International Journal of Mathematics Trends and Technology
ISSN: 2231 - 5373 /doi:10.14445/22315373/IJMTT-V67I3P505
Volume 67 Issue 3, 28-44, March, 2021

GENERALIZED WEINSTEIN AND SOBOLEV SPACES

HASSEN BEN MOHAMED
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES, GABES

MOHAMED MOKTAR CHAFFAR
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY
PARIS-EST UNIVERSITY (UPEC) CRETEIL, FRANCE

ABSTRACT. In this paper we present a brief history and the basic ideas of the generalized Weinstein operator $\triangle_W^{\alpha,d,n}$ which generalizes the Weinstein operator $\triangle_W^{\alpha,d}$. In n=0 we regain the Weinstein operator has several applications in pure and applied mathematics especially in fluid mechanics. We study the Sobolev spaces of exponential type $\mathscr{H}^s_{\alpha,n}\left(\mathbb{R}^{d+1}_+\right)$ associated with the generalized Weinstein and investigate their properties, Sobolev spaces are named after the Russian mathematician Sergei Sobolev. Using the theory of reproducing kernels (which was written in 1942-1943), we introduce a class of symbols of exponential type and their associated pseudodifferential operators related to the generalized Weinstein operator $\triangle_W^{\alpha,d,n}$ and finally, we give some applications to these spaces.

Keywords: Sobolev Spaces, Generalized Weinstein operator, Generalized Weinstein transform, Weinstein, Kernel Reproducing Theory, pseudodifferential operator.

1. INTRODUCTION

The generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ studied by various authors defined on $\mathbb{R}^{d+1}_+ = \mathbb{R}^d \times]0,\ +\infty[$, by :

$$\Delta_W^{\alpha,d,n} = \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_{d+1}}\frac{\partial}{\partial x_{d+1}} - \frac{4n(\alpha+n)}{x_{d+1}^2} \tag{1.1}$$

where $n \in \mathbb{N}$ and $\alpha > -\frac{1}{2}$.
The expression above can also be written in the form $\Delta_W^{\alpha,d,n} = \Delta_d + L_{\alpha,n}$ where Δ_d is the Laplacian for the d first variables and $L_{\alpha,n}$ is the second- order singular differentiel operator on the half line given by :

$$L_{\alpha,n} = \frac{\partial^2}{\partial x_{d+1}^2} + \frac{2\alpha+1}{x_{d+1}}\frac{\partial}{\partial x_{d+1}} - \frac{4n(\alpha+n)}{x_{d+1}^2}.$$

In n=0 we regain the Weinstein operator $\Delta_W^{\alpha,d} = \Delta_W^{\alpha,d,0}$, mostly referred to as the Laplace-Bessel differential operator is now known as an important operator in analysis. The relevant harmonic analysis associated with the Bessel differential operator $L_\alpha = L_{\alpha,0}$ goes back to S. Bochner, J. Delsarte, B.M. Levitan and has been studied by many other authors such as J. Löfström and J. peetre [11], K.Stempak [14], K. Trimèche [15], I.A. Aliev and B. Rubin [8]. (See [2], [3], [4], [5], [6] & [16])
The generalized Weinstein kernel $\Lambda_{\alpha,d,n}$ is the function given by :

$$\forall x, y \in \mathbb{C}^{d+1},\ \Lambda_{\alpha,d,n}(x,y) = x_{d+1}^{2n} e^{-i\langle x',y'\rangle} j_{\alpha+2n}(x_{d+1}y_{d+1}),$$

where $x = (x', x_{d+1}),\ x' = (x_1, x_2, ..., x_d)$ and j_α is the normalized Bessel function of index α defined by :

$$\forall \xi \in \mathbb{C},\ j_\alpha(\xi) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\alpha+1)}(\frac{\xi}{2})^{2n}. \tag{1.2}$$

Using the Weinstein kernel $\Lambda_{\alpha,d,n}$, we define the Weinstein transform $\mathscr{F}_W^{\alpha,d,n}$ by :

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}(f)(\lambda) = \int_{\mathbb{R}_+^{d+1}} f(x)\Lambda_{\alpha,d,n}(x,\lambda)d\mu_{\alpha,d}(x),\ f \in L^1(\mathbb{R}_+^{d+1}, \mu_{\alpha,d}(x))$$

where $\mu_{\alpha,d}$ is the measure on $\mathbb{R}_+^{d+1}$ given by :

$$d\mu_{\alpha,d}(x) = x_{d+1}^{2\alpha+1}dx. \tag{1.3}$$

The Weinstein transform, referred to as the Fourier-Bessel transform, has been investigated by I.A. Aliev [7] and others. (See [2], [3], [4], [5], [9] and [16]).
We denote by $\mathscr{G}_{n,*}(\mathbb{R}^{d+1})$ the space, which is constituted of functions $\varphi \in \mathscr{E}_{n,*}(\mathbb{R}^{d+1})$ such that

$$\forall h,\ k > 0,\ N^{h,k}(\varphi) = \sup_{\substack{x \in \mathbb{R}^{d+1} \\ \mu \in \mathbb{N}^{d+1}}} \left[\frac{e^{k\|x\|}|\partial^\mu \mathscr{M}_n^{-1}\varphi(x)|}{h^{|\mu|}\mu!}\right] < \infty.$$

where $\mathscr{M}_n$, is the map defined by :

$$\forall x \in \mathbb{R}_+^{d+1},\ \mathscr{M}_n(f)(x) = x_{d+1}^{2n} f(x).$$

For $s \in \mathbb{R}$, we define the generalized Sobolev-Weinstein space of exponential type of order s, that will be denoted $\mathscr{H}_{\alpha,n}^s(\mathbb{R}_+^{d+1})$, as the set of all $u \in \mathscr{G}'_{n,*}$ (the dual of $\mathscr{G}_{n,*}$) such that $\mathscr{F}_W^{\alpha,d,n}(u)$ is a function and

$$\|u\|_{\mathscr{H}_{\alpha,n}^s} = \left[\int_{\mathbb{R}_+^{d+1}} e^{2s\|\lambda\|}\left|\mathscr{F}_W^{\alpha,d,n}(u)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda)\right]^{\frac{1}{2}} < \infty.$$

The space $\mathscr{H}_{\alpha,n}^s(\mathbb{R}_+^{d+1})$ provided with the norm $\|.\|_{\mathscr{H}_{\alpha,n}^s}$ is a Banach space.

The contents of this paper is as follows :

In the second section, we recapitulate some results related to the harmonic analysis associated with the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$.

In the section 3, we study the Sobolev spaces of exponential type $\mathscr{H}^s_{\alpha,n}\left(\mathbb{R}^{d+1}_+\right)$ associated with the operator $\Delta_W^{\alpha,d,n}$ and investigate their properties.

In the last section, using the theory of reproducing kernels, some applications are given for the spaces $\mathscr{H}^s_{\alpha,n}\left(\mathbb{R}^{d+1}_+\right)$. Moreover, we introduce certain classes of symbols of exponential type and study their associated pseudodifferential operators related to the operator $\triangle_W^{\alpha,d,n}$.

2. **Preliminaries**

In this section, we shall collect some results and definitions from the theory of the harmonic analysis associated with the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ given by (1.1).

In what follows, we need the following notations :

• $\mathscr{C}_*(\mathbb{R}^{d+1})$, the space of continuous functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.

• $\mathscr{E}_*(\mathbb{R}^{d+1})$, the space of $\mathscr{C}^\infty$-functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.

• $\mathscr{S}_*(\mathbb{R}^{d+1})$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.

• $\mathscr{D}_*(\mathbb{R}^{d+1})$, the space of $\mathscr{C}^\infty$-functions on $\mathbb{R}^{d+1}$ which are of compact support, even with respect to the last variable.

• $\mathscr{H}_*(\mathbb{C}^{d+1})$, the space of entire functions on $\mathbb{C}^{d+1}$, even with respect to the last variable, rapidly decreasing and of exponential type.

• $\mathscr{M}_n$, the map defined by :

$$\forall x \in \mathbb{R}^{d+1}_+,\ \mathscr{M}_n\left(f\right)\left(x\right) = x^{2n}_{d+1} f\left(x\right).$$

where $x = (x', x_{d+1})$ and $x' = (x_1, x_2, ..., x_d)$

• $L^p_{\alpha,n}(\mathbb{R}^{d+1}_+)$, $1 \le p \le +\infty$, the space of measurable functions on $\mathbb{R}^{d+1}_+$ such that

$$\begin{array}{rcl} \|f\|_{\alpha,n,p} & = & \left[\int_{\mathbb{R}^{d+1}_+} |\mathscr{M}_n^{-1} f(x)|^p d\mu_{\alpha+2n,d}(x)\right]^{\frac{1}{p}} < +\infty,\ \text{if } 1 \le p < +\infty, \\ \|f\|_{\alpha,n,\infty} & = & \operatorname*{ess\ sup}_{x\in\mathbb{R}^{d+1}_+} \left|\mathscr{M}_n^{-1} f(x)\right| < +\infty, \end{array}$$

where $\mu_{\alpha,d}$ is the measure given by the relation (1.3).

