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Common Fixed Point Theorems of Self Mappings in non-Newtonian Metric Spaces

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Abstract - *The non-Newtonian metric concept was defined in 2012 [5], and then the non-Newtonian metric spaces and their some topological properties were gived by Binbasioglu, Demiriz, Turkoglu in 2016 [6]. Also, they introduced the fixed point theory in non-Newtonian metric spaces. In this paper, we proved some common fixed point theorems and results for self mappings in the non-Newtonian metric spaces.*

Keywords - *Common fixed point theorem, Contraction mapping, Fixed point theorem, Non-Newtonian calculi, Non-Newtonian metric space*

I. INTRODUCTION

There are many applications of fixed point theory and self-mapping that meet certain contraction conditions and have been an important area of various research activities [1]-[4], [7]-[11].
The non-Newtonian calculi is an alternative to usual calculus. The calculus in various fields including fractal geometry, economic growth, finance, wave theory in physics, quantum physics, information technology, tumor therapy and cancer-chemotherapy in medicine, differential equations (inclusive of multiplicative Lorenz system and Runge–Kutta methods), approximation theory, least-squares methods, complex analysis, functional analysis, probability theory, decision making, dynamical systems and chaos theory has many applications. The non-Newtonian metric concept was defined in 2012 [5] and then Binbaşıoğlu, Demiriz, Türkoğlu gived the non-Newtonian metric spaces and their some topological properties in 2016 [6]. Also, they introduced the fixed point theory in non-Newtonian metric spaces. In this paper, we proved some common fixed point theorems and results for self mappings in the non-Newtonian metric spaces.

II. PRELIMINARIES

We mention that some basic knowledge related to non-Newtonian calculus. Now, we give the non-Newtonian real field and its properties.

Definition: Definition of a *generator* is gived an injective function whose domain $\mathbb{R}$ and the its range is a subset of $\mathbb{R}$ [6].
Remark: Every generator generates an arithmetic [6].

Definition: Let we take the function $\beta: \mathbb{R} \to \mathbb{R}^+$, $x \to \beta(x) = e^x = y$, where $\mathbb{R}^+$ is the positive real numbers set. If $\beta =$ exp, then it generates the geometrical arithmetics.
Define the set $\mathbb{R}(N) := \{\beta(x): x \in \mathbb{R}\}$, as $\mathbb{R}(N)$ is the non-Newtonian real numbers set [6].
Remark: Suppose that this function β is a generator, that is, if $\beta = I,\ I(x) = x$ for all $x \in \mathbb{R}$, then β generates the classical arithmetic.
All the concepts of β- arithmetic are similar to the classical ones [6].

Definition: The β-integers are generated as follows;
β-zero, β-one and all β-integers are showed as
..., $\beta(-1)$, $\beta(0)$, $\beta(1)$, ...
Let's get any generator β with range A. Then for $x, y \in \mathbb{R}$, *the operations* β-addition, β-substraction, β-multiplication, β-division and β-order are defined as in the follow,
β-addition $x \dot{+} y = \beta\{\beta^{-1}(x) + \beta^{-1}(y)\}$,
β-substraction $x \dot{-} y = \beta\{\beta^{-1}(x) - \beta^{-1}(y)\}$,
β-multiplication $x \dot{\times} y = \beta\{\beta^{-1}(x) \times \beta^{-1}(y)\}$,
β-division $x \dot{/} y = \beta\{\beta^{-1}(x) \div \beta^{-1}(y)\}$,
β-order $x \dot{<} y = \beta(x) < \beta(y)$ [6].
Proposition (See [2]). $(\mathrm{R}(N), \dot{+}, \dot{\times})$ *is a complete field [6].*
For $x \in A \subset \mathrm{R}(N)$, a number β-square is described by $x \dot{\times} x$ and denoted by x^{2N}. The symbol $\sqrt{x}^N$ denotes $t = \beta\{\sqrt{\beta^{-1}(x)}\}$ which is the unique β nonnegative number whose β $-$square is equal to x and which means $t^{2N} = x$, for each β nonnegative number t.

