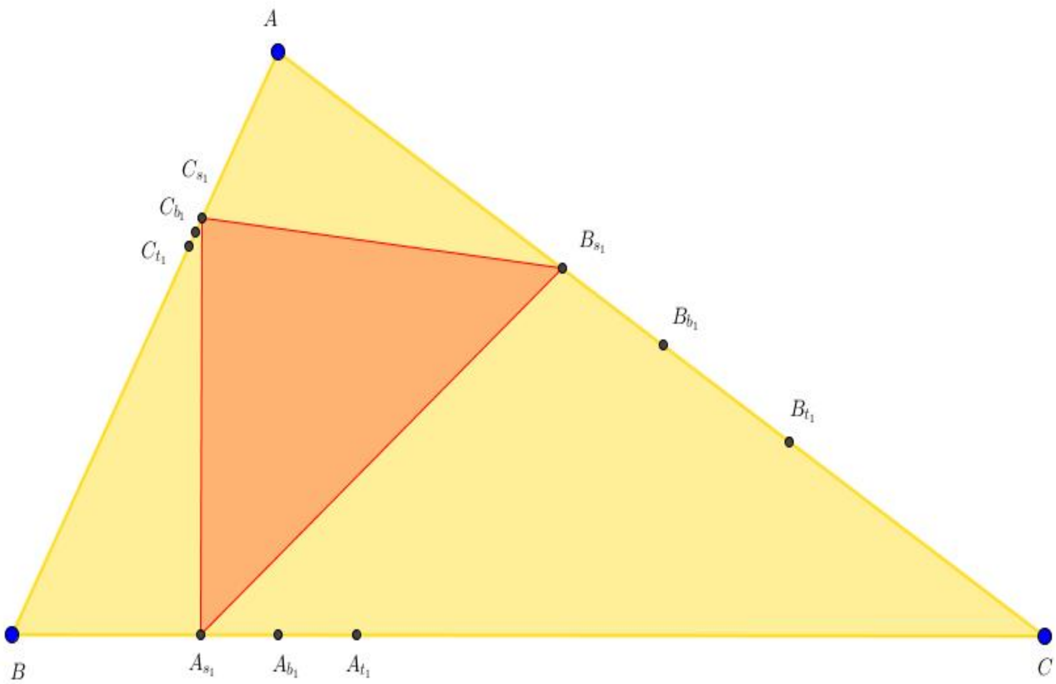

Figure 5: The Area of Symmedian Triangle $A_{s_1}B_{s_1}C_{s_1}$

Proof. To show the area of $A_{s_1}B_{s_1}C_{s_1}$, it is needed to show the area $[\Delta A_{s_1}BC_{s_1}]$, $[\Delta B_{s_1}A_{s_1}C]$ and $[\Delta AB_{s_1}C_{s_1}]$. To show the area $[\Delta A_{s_1}BC_{s_1}]$ is carried out using ΔABC and $\Delta A_{s_1}BC_{s_1}$. Applying the trigonometric concept of sine yields

$$[\Delta ABC] = \tfrac{1}{2}BC.AB.\sin\beta \tag{10}$$

$$[\Delta A_{s_1}BC_{s_1}] = \tfrac{1}{2}BA_{s_1}.BC_{s_1}.\sin\beta \tag{11}$$

Substituting equation (5) and equation (6) into equation (11) gives

$$[\Delta A_{s_1}BC_{s_1}] = \frac{1}{2}.\frac{2ac^2}{2c^2+b^2}.\frac{2ca^2}{2a^2+b^2}.\sin\beta$$

$$\sin\beta = \tfrac{1}{2}.\tfrac{(2c^2+b^2)(2a^2+b^2)}{2a^3b^3}.[\Delta A_{s_1}BC_{s_1}]. \tag{12}$$

Substituting equation (12) into equation (10) yields

$$[\Delta A_{s_1}BC_{s_1}] =.\tfrac{4a^2b^2}{(2c^2+b^2)(2a^2+b^2)}.[\Delta ABC]. \tag{13}$$

To show the area of $\Delta B_{s_1}A_{s_1}C$ is used ΔABC and $\Delta B_{s_1}A_{s_1}C$. Applying the trigonometric concept of sine gives

$$[\Delta ABC] = \frac{1}{2} BC.BC.\sin\gamma \quad (14)$$

$$[\Delta B_{s_1} A_{s_1} C] = \frac{1}{2} CB_{s_1}.CA_{s_1}.\sin\gamma \quad (15)$$

Substituting equation (9) and equation (4) into equation (15) yields

$$\sin\gamma = \frac{2(2a^2+c^2)(2c^2+b^2)}{b^3ac^2}.\left[\Delta B_{s_1} A_{s_1} C\right]. \quad (16)$$

Substituting equation (16) into equation (22) gives

$$\left[\Delta B_{s_1} A_{s_1} C\right] =. \frac{b^2c^2}{(2a^2+c^2)(2c^2+b^2)}.[\Delta ABC]. \quad (17)$$

To show the area $[\Delta B_{s_1} A_{s_1} C]$ is carried out using ΔABC and $\Delta AB_{s_1} C_{s_1}$. Again applying the sine produces

$$[\Delta ABC] = \frac{1}{2} AB.AC.\sin\alpha \quad (18)$$

$$\left[\Delta AB_{s_1} C_{s_1}\right] = \frac{1}{2} AB_{s_1}.AC_{s_1}.\sin\alpha. \quad (19)$$

On substitution of equation (8) and equation (7) into equation (19) gives

$$[\Delta AB_{s_1} C_{s_1}] = \frac{1}{2}.\frac{cb^2}{2a^2+b^2}.\frac{2ba^2}{2a^2+c^2}.\sin\alpha$$

$$\sin\alpha = \frac{(2a^2+b^2)(2a^2+c^2)}{a^2b^3c}.[\Delta AB_{s_1} C_{s_1}] \quad (20)$$

Next, substituting equation (20) into equation (19) yields

$$\left[\Delta AB_{s_1} C_{s_1}\right] = \frac{2a^2b^2}{(2a^2+b^2)(2a^2+c^2)}.[\Delta ABC]. \quad (21)$$

Therefore, the area of $\Delta A_{s_1} B_{s_1} C_{s_1}$ is

$$\left[\Delta A_{s_1} B_{s_1} C_{s_1}\right] = [\Delta ABC] - \left[\Delta A_{s_1} B C_{s_1}\right] - \left[\Delta B_{s_1} A_{s_1} C\right] - \left[\Delta AB_{s_1} C_{s_1}\right],$$

or

$$\left[\Delta A_{s_1} B_{s_1} C_{s_1}\right] = \frac{4a^4b^2+2b^2c^4}{(2a^2+b^2)(2c^2+b^2)(2a^2+c^2)}[\Delta ABC]. \qquad \blacksquare$$

V. CONCLUSIONS

From the above discussion, it can be concluded that the side lengths of the symmedian in triangles can be calculated if the sides of triangles are known, the proof is done by using a very simple concept of geometry. If the lengths of the symmedian are known, the area of the symmedian triangle can be determined using the trigonometri formula for the area of the triangle.

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