• $L^p_\alpha(\mathbb{R}^{d+1}_+) := L^p_{\alpha,0}(\mathbb{R}^{d+1}_+)$ and $\|f\|_{\alpha,p} := \|f\|_{\alpha,0,p}$, $1 \le p \le +\infty$.

• $\mathscr{E}_{n,*}(\mathbb{R}^{d+1})$, $\mathscr{D}_{n,*}(\mathbb{R}^{d+1})$ and $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ repespectively stand for the subspace of $\mathscr{E}_*(\mathbb{R}^{d+1})$, $\mathscr{D}_*(\mathbb{R}^{d+1})$ and $\mathscr{S}_*(\mathbb{R}^{d+1})$ consisting of functions f such that

$$\forall k \in \{1, ..., 2n-1\},\ \frac{\partial^k f}{\partial x^k_{d+1}}(x', 0) = f(x', 0) = 0.$$

For all $f \in L^1_{\alpha,n}(\mathbb{R}^{d+1}_+)$, we define the Weinstein transform $\mathscr{F}_W^{\alpha,d,n}$ by :

$$\forall \lambda \in \mathbb{R}^{d+1}_+,\ \mathscr{F}_W^{\alpha,d,n}\left(f\right)\left(\lambda\right) = \int_{\mathbb{R}^{d+1}_+} f(x)\Lambda_{\alpha,d,n}(x,\lambda)d\mu_{\alpha,d}(x)$$

where $\Lambda_{\alpha,d,n}$ is the generalized Weinstein kernel given by :

$$\forall x, y \in \mathbb{C}^{d+1},\ \Lambda_{\alpha,d,n}(x,y) = x_{d+1}^{2n} e^{-i\langle x',y'\rangle} j_{\alpha+2n}(x_{d+1}y_{d+1}),$$

$x = (x', x_{d+1}),\ x' = (x_1, x_2, ..., x_d)$ and j_α is the normalized Bessel function of index α defined by the relation (1.2).
Let us begin by the following definition and result.

Lemma 1. *(see* [1] *)*
i) The map $\mathscr{M}_n$ is an isomorphism from $\mathscr{E}_*(\mathbb{R}^{d+1})$ $\left(resp.\ \mathscr{S}_*(\mathbb{R}^{d+1})\right)$ onto $\mathscr{E}_{n,*}(\mathbb{R}^{d+1})$ $\left(resp.\ \mathscr{S}_{n,*}(\mathbb{R}^{d+1})\right)$.
ii) For all $f \in \mathscr{E}_*(\mathbb{R}^{d+1})$, we have

$$L_{\alpha,n} \circ \mathscr{M}_n(f) = \mathscr{M}_n \circ L_{\alpha+2n}(f). \tag{2.1}$$

iii) For all $f \in \mathscr{E}_*(\mathbb{R}^{d+1})$, we have

$$\Delta_W^{\alpha,d,n} \circ \mathscr{M}_n(f) = \mathscr{M}_n \circ \Delta_W^{\alpha+2n}(f). \tag{2.2}$$

iv) For all $f \in \mathscr{E}_*(\mathbb{R}^{d+1})$ and $g \in \mathscr{D}_{n,*}(\mathbb{R}^{d+1})$, we have

$$\int_{\mathbb{R}_+^{d+1}} \Delta_W^{\alpha,d,n}(f)(x)\, g(x)\, d\mu_{\alpha,d}(x) = \int_{\mathbb{R}_+^{d+1}} f(x)\, \Delta_W^{\alpha,d,n} g(x)\, d\mu_{\alpha,d}(x). \tag{2.3}$$

Definition 1. *The generalized Weinstein kernel $\Lambda_{\alpha,d,n}$ is the function given by :*

$$\forall x, y \in \mathbb{C}^{d+1},\ \Lambda_{\alpha,d,n}(x,y) = x_{d+1}^{2n} e^{-i\langle x',y'\rangle} j_{\alpha+2n}(x_{d+1}y_{d+1}), \tag{2.4}$$

where $x = (x', x_{d+1}),\ x' = (x_1, x_2, ..., x_d)$ and j_α is the normalized Bessel function of index α defined by the relation (1.2).

It is easy to see that the generalized Weinstein kernel $\Lambda_{\alpha,d,n}$ has a unique extention to $\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ and satisifies the following properties.

Proposition 1. *i) The function $x \mapsto \Lambda_{\alpha,d,n}(x,y)$ satisifies the differentiel equation*

$$\triangle_W^{\alpha,d,n}\left(\Lambda_{\alpha,d,n}(.,y)\right)(x) = -\left\|y\right\|^2 \Lambda_{\alpha,d,n}(x,y). \tag{2.5}$$

ii) For all $x,\ y \in \mathbb{C}^{d+1}$, we have

$$\Lambda_{\alpha,d,d}(x,y) = a_{\alpha+2n} e^{-i\langle x',y'\rangle} x_{d+1}^{2n} \int_0^1 \left(1-t^2\right)^{\alpha+2n-\frac{1}{2}} \cos(t x_{d+1} y_{d+1}) dt \tag{2.6}$$

where a_α is the constant given by :

$$a_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}. \tag{2.7}$$

iii) For all $\beta \in \mathbb{N}^{d+1}$, $x \in \mathbb{R}_+^{d+1}$ and $z \in \mathbb{C}^{d+1}$, we have

$$|D_z^\beta \Lambda_{\alpha,d,n}(x,z)| \leq x_{d+1}^{2n} \|x\|^{|\beta|} \exp(\|x\|\,\|\operatorname{Im} z\|), \tag{2.8}$$

where

$$D_z^\beta = \frac{\partial^\beta}{\partial z_1^{\beta_1} ... \partial z_{d+1}^{\beta_{d+1}}} \text{ and } |\beta| = \beta_1 + ... + \beta_{d+1}.$$

In particular, we have

$$\forall x, y \in \mathbb{R}_+^{d+1},\ |\Lambda_{\alpha,d,n}(x,y)| \leq x_{d+1}^{2n}. \tag{2.9}$$

Definition 2. *The generalized Weinstein transform* $\mathscr{F}_W^{\alpha,d,n}$ *is given for* $f \in L^1_{\alpha,n}(\mathbb{R}_+^{d+1})$ *by :*

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}(f)(\lambda) = \int_{\mathbb{R}_+^{d+1}} f(x)\Lambda_{\alpha,d,n}(x,\lambda)d\mu_{\alpha,d}(x). \tag{2.10}$$

Remark 1. *The generalized Weinstein transform* $\mathscr{F}_W^{\alpha,d,n}$ *can be written in the form :*

$$\mathscr{F}_W^{\alpha,d,n} = \mathscr{F}_W^{\alpha+2n,d} \circ \mathscr{M}_n^{-1} \tag{2.11}$$

where $\mathscr{F}_W^{\alpha,d} = \mathscr{F}_W^{\alpha,d,0}$ *is the classical Weinstein transform.*

Some basic properties of the transform $\mathscr{F}_W^{\alpha,d,n}$ are summarized in the following results. For the proofs, we refer to [1].

Proposition 2. *(see* [1] *)*
i) Let $m \in \mathbb{N}$ *and* $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$*, for all* $x \in \mathbb{R}_+^{d+1}$*, we have*

$$\mathscr{F}_W^{\alpha,d,n}\left[\left(\triangle_W^{\alpha,d,n}\right)^m f\right](x) = (-1)^m \|x\|^{2m} \mathscr{F}_W^{\alpha,d,n}(f)(x). \tag{2.12}$$

ii) Let $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ *and* $m \in \mathbb{N}$*. For all* $\lambda \in \mathbb{R}_+^{d+1}$*,we have*

$$\left(\triangle_W^{\alpha,d,n}\right)^m \left[\mathscr{M}_n \mathscr{F}_W^{\alpha,d,n}(f)\right](\lambda) = \mathscr{M}_n \mathscr{F}_W^{\alpha,d,n}(P_m f)(\lambda) \tag{2.13}$$

where $P_m(x) = (-1)^m \|x\|^{2m}$.

Theorem 1. *(see* [1] *)*

i) Let $f \in L^1_{\alpha,n}(\mathbb{R}^{d+1}_+)$. If $\mathscr{F}^{\alpha,d,n}_W(f) \in L^1_{\alpha+2n}(\mathbb{R}^{d+1}_+)$, then we have

(2.14)
$$f(x) = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(y)\, \Lambda_{\alpha,d,n}(-x,y) d\mu_{\alpha+2n,d}(y),\ a.e\ x \in \mathbb{R}^{d+1}_+$$

where $C_{\alpha,d}$ is the constant given by :

(2.15)
$$C_{\alpha,d} = \frac{1}{(2\pi)^{\frac{d}{2}}\, 2^\alpha \Gamma(\alpha+1)}.$$

ii) The Weinstein transform $\mathscr{F}^{\alpha,d,n}_W$ is a topological isomorphism from $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ onto $\mathscr{S}_*(\mathbb{R}^{d+1})$ and from $\mathscr{D}_{n,*}(\mathbb{R}^{d+1})$ onto $\mathscr{H}_*(\mathbb{C}^{d+1})$.