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Throughout this paper, x^{pN} denotes the pth non-Newtonian exponent.
Thus we have

$$x^{2N} = x \dot{\times} x = \beta\{\beta^{-1}(x) \times \beta^{-1}(x)\} = \beta\{[\beta^{-1}(x)]^2\},$$
$$x^{3N} = x^{2N} \dot{\times} x = \beta\{\beta^{-1}\{\beta[\beta^{-1}(x) \times \beta^{-1}(x)]\} \times \beta^{-1}(x)\} = \beta\{[\beta^{-1}(x)]^3\},$$

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$$x^{pN} = x^{(p-1)N} \dot{\times} x = \beta\{[\beta^{-1}(x)]^p\},$$

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We denote by $|x|_N$ the $\beta -$absolute value of a number $x \in A \subset \mathrm{R}(N)$ defined as $\beta(|\beta^{-1}(x)|)$ and also
$\sqrt{x^{2N}}^{N} = |x|_N = \beta(|\beta^{-1}(x)|)$.
Thus,

$$|x|_N = \beta(|\beta^{-1}(x)|) = \begin{cases} x, & \beta(0) \dot{<} x, \\ \beta(0), & \beta(0) = x, \\ \beta(0) \dot{-} x, & \beta(0) \dot{>} x. \end{cases}$$

For $x_1, x_2 \in A \subset \mathrm{R}(N)$, the non-Newtonian distance $|.|_N$ is defined as
$|x_1 \dot{-} x_2|_N = \beta\{|\beta^{-1}(x_1) - \beta^{-1}(x_2)|\}$.
This distance is commutative; i.e., $|x_1 \dot{-} x_2|_N = |x_2 \dot{-} x_1|_N$.
Take any $z \in \mathrm{R}(N)$, if $(z \dot{>} \beta(0))$, then z is called a *positive non-Newtonian real number* ; if $z \dot{<} \beta(0)$, then z is called a *non-Newtonian negative real number* and if $z = \beta(0)$, then z is called an *unsigned non-Newtonian real number*. Non-Newtonian positive real numbers are denoted by $\mathrm{R}^+(N)$ and non-Newtonian negative real numbers by $\mathrm{R}^-(N)$.
The fundamental properties provided in the classical calculus is provided in non-Newtonian calculus, too [6].

Proposition 1.2 $|x \dot{\times} y|_N = |x|_N \dot{\times} |y|_N$ for any $x, y \in \mathrm{R}(N)$ [6].
Proposition 1.3 *The triangle inequality with respect to non-Newtonian distance* $|.|_N$, *for any* $x, y \in \mathrm{R}(N)$ *is given by*

$$|x \dot{+} y|_N \leq |x|_N \dot{+} |y|_N.$$

The non-Newtonian metric spaces provide an alternative to the metric spaces introduced in [6].

Definition 1.4 Let $X \neq \emptyset$ be a set. If a function $d_N: X \times X \to \mathrm{R}^+(N)$ satisfies the following axioms for all $x, y, z \in \mathrm{X}$:
(NM1) $d_N(x, y) = \beta(0) = \dot{0}$ if and only if $x = y$,
(NM2) $d_N(x, y) = d_N(y, x)$,
(NM3) $d_N(x, y) \dot{<} d_N(x, z) \dot{+} d_N(z, y)$,
then it is called a non-Newtonian metric on X and the pair (X, d_N) is called a non-Newtonian metric space[6].

Proposition 1.5 *Suppose that the non-Newtonian metric* d_N *on* $\mathrm{R}(N)$ *is such that* $d_N(x, y) = |x \dot{-} y|_N$ *for all* $x, y \in \mathrm{R}(N)$, *then* $(\mathrm{R}(N), d_N)$ *is a non-Newtonian metric space* [6].