Theorem 2. *(see* [1] *).*

i) For all $f, g \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$, we have the following Parseval formula

(2.16)
$$\int_{\mathbb{R}^{d+1}_+} f(x)\overline{g(x)} d\mu_{\alpha,d}(x) = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\overline{\mathscr{F}^{\alpha,d,n}_W(g)(\lambda)} d\mu_{\alpha+2n,d}(\lambda)$$

where $C_{\alpha,d}$ is the constant given by the relation (2.15).

ii) (Plancherel formula).

For all $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$, we have :

(2.17)
$$\int_{\mathbb{R}^{d+1}_+} |f(x)|^2 d\mu_{\alpha,d}(x) = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \left|\mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda).$$

iii) (Plancherel Theorem) :

The transform $\mathscr{F}^{\alpha,d,n}_W$ extends uniquely to an isometric isomorphism from $L^2(\mathbb{R}^{d+1}_+,\ d\mu_{\alpha,d}(x))$ onto $L^2(\mathbb{R}^{d+1}_+,\ C^2_{\alpha+2n,d} d\mu_{\alpha+2n,d}(x))$.

Definition 3. *The translation operator $T^{\alpha,d,n}_x$, $x \in \mathbb{R}^{d+1}_+$, associated with the operator $\Delta^{\alpha,d,n}_W$ is defined on $\mathscr{E}_{n,*}(\mathbb{R}^{d+1}_+)$ by :*

(2.18)
$$\forall y \in \mathbb{R}^{d+1}_+,\ T^{\alpha,d,n}_x f(y) = x^{2n}_{d+1} \mathscr{M}_n T^{\alpha+2n,d}_x \mathscr{M}^{-1}_n f(y)$$

where

(2.19)
$$T^{\alpha,d}_x f(y) = \frac{a_\alpha}{2} \int_0^\pi f\left(x'+y',\ \sqrt{x^2_{d+1}+y^2_{d+1}+2x_{d+1}y_{d+1}\cos\theta}\right)(\sin\theta)^{2\alpha} d\theta$$

$x'+y' = (x_1+y_1, ..., x_d+y_d)$ and a_α is the constant given by (2.7).

The following proposition summarizes some properties of the generalized Weinstein translation operator.

Proposition 3. *(see* [1] *)*

i) Let $f \in L^p_{\alpha,n}(\mathbb{R}^{d+1}_+)$, $1 \le p \le +\infty$ and $x \in \mathbb{R}^{d+1}_+$. Then $T^{\alpha,d,n}_x f$ belongs to $L^p_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and we have

(2.20)
$$\|T^{\alpha,d,n}_x f\|_{\alpha,n,p} \le x^{2n}_{d+1} \|f\|_{\alpha,n,p}.$$

ii) The function $t \mapsto \Lambda_{\alpha,d,n}(t,\lambda)$, $\lambda \in \mathbb{C}^{d+1}$, satisfies on $\mathbb{R}^{d+1}_+$ the following product formula:

(2.21)
$$\forall x, y \in \mathbb{R}^{d+1}_+,\ \Lambda_{\alpha,d,n}(x,\lambda)\, \Lambda_{\alpha,d,n}(y,\lambda) = T^{\alpha,d,n}_x [\Lambda_{\alpha,d,n}(.,\lambda)](y).$$

iii) Let $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ *and* $x \in \mathbb{R}_+^{d+1}$*, we have*

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}\left(T_x^{\alpha,d,n} f\right)(\lambda) = \Lambda_{\alpha,d,n}(-x,\lambda)\,\mathscr{F}_W^{\alpha,d,n}(f)(\lambda). \tag{2.22}$$

iv) Let $f \in \mathscr{S}_{n,*}\left(\mathbb{R}^{d+1}\right)$*, for all* $x, y \in \mathbb{R}_+^{d+1}$*, we have*

$$T_x^{\alpha,d,n} f(y) = C_{\alpha+2n,d}^2 \int_{\mathbb{R}_+^{d+1}} \Lambda_{\alpha,d,n}(-x,\lambda)\,\Lambda_{\alpha,d,n}(-y,\lambda)\,\mathscr{F}_W^{\alpha,d,n}(f)(\lambda)\, d\mu_{\alpha+2n,d}(\lambda). \tag{2.23}$$

Definition 4. *The generalized Weinstein convolution product of* $f, g \in L^1_{\alpha,n}(\mathbb{R}_+^{d+1})$ *is given by :*

$$\forall x \in \mathbb{R}_+^{d+1},\ f *_{\alpha,n} g(x) = \int_{\mathbb{R}_+^{d+1}} T_x^{\alpha,d,n} f(-y)\, g(y)\, d\mu_{\alpha,d}(y). \tag{2.24}$$

Proposition 4. *(see* [1] *)*
i) Let $p, q, r \in [1, +\infty]$ *such that* $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$*. Then for all* $f \in L^p_{\alpha,n}(\mathbb{R}_+^{d+1})$ *and* $g \in L^q_{\alpha,n}(\mathbb{R}_+^{d+1})$*, the function* $f *_{\alpha,n} g \in L^r_{\alpha,n}(\mathbb{R}_+^{d+1})$ *and we have*

$$\|f *_{\alpha,n} g\|_{\alpha,n,r} \le \|f\|_{\alpha,n,p}\|g\|_{\alpha,n,q}. \tag{2.25}$$

ii) For all $f, g \in L^1_{\alpha,n}(\mathbb{R}_+^{d+1}),\ f *_{\alpha,n} g \in L^1_{\alpha,n}(\mathbb{R}_+^{d+1})$ *and we have*

$$\mathscr{F}_W^{\alpha,d,n}(f *_{\alpha,n} g) = \mathscr{F}_W^{\alpha,d,n}(f)\mathscr{F}_W^{\alpha,d,n}(g). \tag{2.26}$$

Notation. We denoted by $\mathscr{S}'_*$, $\left(\text{resp. } \mathscr{S}'_{n,*}\right)$ the strong dual of the space $\mathscr{S}_*(\mathbb{R}^{d+1})$, $\left(\text{resp. } \mathscr{S}_{n,*}(\mathbb{R}^{d+1})\right)$.

Definition 5. *The generalized Fourier-Weinstein transform of a distribution* $u \in \mathscr{S}'_*$ *is defined by :*

$$\forall \phi \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1}),\ \langle \mathscr{F}_W^{\alpha,d,n}(u), \phi\rangle = \langle u, \mathscr{F}_W^{\alpha,d,n}(\phi)\rangle. \tag{2.27}$$

The following proposition is as an immediate consequence of Theorem 1.

Proposition 5. *The transform* $\mathscr{F}_W^{\alpha,d,n}$ *is a topological isomorphism from* $\mathscr{S}'_*$ *onto* $\mathscr{S}'_{n,*}$.

Remark 2. *Let* $m \in \mathbb{N}$ *and* $u \in \mathscr{S}'_{n,*}$*, we have*

$$\mathscr{F}_W^{\alpha,d,n}\left[\mathscr{M}_n(\Delta_W^{\alpha,d,n})^m u\right] = (-1)^m \|x\|^{2m} \mathscr{F}_W^{\alpha,d,n}(\mathscr{M}_n u) \tag{2.28}$$

where

$$\forall \phi \in \mathscr{S}_{n,*}\left(\mathbb{R}^{d+1}\right),\ \langle \Delta_W^{\alpha,d,n} u,\ \phi\rangle = \langle u,\ \Delta_W^{\alpha,d,n}\phi\rangle. \tag{2.29}$$

3. The generalized Weinstein-Sobolev spaces of exponential type

In this section, we introduce and study the Sobolev spaces of exponential type associated with the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$.
Notation. We denote by :
$\mathscr{G}_{n,*}(\mathbb{R}^{d+1})$ the set of all functions $\varphi \in \mathscr{E}_{n,*}(\mathbb{R}^{d+1})$ such that

$$\forall h,\ k > 0,\ N^{h,k}(\varphi) = \sup_{\substack{x \in \mathbb{R}^{d+1} \\ \mu \in \mathbb{N}^{d+1}}} \left[\frac{e^{k\|x\|}|\partial^\mu \mathscr{M}_n^{-1}\varphi(x)|}{h^{|\mu|}\mu!}\right] < +\infty.$$

The topology of $\mathscr{G}_{n,*}(\mathbb{R}^{d+1})$ is defined by the above seminorms.
We have the following useful result.

Theorem 3. *The transform $\mathscr{F}_W^{\alpha,d,n}$ is a topological isomorphism from $\mathscr{G}_{n,*}(\mathbb{R}^{d+1})$ onto $\mathscr{G}_*(\mathbb{R}^{d+1}) := \mathscr{G}_{0,*}(\mathbb{R}^{d+1})$.*

Proof. The result follows from the relations (2.11) and the fact that $\mathscr{F}_W^{\alpha+2n,d}$ is an isomorphism from $\mathscr{G}_*(\mathbb{R}^{d+1})$ onto itself. □

Notation. We denote by $\mathscr{G}'_{n,*}$ the strong dual of the space $\mathscr{G}_{n,*}(\mathbb{R}^{d+1})$.

Definition 6. *The Weinstein transform of a distribution $S \in \mathscr{G}'_*$ is defined by :*

$$\forall \phi \in \mathscr{G}_{n,*}(\mathbb{R}^{d+1}),\ \langle \mathscr{F}_W^{\alpha,d,n}(S),\ \phi\rangle = \langle S,\ \mathscr{F}_W^{\alpha,d,n}(\phi)\rangle. \tag{3.1}$$

Proposition 6. *Let $m \in \mathbb{N}$ and $T \in \mathscr{G}'_*$, we have*

$$\mathscr{F}_W^{\alpha,d,n}\left[(\Delta_W^{\alpha,d,n})^m T\right] = (-1)^m \|\xi\|^{2m} \mathscr{F}_W^{\alpha,d,n}(T).$$

Proof. The result is a direct consequence of the relations (2.12) and (3.1). □

Definition 7. *For $s \in \mathbb{R}$ and $1 \le p < +\infty$, we define the space $\mathscr{W}_{\alpha,n}^{s,p}(\mathbb{R}_+^{d+1})$ as the set of all $u \in \mathscr{G}'_*$ such that $\mathscr{F}_W^{\alpha,d,n}(u)$ is a function and*

$$\|u\|_{\mathscr{W}_{\alpha,n}^{s,p}} = \left[C_{\alpha+2n,d}^2 \int_{\mathbb{R}_+^{d+1}} e^{ps\|\lambda\|} \left|\mathscr{F}_W^{\alpha,d,n}(u)(\lambda)\right|^p d\mu_{\alpha+2n,d}(\lambda)\right]^{\frac{1}{p}} < +\infty. \tag{3.2}$$

The norm on $\mathscr{W}_{\alpha,n}^{s,p}(\mathbb{R}_+^{d+1})$ is given by $\|u\|_{\mathscr{H}_{\alpha,n}^s}$.
For $p = 2$, we provide the space $\mathscr{H}_{\alpha,n}^s(\mathbb{R}_+^{d+1}) := \mathscr{W}_{\alpha,n}^{s,2}(\mathbb{R}_+^{d+1})$ with the scalar product

$$\langle u, v\rangle_{s,\alpha,n} = C_{\alpha+2n,d}^2 \int_{\mathbb{R}_+^{d+1}} e^{2s\|\xi\|} \mathscr{F}_W^{\alpha,d,n}(u)(\xi)\overline{\mathscr{F}_W^{\alpha,d,n}(v)(\xi)} d\mu_{\alpha+2n,d}(\xi) \tag{3.3}$$

and the norm

$$\|u\|_{\mathscr{H}_{\alpha,n}^s} = \langle u, u\rangle_{s,\alpha,n}^{\frac{1}{2}}.$$

$\mathscr{H}_{\alpha,n}^s(\mathbb{R}_+^{d+1})$ is the generalized Sobolev-Weinstein space of exponential type of order s. For $n = 0$, we regain the classical Sobolev-Weinstein space $\mathscr{H}_{\mathscr{G}_*}^{s,\alpha}(\mathbb{R}_+^{d+1})$ given in [3] and $\mathscr{F}_W^{\alpha,d} = \mathscr{F}_W^{\alpha,d,0}$ is the classical Weinstein transform.(See [2], [3], [10], [12] and [13]).

Proposition 7. *Let $s \in \mathbb{R}$ and $1 \le p < +\infty$. The space $\mathscr{W}_{\alpha,n}^{s,p}(\mathbb{R}_+^{d+1})$ provided with the norm $\|.\|_{\mathscr{W}_{\alpha,n}^{s,p}}$ is a Banach space.*

Proof. It is clear that the space $L^p(\mathbb{R}_+^{d+1},\ e^{ps||\lambda||}d\mu_{\alpha+2n,d}(x))$ is complete. On the other hand $\mathscr{F}_W^{\alpha,d,n}$ is a topological isomorphism from $\mathscr{G}'_*$ onto itself $\mathscr{G}'_{n,*}$. This achieves the proof. □

We proceed as [3], we obtain the following results.

Proposition 8. *i) For all $s \in \mathbb{R}$, we have*

$$\mathscr{G}_{n,*}(\mathbb{R}^{d+1}) \subset \mathscr{H}_{\alpha,n}^s(\mathbb{R}_+^{d+1}).$$

ii) We have

$$\mathscr{H}_{\alpha,n}^0(\mathbb{R}_+^{d+1}) = L_{\alpha+2n}^2(\mathbb{R}_+^{d+1}).$$

iii) For all $s, t \in \mathbb{R}$, $t > s$, the space $\mathscr{W}_{\alpha,n}^{t,p}(\mathbb{R}_+^{d+1})$ is continuously contained in $\mathscr{W}_{\alpha,n}^{s,p}(\mathbb{R}_+^{d+1})$.

iv) Let P be a linear partial differential operator with constant coefficients, $s \in \mathbb{R}$, $u \in \mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and $t < s$.
Then $P(u) \in \mathscr{H}^t_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and the map $v \mapsto P(v)$ is continuous on $\mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)$.
v) Let $Q(D) = \sum_{m\in\mathbb{N}} a_m D^m$ be a differential operator of infinite order such that there exist constants $C > 0$ and $r > 0$ satisfying :

$$\forall m \in \mathbb{N},\ |a_m| \leq C\frac{r^m}{m!}. \tag{3.4}$$

If $u \in \mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)$, then $Q(u) \in \mathscr{H}^{s-r}_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and the map :
$Q : \mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+) \to \mathscr{H}^{s-r}_{\alpha,n}(\mathbb{R}^{d+1}_+)$ is continuous.

Proposition 9. *Let $t \in \mathbb{R}$. The operator $\nabla_t : \mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+) \to \mathscr{H}^{s-t}_{\alpha,n}(\mathbb{R}^{d+1}_+)$ defined for all $x \in \mathbb{R}^{d+1}_+$ by :*

$$\nabla_t u(x) = C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} e^{t\sqrt{1+\|\xi\|^2}}\Lambda_{\alpha,d,n}(-x,\xi)\mathscr{F}^{\alpha,d,n}_W(u)(\xi)\,d\mu_{\alpha+2n,d}(\xi) \tag{3.5}$$

is an isomorphism.

Proof. Let $u \in \mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)$. Then, the function $\xi \mapsto e^{(s-t)\|\xi\|}e^{t\sqrt{1+\|\xi\|^2}}\mathscr{F}^{\alpha,d,n}_W(u)(\xi)$ belongs to $L^2_{\alpha+2n}(\mathbb{R}^{d+1}_+)$ and we have

$$\forall\xi \in \mathbb{R}^{d+1}_+,\ \mathscr{F}^{\alpha,d,n}_W(\nabla_t u)(\xi) = e^{t\sqrt{1+\|\xi\|^2}}\mathscr{F}^{\alpha,d,n}_W(u)(\xi).$$

Thus

$$\begin{aligned}\int_{\mathbb{R}^{d+1}_+} & e^{2(s-t)\|\lambda\|}\left|\mathscr{F}^{\alpha,d,n}_W(\nabla u)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda)\\ &= \int_{\mathbb{R}^{d+1}_+} e^{2(s-t)\|\lambda\|+2t\sqrt{1+\|\lambda\|^2}}\left|\mathscr{F}^{\alpha,d,n}_W(u)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda)\\ &\leq k_t\int_{\mathbb{R}^{d+1}_+} e^{2s\|\lambda\|}\left|\mathscr{F}^{\alpha,d,n}_W(u)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda),\end{aligned}$$

with $k_t = \sup_{\lambda\in\mathbb{R}^{d+1}_+}\left[e^{2t\left(\sqrt{1+\|\lambda\|^2}-\|\lambda\|\right)}\right] \leq e^{2|t|}$.
Thenwe deduce that $\nabla_t u \in \mathscr{H}^{s-t}_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and we have

$$\|\nabla_t u\|_{\mathscr{H}^{s-t}_{\alpha,n}} \leq e^{|t|}\|u\|_{\mathscr{H}^s_{\alpha,n}}.$$

On the other hand, let $v \in \mathscr{H}^{s-t}_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and put

$$u = \left[\mathscr{F}^{\alpha,d,n}_W\right]^{-1}\left(e^{-t\sqrt{1+\|\lambda\|^2}}\mathscr{F}^{\alpha,d,n}_W(v)\right).$$

From the definition of the operator ∇_t, we have $\nabla_t u = v$ and we get

$$\int_{\mathbb{R}^{d+1}_+} e^{2s\|\lambda\|}\left|\mathscr{F}^{\alpha,d,n}_W(u)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda) = \int_{\mathbb{R}^{d+1}_+} e^{2\left(s\|\lambda\|-t\sqrt{1+\|\lambda\|^2}\right)}\left|\mathscr{F}^{\alpha,d,n}_W(v)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda)$$

$$\leq \sup_{\lambda\in\mathbb{R}^{d+1}_+}\left[e^{2t\left(\|\lambda\|-\sqrt{1+\|\lambda\|^2}\right)}\right]\times\int_{\mathbb{R}^{d+1}_+} e^{2(s-t)\|\lambda\|}\left|\mathscr{F}^{\alpha,d,n}_W(v)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda) < \infty.$$

Then, $u \in \mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and we obtain

$$\|u\|_{\mathscr{H}^s_{\alpha,n}} \leq e^{|t|} \|\nabla_t u\|_{\mathscr{H}^{s-t}_{\alpha,n}}.$$

Hence the operator ∇_t is an isomorphism. □

The following theorem deals with the dual $\left(\mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)\right)'$ of $\mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and gives a relation between $\left(\mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)\right)'$ and $\mathscr{H}^{-s}_{\alpha,n}(\mathbb{R}^{d+1}_+)$.