Definition 1.6 Let (X, d_N) be a non-Newtonian metric space, $x \in \mathrm{X}$ and $\varepsilon \dot{>} \beta(0)$, we now define a set

$$B_\varepsilon^N(x) = \{y \epsilon X: d_N(x, y) \dot{<} \varepsilon\},$$

which is called a non-Newtonian open ball of radius ε with center x. Similarly, one describes the non-Newtonian open ball of radius ε with center x. Similarly, one describes the non-Newtonian closed ball as

$$D_\varepsilon^N(x) = \{y \epsilon X: d_N(x, y) \dot{\leq} \varepsilon\} \text{ [5].}$$

Example 2.3. Consider the non-Newtonian metric space *then* $(\mathrm{R}^+(N), d^*{}_N)$. From the definition of $d^*{}_N$, we can verify that the non-Newtonian open ball of radius $\varepsilon \dot{>} \beta(1)$ with center x_0 appears as

$$(x_0 \dot{-} \varepsilon, x_0 \dot{-} \varepsilon) \subset \mathrm{R}^+(N) \text{ [5].}$$

Definition 2.4 Let (X, d_N) be a non-Newtonian metric space and $A \subset X$. Then we call $x \in A$ a non-Newtonian interior point of A if there exists an $\varepsilon \dot{>} \beta(0)$ such that $B_\varepsilon^N(x) \subset A$. The collection of all interior points of A is called the non-Newtonian interior of A and is denoted by $int_N(A)$ [5].

Definition 2.5 Let (X, d_N) be a non-Newtonian metric space and $A \subset X$. If every point of A is a non-Newtonian interior point of A, i.e. $A = int_N(A)$, then A is called a non-Newtonian open set [5].

Definition 2.6 Let (X, d_N^X) and (Y, d_N^Y) be two non-Newtonian metric spaces and let $f: X \to Y$ be a function. If f satisfies the requirement that, for every $\varepsilon \dot{>} \beta(0)$ there exists an $\delta \dot{>} \beta(0)$ such that $f(B_\delta^N(x)) \subset B_\varepsilon^N(f(x))$ then f is said to be non-Newtonian continuous at $x \in X$ [5].

Definition 2.7 A sequence (x_n) in a non − Newtonian metric space $X = (X, d_N)$ is said to be non-Newtonian convergent if for every given $\varepsilon \dot{>} \beta(0)$ there exists an $n_0 = n_0(\varepsilon) \in$ N and $x \in X$ such that $d_N(x_n, x) \dot{<} \varepsilon$ for all $n > n_0$ and it is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \overset{N}{\to} x$, as $n \to \infty$ [5].

Definition 2.8 A sequence (x_n) in a non-Newtonian metric space $X = (X, d_N)$ is said to be non-Newtonian Cauchy if for every given $\varepsilon \dot{>} \beta(0)$ there exists an $n_0 = n_0(\varepsilon) \in$ N such that $d_N(x_n, x_m) \dot{<} \varepsilon$ for all $m, n > n_0$.
Similarly, if for every non-Newtonian open ball $B_\varepsilon^N(x)$, there exists a natural number n_0 such that $n > n_0$, $x_n \in B_\varepsilon^N(x)$, then the sequence (x_n) is said to be non-Newtonian convergent to x.
The space X is said to be non-Newtonian complete if every non-Newtonian Cauchy sequence in X converges [5].

III. MAIN RESULTS

Theorem 3.3. *Let (X, d_N) be a non-Newtonian complete metric space. Suppose T is a continuous self map on X and S be any self map on X that commutes with T. Assume the following conditions are satisfied;*

i) $S(X) \subset T(X)$

ii) *For all $x, y \in X$, $d_N(Sx, Sy) \dot{\le} t \dot{\times} v(x,y)$ where $t \in \left(\beta(0), \beta\left(\frac{1}{2}\right)\right)$ is a non-Newtonian positive real number and*

$$v(x,y) \in \{d_N(Tx,Ty), d_N(Tx,Sx), d_N(Ty,Sy), d_N(Tx,Sy), d_N(Ty,Sx)\}$$

iii) *$T(X)$ or $S(X)$ is E-complete subspace of X.*

Then S and T have the unique common fixed point.