Theorem 4. *The dual of $\mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)$ can be identified with $\mathscr{H}^{-s}_{\alpha,n}(\mathbb{R}^{d+1}_+)$. The relation of the identification is as follows :*

$$\langle u, v\rangle_{0,\alpha,n} = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(u)(\xi)\overline{\mathscr{F}^{\alpha,d,n}_W(v)(\xi)} d\mu_{\alpha+2n,d}(\xi), \tag{3.6}$$

with $u \in \mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and $v \in \mathscr{H}^{-s}_{\alpha,n}(\mathbb{R}^{d+1}_+)$.

Proof. Using the same technique as in Theorem 3.10 of [3], we obtain the result. □

Proposition 10. *Let $s_1, s, s_2 \in \mathbb{R}$, satisfying $s_1 < s < s_2$. Then, for all $\varepsilon > 0$, there exists a nonnegative constant C_ε such that for all $u \in \mathscr{W}^{s,p}_{\alpha,n}(\mathbb{R}^{d+1}_+)$, we have*

$$\|u\|_{\mathscr{W}^{s,p}_{\alpha,n}} \leq C_\varepsilon \|u\|_{\mathscr{W}^{s_1,p}_{\alpha,n}} + \varepsilon \|u\|_{\mathscr{W}^{s_2,p}_{\alpha,n}}. \tag{3.7}$$

Proof. Let $s_1, s_2 \in \mathbb{R},\ s_1 < s_2$ and $s \in]s_1, s_2[$. Then there exists $t \in]0,\ 1[$ such that $s = (1-t)\, s_1 + t s_2$.
Using the Hölder's inequlity, we get

$$\begin{aligned}
\|u\|_{\mathscr{W}^{s,p}_{\alpha,n}} &\leq \|u\|^{1-t}_{\mathscr{W}^{s_1,p}_{\alpha,n}} \times \|u\|^{t}_{\mathscr{W}^{s_2,p}_{\alpha,n}} \\
&\leq \left(\varepsilon^{\frac{-t}{1-t}} \|u\|_{\mathscr{W}^{s_1,p}_{\alpha,n}}\right)^{1-t} \times \left(\varepsilon \|u\|_{\mathscr{W}^{s_2,p}_{\alpha,n}}\right)^{t} \\
&\leq C_\varepsilon \|u\|_{\mathscr{W}^{s_1,p}_{\alpha,n}} + \varepsilon \|u\|_{\mathscr{W}^{s_2,p}_{\alpha,n}}
\end{aligned}$$

where $C_\varepsilon = \varepsilon^{\frac{-t}{1-t}}$. □

Proposition 11. *Let $s \in \mathbb{R}$, $m \in \mathbb{N}$ and $\varepsilon > 0$. If $\mathscr{M}_n u \in \mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)$ then $\mathscr{M}_n \left(\Delta^{\alpha,d,n}_W\right)^m u \in \mathscr{H}^{s-\varepsilon}_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and we have*

$$\|\mathscr{M}_n \left(\Delta^{\alpha,d,n}_W\right)^m (u)\|_{\mathscr{H}^{s-\varepsilon}_{\alpha,n}} \leq \left(\frac{2m}{\varepsilon e}\right)^{2m} \|\mathscr{M}_n u\|_{\mathscr{H}^s_{\alpha,n}}.$$

Proof. Let $\varepsilon > 0$, $m \in \mathbb{N}$, $s \in \mathbb{R}$ and $u \in \mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)$.
From (2.28), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}_+} e^{2(s-\varepsilon)\|\lambda\|} \left|\mathscr{F}^{\alpha,d,n}_W \left[\mathscr{M}_n \left(\Delta^{\alpha,d,n}_W\right)^m u\right](\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda) \\
&\qquad = \int_{\mathbb{R}^{d+1}_+} \|\lambda\|^{4m} e^{2(s-\varepsilon)\|\lambda\|} \left|\mathscr{F}^{\alpha,d,n}_W(\mathscr{M}_n u)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda) \\
&\qquad \leq \left(\frac{2m}{\varepsilon e}\right)^{4m} \int_{\mathbb{R}^{d+1}_+} e^{2s\|\lambda\|} \left|\mathscr{F}^{\alpha,d,n}_W(\mathscr{M}_n u)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda) < +\infty.
\end{aligned}$$

Then $\mathscr{M}_n\left(\Delta_W^{\alpha,d,n}\right)^m(u)\in\mathscr{H}^{s-\varepsilon}_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and we have

$$\|\mathscr{M}_n\left(\Delta_W^{\alpha,d,n}\right)^m(u)\|_{\mathscr{H}^{s-\varepsilon}_{\alpha,n}}\leq\left(\frac{2m}{\varepsilon e}\right)^{2m}\|\mathscr{M}_n u\|_{\mathscr{H}^{s}_{\alpha,n}}.$$

□

Definition 8. *Let* $u\in\mathscr{S}_{n,*}\left(\mathbb{R}^{d+1}\right)$*,we define the operator* $(-\Delta_W^{\alpha,d,n})^{\frac{1}{2}}$ *by :*
(3.8)

$$\forall x\in\mathbb{R}^{d+1}_+,(-\Delta_W^{\alpha,d,n})^{\frac{1}{2}}u(x)=C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}\|\xi\|\Lambda_{\alpha,d,n}(-x,\xi)\mathscr{F}_W^{\alpha,d,n}(u)(\xi)d\mu_{\alpha+2n,d}(\xi).$$

Proposition 12. *Let* $P((-\Delta_W^{\alpha,d,n})^{\frac{1}{2}})=\sum_{m\in\mathbb{N}}a_m\left[(-\Delta_W^{\alpha,d,n})^{\frac{1}{2}}\right]^m$ *be a fractional Weinstein Laplace operators of infinite order satisfying : there exist positive constants* C *and* r *such that*

$$\forall m\in\mathbb{N},\ |a_m|\leq C\frac{r^m}{m!}. \tag{3.9}$$

If $u\in\mathscr{W}^{s,p}_{\alpha,n}(\mathbb{R}^{d+1}_+)$*, then* $P((-\Delta_W^{\alpha,d,n})^{\frac{1}{2}})u\in\mathscr{W}^{s-r,p}_{\alpha,n}(\mathbb{R}^{d+1}_+)$ *and we have*

$$\|P((-\Delta_W^{\alpha,d,n})^{\frac{1}{2}})u\|_{\mathscr{W}^{s-r,p}_{\alpha,n}}\leq C\|u\|_{\mathscr{W}^{s,p}_{\alpha,n}}.$$

Proof. As an immediate consequence of the the condition (3.9), we have

$$\forall\xi\in\mathbb{R}^{d+1}_+,\ |P\left(\|\xi\|\right)|\leq Ce^{r\|\xi\|}.$$

Thus we deduce the desired result. □

Proposition 13. *Let* $t,s\in\mathbb{R}$*. The operator* $\exp(t(-\Delta_W^{\alpha,d,n})^{\frac{1}{2}})$ *defined by :*

$$\forall x\in\mathbb{R}^{d+1}_+,\ \exp(t(-\Delta_W^{\alpha,d,n})^{\frac{1}{2}})u(x)=C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}e^{t\|\xi\|}\mathscr{F}_W^{\alpha,d,n}(u)(\xi)\Lambda_{\alpha,d,n}(-x,\xi)d\mu_{\alpha+2n,d}(\xi)$$

is an isomorphism from $\mathscr{W}^{s,p}_{\alpha,n}(\mathbb{R}^{d+1}_+)$ *onto* $\mathscr{W}^{s-t,p}_{\alpha,n}(\mathbb{R}^{d+1}_+)$.

Proof. Let $t,s\in\mathbb{R}$ and $u\in\mathscr{W}^{s,p}_{\alpha,n}(\mathbb{R}^{d+1}_+)$. It is easy to see that

$$\|\exp(t(-\Delta_W^{\alpha,d,n})^{\frac{1}{2}})u\|_{\mathscr{W}^{s-t,p}_{\alpha,n}}=\|u\|_{\mathscr{W}^{s,p}_{\alpha,n}}.$$

Thus the proof is immediate. □

Proposition 14. *Let* $s>0$*. Then each* $u\in\mathscr{H}^{-s}_{\alpha,n}(\mathbb{R}^{d+1}_+)$ *can be represented as an infinite sum of fractional Weinstein Laplace operators of square integrable function* v*, in other words,*

$$u=\sum_{m\in\mathbb{N}}\frac{s^m}{m!}\left[(-\Delta_W^{\alpha,d,n})^{\frac{1}{2}}\right]^m v.$$

Proof. Let $u\in\mathscr{H}^{-s}_{\alpha,n}(\mathbb{R}^{d+1}_+),\ s>0$. Then, the function $\xi\mapsto e^{-s\|\xi\|}\mathscr{F}_W^{\alpha,d,n}(u)\left(\xi\right)$ belongs to $L^2_\alpha(\mathbb{R}^{d+1}_+)$.
So, from the Plancheral theorem, there exists $v\in L^2_{\alpha+2n}(\mathbb{R}^{d+1}_+)$, such that

$$\forall\xi\in\mathbb{R}^{d+1}_+,\ \mathscr{F}_W^{\alpha,d,n}(v)\left(\xi\right)=\frac{\mathscr{F}_W^{\alpha,d,n}(u)\left(\xi\right)}{\sum_{m\in\mathbb{N}}\frac{s^m}{m!}\|\xi\|^m}.$$

Then

$$\begin{aligned}\mathscr{F}_W^{\alpha,d,n}(u)(\xi) &= \sum_{m\in\mathbb{N}} \frac{s^m}{m!}\|\xi\|^m \mathscr{F}_W^{\alpha,d,n}(v)(\xi)\\ &= \sum_{m\in\mathbb{N}} \frac{s^m}{m!}\mathscr{F}_W^{\alpha,d,n}\left[\left((-\Delta_W^{\alpha,d})^{\frac{1}{2}}\right)^m v\right](\xi),\end{aligned}$$

which achieves the proof. □

4. Applications

4.1. **The reproducing kernels.**

Proposition 15. *For $s>0$, the Hilbert space $\mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)$ admits the reproducing kernel :*

(4.1)

$$\forall x,y\in\mathbb{R}^{d+1}_+, \Theta_s^{\alpha,d,n}(x,y) = C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} e^{-2s||\lambda||}\Lambda_{\alpha,d,n}(-x,\lambda))\Lambda_{\alpha,d,n}(-y,\lambda)d\mu_{\alpha+2n,d}(\lambda).$$

That is :

i) For every $y\in\mathbb{R}^{d+1}_+$, the distribution given by the function $x\mapsto\Theta_s^{\alpha,d,n}(x,y)$ belongs to $\mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)$.

ii) For every $f\in\mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)$, we have

$$\forall y\in\mathbb{R}^{d+1}_+,\ \langle f,\ \Theta_s^{\alpha,d,n}(.,y)\rangle_{s,\alpha} = f(y).$$

Proof. i) Let $y\in\mathbb{R}^{d+1}_+$ and $s>0$, the function $\lambda\mapsto e^{-2s||\lambda||}\Lambda_{\alpha,d,n}(y,\lambda)$ belongs to $L^1_{\alpha,n}(\mathbb{R}^{d+1}_+)\cap L^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$. Then, from the relation (2.17), the function $x\mapsto \Theta_s^{\alpha,d,n}(x,y)$ belongs to $L^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and we have

$$\forall\lambda\in\mathbb{R}^{d+1}_+,\ \mathscr{F}_W^{\alpha,d,n}\left[\Theta_s^{\alpha,d,n}(.,y)\right](\lambda) = e^{-2s||\lambda||}\Lambda_{\alpha,d,n}(-y,\lambda). \tag{4.2}$$

Then $\Theta_s^{\alpha,d,n}(.,y)\in\mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)$.

ii) Let $f\in\mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and $y\in\mathbb{R}^{d+1}_+$. Using the relations (3.3), (4.2) and (2.14), we obtain

$$\begin{aligned}\langle f,\Theta_s^{\alpha,d,n}(.,y)\rangle_{s,\alpha,n} &= C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}\mathscr{F}_W^{\alpha,d,n}(f)(\lambda)\Lambda_{\alpha,d,n}(-y,\lambda)d\mu_{\alpha+2n,d}(\lambda)\\ &= f(y).\end{aligned}$$

□

Definition 9. *The generalized heat kernel $G^{\alpha,n,d}$ is given by :*

(4.3)

$$\forall t>0,\forall x,y\in\mathbb{R}^{d+1},\ G^{\alpha,n,d}(t,x,y) = C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} e^{-t\|\xi\|^2}\Lambda_{\alpha,d,n}(x,\xi)\Lambda_{\alpha,d,n}(-y,\xi)d\mu_{\alpha+2n,d}(\xi).$$

The following Lemma will be useful later

Lemma 2. *Let $t>0$, we define the function $\phi_t^{\alpha,d,n}$ by :*

$$\forall x\in\mathbb{R}^{d+1},\ \phi_t^{\alpha,d,n}(x) = \frac{C_{\alpha+2n,d}}{(2t)^{\alpha+2n+\frac{d}{2}+1}}x^{2n}_{d+1}e^{-\frac{\|x\|^2}{4t}}. \tag{4.4}$$

i) We have

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}(\phi_t^{\alpha,d,n})(\lambda) = e^{-t\|\lambda\|^2}. \tag{4.5}$$

ii) We have

$$\forall x, y \in \mathbb{R}^{d+1},\ T_x^{\alpha,d,n}\left(\phi_t^{\alpha,d,n}\right)(y) = \frac{C_{\alpha+2n,d}}{(2t)^{\alpha+2n+\frac{d}{2}+1}} y_{d+1}^{2n} e^{-\frac{\|x\|^2+\|y\|^2}{4t}} \Lambda_{\alpha,d,n}(x, -\frac{iy}{2t}). \tag{4.6}$$

Proof. i) To see the result, we have to show that

$$\forall t > 0,\ \forall x \in \mathbb{R},\ \int_0^{+\infty} j_\alpha(x\xi)\, e^{-t\xi^2} \xi^{2\alpha+1} d\xi = \frac{\Gamma(\alpha+1)}{2t^{\alpha+1}} e^{-\frac{x^2}{4t}}.$$

ii) We obtain the result using the following relation :

$$\forall \lambda \in \mathbb{R},\ \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \int_0^\pi e^{\lambda \cos\theta} \sin^{2\alpha}(\theta)\, d\theta = j_\alpha(i\lambda).$$

□

The following properties of the generalized heat kernel $G^{\alpha,n,d}$ can be easily established using the Lemma 2.

Proposition 16. *i) We have*

$$\forall x, y \in \mathbb{R}^{d+1},\ G^{\alpha,n,d}(t,x,y) = T_x^{\alpha,d,n}\left(\phi_t^{\alpha,d,n}\right)(-y) \tag{4.7}$$

where $\phi_t^{\alpha,d,n}$, $t > 0$, be the function defined by the relation (4.4).
ii) For all $t > 0$ and $x, y \in \mathbb{R}^{d+1}$, we have

$$G^{\alpha,n,d}(t,x,y) = \frac{C_{\alpha+2n,d}}{(2t)^{\alpha+2n+\frac{d}{2}+1}} y_{d+1}^{2n} e^{-\frac{\|x\|^2+\|y\|^2}{4t}} \Lambda_{\alpha,d,n}(x, \frac{iy}{2t}). \tag{4.8}$$

iii) We have

$$\forall t > 0,\ \forall y \in \mathbb{R}_+^{d+1},\ \int_{\mathbb{R}_+^{d+1}} G^{\alpha,n,d}(t,x,y)\, d\mu_{\alpha+2n,d}(x) = y_{d+1}^{2n}. \tag{4.9}$$

iv) For a fixed $y \in \mathbb{R}_+^{d+1}$, the function $u : (x,t) \mapsto G^{\alpha,n,d}(t,x,y)$ solves on $\mathbb{R}_+^{d+1} \times]0, +\infty[$ the generalized heat equation :

$$\Delta_W^{\alpha,d} u(x,t) = \frac{\partial}{\partial t} u(x,t).$$

Definition 10. *The generalized heat semigroup $\mathscr{H}_t^{\alpha,d,n}, t > 0$, is the integral operator given for f in $L_\alpha^2(\mathbb{R}_+^{d+1})$ by :*

$$\forall x \in \mathbb{R}_+^{d+1},\ \mathscr{H}_t^{\alpha,d,n} f(x) := \begin{cases} \int_{\mathbb{R}_+^{d+1}} G^{\alpha,n,d}(t,x,y) f(y)\, d\mu_{\alpha,d}(y), & \text{if } t > 0 \\ f(x), & \text{if } t = 0. \end{cases}$$

Proposition 17. *i) Let $t > 0$. We have*

$$\forall x \in \mathbb{R}_+^{d+1},\ \mathscr{H}_t^{\alpha,d,n} f(x) = f *_{\alpha,n} \phi_t^{\alpha,d,n}(x). \tag{4.10}$$

where $\phi_t^{\alpha,d,n}$ is the function given by the relation (4.4).
ii) Let $f \in \mathscr{H}_{\alpha,n}^s(\mathbb{R}_+^{d+1})$. We have

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}\left[\mathscr{H}_t^{\alpha,d,n} f\right](\lambda) = e^{-t\|\lambda\|^2} \mathscr{F}_W^{\alpha,d,n}(f)(\lambda). \tag{4.11}$$

Proof. i) It is an immediate consequence of the relation (4.7).
ii) Let $f \in \mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)$. Using the relations (4.5), (4.10) and (2.26), we obtain the result. □