Proof. *From the first condition, implies that by taking an arbitrary $x_0 \in X$, so we can construct a sequence $\{y_n\}$ of points in X such that $y_n = Sx_n = Tx_{n+1}$, for all $n \ge 0$.*
Now, we prove that $\{y_n\}$ is a non-Newtonian Cauchy sequence.
We show that $d_N(y_n, y_{n+1}) \dot{\le} \frac{t}{1-t} \dot{\times} d_N(y_{n-1}, y_n)$ for all $n \ge 1$.
In real,
$d_N(y_n, y_{n+1}) = d_N(Sx_n, Sx_{n+1}) \dot{\le} t \dot{\times} v_n$ *where*

$$v_n \in \{d_N(Tx_n, Tx_{n+1}), d_N(Tx_n, Sx_n), d_N(Tx_{n+1}, Sx_{n+1}), d_N(Tx_n, Sx_{n+1}), d_N(Tx_{n+1}, Sx_n)\}$$

$=\{d_N(y_{n-1}, y_n), d_N(y_{n-1}, y_n), d_N(y_n, y_{n+1}), d_N(y_{n-1}, y_{n+1}), d_N(y_n, y_n)\}$
$=\{d_N(y_{n-1}, y_n), d_N(y_n, y_{n+1}), d_N(y_{n-1}, y_{n+1}), 0\}$.
There exist four cases;
Case 1: $d_N(y_n, y_{n+1}) \dot{\le} t \dot{\times} d_N(y_{n-1}, y_n) \dot{\le} \frac{t}{1-t} \dot{\times} d_N(y_{n-1}, y_n)$.
Case 2: $d_N(y_n, y_{n+1}) \dot{\le} t \dot{\times} d_N(y_n, y_{n+1})$ *and so* $d_N(y_n, y_{n+1}) = 0$. *Then the above inequality is provided, because* $t \dot{<} \frac{t}{1-t}$.
Case 3: $d_N(y_n, y_{n+1}) \dot{\le} t \dot{\times} d_N(y_{n-1}, y_{n+1}) \dot{\le} t \dot{\times} d_N(y_{n-1}, y_n) \dot{+} t \dot{\times} d_N(y_n, y_{n+1})$.
Case 4: $d_N(y_n, y_{n+1}) \dot{\le} t \dot{\times} \beta(0) = \beta(0)$ *and so* $d_N(y_n, y_{n+1}) = \beta(0)$.
Thus by putting $= \frac{t}{1-t}$, $d_N(y_n, y_{n+1}) \dot{\le} \delta \dot{\times} d_N(y_{n-1}, y_n)$. *Now, we have*
$d_N(y_n, y_{n+1}) \dot{\le} \delta \dot{\times} d_N(y_{n-1}, y_n) \dot{\le} \dots \dot{\le} \delta^n d_N(y_0, y_1)$, *for all* $n \ge 1$.
Now, for $n > m$ we have

$$\begin{aligned} d_N(y_n, y_m) &\dot{\le} d_N(y_n, y_{n-1}) \dot{+} d_N(y_{n-1}, y_{n-2}) \dot{+} \dots \dot{+} d_N(y_{m+1}, y_m) \\ &\dot{\le} (\delta^{n-1} \dot{+} \delta^{n-2} \dot{+} \dots \dot{+} \delta^m) \dot{\times} d_N(y_0, y_1) \\ &\dot{\le} \frac{\delta^m}{1-\delta} d_N(y_0, y_1). \end{aligned}$$

Therefore, (y_n) is a non-Newtonian Cauchy sequence. Since the range of T contains the range of S and the range of at least one is non-Newtonian complete, there exists a $z \in T(X)$ such that $Tx_n \overset{N}{\to} z$. So there exists a sequence (a_n) in X such that $(a_n) \overset{N}{\to} \beta(0)$ and $d_N(Tx_n, z) \dot{\le} a_n$.
Hence, $y_n = Sx_n = Tx_{n+1} \overset{N}{\to} z$.
Now, we show that $Sz = Tz = z$. In this way, note that

$d_N(Tz, Sz) \dot{\le} d_N(Tz, STx_n) \dot{+} d_N(STx_n, Sz)$, *for all* $n \ge 1$.

Also we have $d_N(STx_n, Sz) \dot{\le} t \dot{\times} v_n$ *for all* $n \ge 1$, *where*

$$\nu_n \in \{d_N(T^2x_n,Tz), d_N(T^2x_n,STx_n), d_N(Tz,Sz), d_N(T^2x_n,Sz), d_N(Tz,STx_n)\}.$$

Choose a natural number n_0 such that for all $n \geq n_0$, because

$TSx_n = STx_n \overset{N}{\rightarrow} Tz$ *and* $T^2x_n \overset{N}{\rightarrow} Tz$, *then there exists sequences* (a_n) *and* (b_n) *in* X *such that* $(a_n) \overset{N}{\rightarrow} \beta(0)$ *and* $(b_n) \overset{N}{\rightarrow} \beta(0)$, *we have* $d_N(Tz,STx_n) \dot{\leq} a_n$ *and* $d_N(T^2x_n,Tz) \dot{\leq} b_n$.

Thus we obtain the following cases;

Case 1: $d_N(Tz,Sz) \dot{\leq} d_N(Tz,STx_n) \dot{+} t \dot{\times} d_N(T^2x_n,Tz) \dot{\leq} a_n \dot{+} t \dot{\times} b_n$

Case 2: $d_N(Tz,Sz) \dot{\leq} d_N(Tz,STx_n) \dot{+} t \dot{\times} d_N(T^2x_n,STx_n)$

$$\dot{\leq} d_N(Tz,STx_n) \dot{+} t \dot{\times} (d_N(T^2x_n,Tz) \dot{+} d_N(Tz,STx_n))$$

$$\dot{\leq} a_n \dot{+} t \dot{\times} (b_n \dot{+} a_n)$$

Case 3: $d_N(Tz,Sz) \dot{\leq} d_N(Tz,STx_n) \dot{+} t \dot{\times} d_N(Tz,Sz) \dot{\leq} \frac{a_n}{\beta(1) \dot{-} t}$

Case 4: $d_N(Tz,Sz) \dot{\leq} d_N(Tz,STx_n) \dot{+} t \dot{\times} d_N(T^2x_n,Sz)$

$$\dot{\leq} d_N(Tz,STx_n) \dot{+} t \dot{\times} (d_N(T^2x_n,Tz) \dot{+} d_N(Tz,Sz))$$

$$\dot{\leq} \frac{a_n \dot{+} t \dot{\times} b_n}{\beta(1) \dot{-} t}$$

Case 5: $d_N(Tz,Sz) \dot{\leq} d_N(Tz,STx_n) \dot{+} t \dot{\times} d_N(Tz,TSx_n)$

$$\dot{\leq} (\beta(1) \dot{+} t) \dot{\times} d_N(Tz,STx_n)$$

$$\dot{\leq} (\beta(1) \dot{+} t) \dot{\times} a_n$$

$$\dot{\leq} \beta(2) \dot{\times} a_n.$$

Therefore $d_N(Tz,Sz) = \beta(0)$ *i.e.* $Tz = Sz$. *So,* $d_N(Tz,z) \dot{\leq} d_N(Sz,Sx_n) \dot{+} d_N(Sx_n,z) \dot{\leq} d_N(Sx_n,z) \dot{+} t \dot{\times} \nu_n$, *where*

$$\nu_n \in d_N(Tx_n,Tz), d_N(Tx_n,Sx_n), d_N(Tz,Sz), d_N(Tx_n,Sz), d_N(Tz,Sx_n)$$

$= \{d_N(Tx_n,Tz), d_N(Tx_n,Sx_n), 0, d_N(Tz,Sx_n)\}$.

Choose a natural number n_0 *such that for all* $n \geq n_0$ *we have* $d_N(Tx_n,z) \dot{\leq} c_n$ *and* $d_N(Tx_n,z) \dot{\leq} d_n$, *as* $(c_n) \overset{N}{\rightarrow} \beta(0)$ *and* $(d_n) \overset{N}{\rightarrow} \beta(0)$.