Proposition 18. *i) Let $f \in \mathscr{G}_{n,*}$. Then $u(x,t) = \mathscr{H}^{\alpha,d,n}_t f(x)$ solves on $\mathbb{R}^{d+1}_+ \times]0,+\infty[$, the following system :*

$$\begin{cases} (\Delta^{\alpha,d,n}_W - \frac{\partial}{\partial t})u(x,t) &= 0 \\ u(x,0) &= f(x). \end{cases}$$

ii) Let $s \in \mathbb{R}$. The integral transform $\mathscr{H}^{\alpha,d,n}_t$, $t > 0$, is a bounded linear operator from $\mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)$ into $L^2_\alpha(\mathbb{R}^{d+1}_+)$ and we have :

$$\|\mathscr{H}^{\alpha,d,n}_t f\|_{\alpha,n,2} \le e^{\frac{s^2}{4t}} \|f\|_{\mathscr{H}^s_{\alpha,n}}, \; f \in \mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+). \tag{4.12}$$

Proof. i) The assertion follows from Proposition 16 iv).
ii) Let $f \in \mathscr{H}^s_{\alpha,n}(\mathbb{R}^{d+1}_+)$. Using the relations (2.17) and (4.11), we have

$$\begin{aligned} \|\mathscr{H}^{\alpha,d,n}_t f\|^2_{\alpha,0,2} &= C^2_{\alpha+2n,d} \|\mathscr{F}^{\alpha,d,n}_W \left(\mathscr{H}^{\alpha,d,n}_t f\right) \|^2_{\alpha+2n,0,2} \\ &= C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} e^{-2t\|\lambda\|^2} \left|\mathscr{F}^{\alpha,d,n}_W (f)(\lambda)\right|^2 d\mu_{\alpha+2n,d}(\lambda) \\ &\le \sup_{\lambda \in \mathbb{R}^{d+1}_+} \left[e^{-2s\|\lambda\| - 2t\|\lambda\|^2}\right] \|f\|^2_{\mathscr{H}^s_{\alpha,n}} = e^{\frac{s^2}{2t}} \|f\|^2_{\mathscr{H}^s_{\alpha,n}}. \end{aligned}$$

Thus the proof is finished. □

4.2. **Pseudo-differential associated with the Generalized Weinstein operator. Notations.** We need the following notations :

- For $r \ge 0$, we designate by $\mathscr{S}^r$, the space of $C^\infty-$function $a : \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to \mathbb{C}$ such that for each compact set $K \subset \mathbb{R}^{d+1}$ and each $\beta, \gamma \in \mathbb{N}$, there exists a constant $C = C(K, \beta, \gamma)$ satisfying :

$$\forall (x,\xi) \in K \times \mathbb{R}^{d+1}, \; \left|D^\beta_\xi D^\gamma_x a(x, \; \xi)\right| \le C e^{r\|\xi\|}. \tag{4.13}$$

- For $r, l \in \mathbb{R}$ with $l > 0$, we denote by $\mathscr{S}^{r,l}$, the space consits of all $C^\infty-$function $a : \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to \mathbb{C}$ such that for each $L > 0$ and $\beta, \gamma \in \mathbb{N}$, there exist a positive constant $C = C(r, l, \gamma)$ satisfying the relation :

$$\forall (x,\xi) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}, \; \left|D^\beta_\xi D^\gamma_x a(x, \; \xi)\right| \le C L^{|\beta|} |\beta|! e^{r\|\xi\|} e^{-l\|x\|}. \tag{4.14}$$

Definition 11. *The pseudo-differential operator $A\left(a, \Delta^{\alpha,d,n}_W\right)$ associated with $a(x, \; \xi) \in \mathscr{S}^r$ is defined for $u \in \mathscr{G}_{n,*}(\mathbb{R}^{d+1})$ by :*

$$\left[A\left(a, \Delta^{\alpha,d,n}_W\right) u\right](x) = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \Lambda_{\alpha,d,n}(-x,\xi) a(x,\xi) \mathscr{F}^{\alpha,d,n}_W(u)(\xi) d\mu_{\alpha+2n,d}(\xi). \tag{4.15}$$

Theorem 5. *If $a(x,\xi) \in \mathscr{S}^r$, then its associated pseudo-differential operator $A\left(a, \; \Delta^{\alpha,d}_W\right)$ is a well-defined mapping from $\mathscr{G}_{n,*}(\mathbb{R}^{d+1})$ into $C^\infty\left(\mathbb{R}^{d+1}\right)$.*

Proof. Let $a(x,\xi) \in \mathscr{S}^r$. From the relation (4.13), for any compact set $K \subset \mathbb{R}^{d+1}$ and any $\gamma \in \mathbb{N}$, we have

$$\forall (x,\xi) \in K \times \mathbb{R}^{d+1},\ |D_x^\gamma a(x,\ \xi)| \le Ce^{r\|\xi\|}. \tag{4.16}$$

Let $u \in \mathscr{G}_{n,*}(\mathbb{R}^{d+1})$ and $x \in K$, using the relations (4.16), (2.9) and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+^{d+1}} \left| a(x,\xi)\, \Lambda_{\alpha,d,n}(-x,\xi) \mathscr{F}_W^{\alpha,d,n}(u)(\xi) \right| d\mu_{\alpha+2n,d}(\xi) \\
&\le C x_{d+1}^{2n} \int_{\mathbb{R}_+^{d+1}} e^{r\|\xi\|} \left| \mathscr{F}_W^{\alpha,d,n}(u)(\xi) \right| d\mu_{\alpha+2n,d}(\xi) \\
&\le \frac{C}{C_{\alpha+2n,d}} x_{d+1}^{2n} \left(\int_{\mathbb{R}_+^{d+1}} e^{2(r-s)\|\xi\|} d\mu_{\alpha+2n,d}(\xi) \right)^{\frac{1}{2}} \|u\|_{\mathscr{H}^s_{\alpha,n}} < +\infty
\end{aligned}$$

where $s > r$.
This relation proves that $A\left(a,\ \Delta_W^{\alpha,d}\right)(u)$ is well-defined and continuous on $\mathbb{R}_+^{d+1}$. Consequently, in vertue of Leibniz formula, we obtain the result. □

The next lemma plays an important role in this section.

Lemma 3. *Let $a(x,\xi) \in \mathscr{S}^{r,l}$. For $L > 0$ there exist $C > 0$ and $0 < t < \dfrac{1}{Ld}$ such that :*

$$\left| \mathscr{F}_W^{\alpha,d,n} \left(\mathscr{M}_n a(.,y) \right)(\xi) \right| \le Ce^{r\|y\|} e^{-t\|\xi\|}, \tag{4.17}$$

where C is a constant depending on r, t, α, d, n and l.

Proof. The result can be obtained by a simple calculation by using the same technique as in Theorem 3.4 of [10]. □

The following theorem gives an alternative form of $A\left(a,\ \Delta_W^{\alpha,d}\right)$ which will be useful in the sequel.

Theorem 6. *Let $a(x,\ \lambda) \in \mathscr{S}^{r,l}$. Then, the pseudo-differential operator $A\left(a,\ \Delta_W^{\alpha,d}\right)$ admits the following representation :*

$$\left[A\left(a, \Delta_W^{\alpha,d}\right) u \right](x) = C^2_{\alpha+2n,d} \int_{\mathbb{R}_+^{d+1}} \Lambda_{\alpha,d,n}(-x,z) \times \tag{4.18}$$

$$\left[\int_{\mathbb{R}_+^{d+1}} \mathscr{M}_{n,z}^{-1} T_y^{\alpha,d,n} \mathscr{M}_n \mathscr{F}_W^{\alpha,d,n} \left(\mathscr{M}_n a(.,y) \right)(z)\, \mathscr{F}_W^{\alpha,d,n}(u)(y)\, y_{d+1}^{2n} d\mu_{\alpha,d}(y) \right] d\mu_{\alpha+2n,d}(z)$$

for all $u \in \mathscr{G}_{n,*}(\mathbb{R}^{d+1})$ where all involved integrals are absolutely convergent.

Proof. From the relation (4.17), for all $y, z \in \mathbb{R}_+^{d+1}$, we obtain :

$$\left| \mathscr{M}_{n,z}^{-1} T_y^{\alpha,d,n} \mathscr{M}_n \mathscr{F}_W^{\alpha,d,n} \left(\mathscr{M}_n a(.,y) \right)(z) \right| \le C_1 e^{r\|y\|} y_{d+1}^{2n} T_y^{\alpha+2n,d} \left(e^{-t\|\xi\|} \right)(z) \tag{4.19}$$

where C_1 is a constant depending on r, t, α, d, n and l.
On the other hand since $u \in \mathscr{G}_{n,*}(\mathbb{R}^{d+1})$, we have

$$\forall y \in \mathbb{R}_+^{d+1},\ \left| \mathscr{F}_W^{\alpha,d,n}(u)(y) \right| \le C_2 e^{-k\|y\|},\ k > 0. \tag{4.20}$$

Now using the relations (4.19) and (4.20), we get :

$$\left|\mathscr{M}_{n,z}^{-1}T_y^{\alpha,d,n}\mathscr{M}_n\mathscr{F}_W^{\alpha,d,n}\left(\mathscr{M}_na(.,y)\right)(z)\,\mathscr{F}_W^{\alpha,d,n}(u)(y)\right| \leq C_3e^{(r-k)\|y\|}y_{d+1}^{2n}T_y^{\alpha+2n,d}\left(e^{-t\|\xi\|}\right)(z).$$

Then for $k > r$ and $t > 0$, we have

$$\begin{aligned}&\int_{\mathbb{R}_+^{d+1}}\left|\mathscr{M}_{n,z}^{-1}T_y^{\alpha,d,n}\mathscr{M}_n\mathscr{F}_W^{\alpha,d,n}\left(\mathscr{M}_na(.,y)\right)(z)\,\mathscr{F}_W^{\alpha,d,n}(u)(y)\right|y_{d+1}^{2n}d\mu_{\alpha,d}(y)\\ &\leq C_3\int_{\mathbb{R}_+^{d+1}}e^{(r-k)\|y\|}T_y^{\alpha+2n,d}\left(e^{-t\|\xi\|}\right)(z)\,d\mu_{\alpha+2n,d}(y)\\ &\leq C_3\varphi *_{\alpha+2n,0}\psi(z)\end{aligned}$$

where

$$\forall x\in\mathbb{R}_+^{d+1},\ \varphi(x) = e^{(r-k)\|x\|} \text{ and } g(x) = e^{-t\|x\|}.$$

Therefore the function :

$$z\mapsto\int_{\mathbb{R}_+^{d+1}}\left|\mathscr{M}_{n,z}^{-1}T_y^{\alpha,d,n}\mathscr{M}_n\mathscr{F}_W^{\alpha,d,n}\left(\mathscr{M}_na(.,y)\right)(z)\,\mathscr{F}_W^{\alpha,d,n}(u)(y)\right|y_{d+1}^{2n}d\mu_{\alpha,d}(y)$$

belongs to $L^1_{\alpha+2n}(\mathbb{R}_+^{d+1})$. So, the result follows by applying the inverse theorem. □

Now, we are in a situation to establish the fundamental result of this section given by the following result.

Theorem 7. *Let $s > 0$, $a(x,\ \lambda) \in \mathscr{S}^{r,l}$ and $A\left(x,\ \Delta_W^{\alpha,d,n}\right)$ be the associated pseudo-differential operator. Then $A\left(a,\ \Delta_W^{\alpha,d,n}\right)$ maps continuously from $\mathscr{H}_{\alpha,n}^{s+r}(\mathbb{R}_+^{d+1})$ to $\mathscr{H}_{\alpha,n}^{s}(\mathbb{R}_+^{d+1})$. Moreover, for all $u\in\mathscr{G}_{n,*}(\mathbb{R}^{d+1})$, we have*

$$\left\|A\left(a,\ \Delta_W^{\alpha,d,n}\right)u\right\|_{\mathscr{H}_{\alpha,n}^{s}} \leq k_s\|u\|_{\mathscr{H}_{\alpha,n}^{s+r}}. \tag{4.21}$$

Proof. Let $s > 0$. We consider the function φ_s given by :

$$\varphi_s(z) = e^{s||z||}\int_{\mathbb{R}_+^{d+1}}\mathscr{M}_{n,z}^{-1}\mathscr{M}_{n,y}T_y^{\alpha,d,n}\mathscr{M}_n\mathscr{F}_W^{\alpha,d,n}\left(\mathscr{M}_na(.,y)\right)(z)\,\mathscr{F}_W^{\alpha,d,n}(u)(y)\,d\mu_{\alpha,d}(y).$$

Using the relation (2.18), we obtain

$$\varphi_s(z) = e^{s||z||}\int_{\mathbb{R}_+^{d+1}}T_y^{\alpha+2n,d}\left(\mathscr{F}_W^{\alpha,d,n}\left(\mathscr{M}_na(.,y)\right)\right)(z)\,\mathscr{F}_W^{\alpha,d,n}(u)(y)\,d\mu_{\alpha+2n,d}(y)$$

Now, from the relations (2.19) and (4.17), we have

$$\begin{aligned}|\varphi_s(z)| &\leq Ce^{s||z||}\int_{\mathbb{R}_+^{d+1}}e^{r\|y\|}T_y^{\alpha+2n,d}\left(e^{-t\|\xi\|}\right)(z)\left|\mathscr{F}_W^{\alpha,d,n}(u)(y)\right|d\mu_{\alpha+2n,d}(y)\\ &\leq C\int_{\mathbb{R}_+^{d+1}}e^{(r+s)||y||}\left|\mathscr{F}_W^{\alpha,d,n}(u)(y)\right|T_y^{\alpha+2n,d}\left(e^{(s-t)\|\xi\|}\right)(z)\,d\mu_{\alpha+2n,d}(y)\\ &\leq Cf *_{\alpha+2n,0} g(z)\end{aligned}$$

where for all $x\in\mathbb{R}_+^{d+1}$

$$f(x) = e^{(s-t)||x||} \text{ and } g(x) = e^{(r+s)||x||}\left|\mathscr{F}_W^{\alpha,d,n}(u)(x)\right|.$$

It is clear that $g \in L^2_{\alpha+2n}\left(\mathbb{R}^{d+1}_+\right)$ and for $t > s$, $f \in L^1_{\alpha+2n}\left(\mathbb{R}^{d+1}_+\right)$. Then from the relation (2.25), we deduce that $f *_{\alpha+2n,0} g \in L^2_{\alpha+2n}\left(\mathbb{R}^{d+1}_+\right)$ and we have

$$\|f *_{\alpha+2n,0} g\|_{\alpha+2n,2} \leq \|f\|_{\alpha+2n,1}\|g\|_{\alpha+2n,2}.$$

So, we get

$$\left\|A\left(a,\ \Delta_W^{\alpha,d,n}\right)u\right\|_{\mathscr{H}^s_{\alpha,n}} = C_{\alpha+2n,d}\|\varphi_s\|_{\alpha+2n,2} \leq CC_{\alpha+2n,d}\|f *_{\alpha,n} g\|_{\alpha+2n,2} \leq k_s\,\|u\|_{\mathscr{H}^{s+r}_{\alpha,n}}$$

where

$$k_s = C\|f\|_{\alpha+2n,1} = C\int_{\mathbb{R}^{d+1}_+} e^{(s-t)||y||}d\mu_{\alpha+2n,d}\,(y)\,.$$

Which achieves the proof. □

References

[1] A. Aboulez, A. Achak, R. Daher and E. Loualid, Harmonic analysis associated with the generalized *Weinstein operator. International.J. of Analysis and Applications. Vol.9,Nr 1, (2015), p. 19-28.*

[2] H. Ben Mohamed, N. Bettaibi and S.H. Jah. *Sobolev type spaces associated with the Weinstein operator* , Int. Journal of Math. Analysis, Vol.5, Nr.28, (2011), p.1353-1373.

[3] H. Ben Mohamed, B. Ghribi *Weinstein-Sobolev spaces of exponential type and applications* . Acta Mathematica Sinica, English Series,Vol.29, Nr.3,(2013), p.591-608.

[4] H. Ben Mohamed, A. Gasmi and N. Bettaibi. Inversion of the *Weinstein intertwining operator and its dual using Weinstein wavelets. An. St. Univ. Ovidius Constanta. Vol. 1, Nr1, (2016), p. 1-19.*

[5] H. Ben Mohamed. On the *Weinstein* equations in spaces $\mathscr{D}^p_{\alpha,d}$ type. International Journal of Open Problems in Complex Analysis. Vol.9, Nr 1, (2017), p. 39-59.

[6] K. El-Hussein. Fourier transform and Plancherel Theorem for Nilpotent Lie Group. International Journal of Mathematics Trends and Technology. Vol.4 Issue 11, (2013), p. 288-294.

[7] I. Aliev. Investigation on the Fourier-Bessel harmonic analusis, Doctoral Dissertation, Baku 1993 (in Russian).

[8] I.A. Aliev and B. Rubin. Parabolic potentials and wavet transform with the generalized translation. Studia Math. 145 (2001) Nr1, p. 1-16.

[9] I.A. Aliev and B. Rubin. Spherical harmonics associated to the Laplace-Bessel operator and generalized spherical convolutions. Anal. Appl. (Singap) Nr 1 (2003), p. 81-109.

[10] S. Lee, *Generalized Sobolev spaces of exponential type*, Kangweon-Kyungki Math. J. Vol.8, Nr 1 (2000) p. 73-86.

[11] J. Löfström and J. Peetre. Approximation theorems connected with generalized translations. Math. Ann. Nr 181 (1969), p. 255-268.

[12] D. H. Pahk and B. H. Kang, *Sobolev spaces in the generlized distribution spaces of Beurling type*, Tsukuba J. Math. Vol. 15, (1991) p.325-334.

[13] R. S. Pathak, *Generalized Sobolev spaces and pseudo-differential operators on spaces of ultradistributions, Structure of solutions of diffential equations*, Edited y M.Morimoto and T. Kawai, World Scientific, Singafore (1996) p. 343-368.

[14] K. Stempak. La théorie de Littlewood-Paley pour la transformation de Fourier-Bessel. C.R. Acad.Sci. Paris Sér. I303 (1986), p. 15-18.

[15] K. Trimèche. Generalized Wavet and Hypergroups. Gordon and Breach, New York, 1997.

[16] A.Saoudi. A Variation of L^p Uncertainty Principles in Weinstein Setting. Indian J.Pure Appl. Math., 51(4),(2020), p. 1697-1712.