Again, we obtain the following cases;

Case 1: $d_N(Tz,z) \dot{\leq} d_N(z,Sx_n) \dot{+} t \dot{\times} d_N(Tx_n,Tz)$

$$\dot{\leq} d_N(z,Sx_n) \dot{+} t \dot{\times} (d_N(Tx_n,z) \dot{+} d_N(z,Tz))$$

$$\dot{\leq} \frac{d_n \dot{+} t \dot{\times} c_n}{\beta(1) \dot{-} t}$$

Case 2: $d_N(Tz,z) \dot{\leq} d_N(Sx_n,z) \dot{+} t \dot{\times} d_N(Tx_n,Sx_n)$

$$\dot{\leq} d_N(Sx_n,z) \dot{+} t \dot{\times} (d_N(Tx_n,z) \dot{+} d_N(z,Sx_n))$$

$$\dot{\leq} d_n \dot{+} t \dot{\times} (c_n \dot{+} d_n)$$

Case 3: $d_N(Tz,z) \dot{\leq} d_N(Sx_n,z) \dot{+} t \dot{\times} \beta(0) = d_N(Sx_n,z) \dot{\leq} d_n$

Case 4: $d_N(Tz,z) \dot{\leq} d_N(Sx_n,z) \dot{+} t \dot{\times} d_N(Tz,Sx_n)$

$$\dot{\leq} d_N(Sx_nz,) \dot{+} t \dot{\times} (d_N(Tz,z) \dot{+} d_N(z,Sx_n))$$

$$\dot{\leq} d_n \dot{+} t \dot{\times} (c_n \dot{+} d_n).$$

Therefore $d_N(Tz,z) = \beta(0)$ *i.e.* $Tz = z$.

Finally,

$Tz = Sz = z$ *and so* z *is a common fixed point for* T *and* S.

Now, we show that, the common fixed point is unique.

If z_1 *is another common fixed point, then* $z_1 = Sz_1 = Tz_1$. *So,*

So, $d_N(z,z_1) = d_N(Sz,Sz_1) \dot{\leq} t \dot{\times} \nu(z,z_1)$ *where*

$$\nu(z,z_1) \in \{d_N(Tz,Tz_1), d_N(Tz,Sz), d_N(Tz_1,Sz_1), d_N(Tz,Sz_1), d_N(Tz_1,Sz)\}$$

$= \{d_N(z,z_1), 0\}$.

Then $d_N(z,z_1) = 0$ *that is* $z = z_1$. *Hence* z *is an unique common fixed point for* T *and* S.

Corollary. *Let* (X,d_N) *be a non-Newtonian complete metric space. Suppose* T *is a continuous self map on* X *and* S *be any self map on* X *that commutes with* T. *Also* $S(X) \subset T(X)$ *and for all* $x,y \in X$, $d_N(Sx,Sy) \dot{\leq} t \dot{\times} d_N(Tx,Ty)$ *where* $t \in (\beta(0),\beta(1))$*is a non-Newtonian positive real number. Then* S *and* T *have an unique common fixed point.*

Theorem 3.3. *Let* (X,d_N) *be a non-Newtonian complete metric space. Suppose* T^2 *is a continuous self map on* X *and* S *be any self map on* X *that commutes with* T. *Assume the following conditions are satisfied;*

i) $ST(X) \subset T^2(X)$

ii) *For all* $x, y \in X$, $d_N(Sx, Sy) \dot{\leq} t \dot{\times} \nu(x, y)$ *where* $t \in \left(\beta(0), \beta\left(\frac{1}{2}\right)\right)$ *is a non-Newtonian positive real number and*

$$\nu(x, y) \in \{d_N(Tx, Ty), d_N(Tx, Sx), d_N(Ty, Sy), d_N(Tx, Sy), d_N(Ty, Sx)\}$$

iii) $T(X)$ *or* $S(X)$ *is* non-Newtonian *complete subspaces of* X.

Then S *and* T *have the unique common fixed point.*

Proof. *From the first condition, implies that by taking an arbitrary* $x_0 \in TX$, *so we can construct a sequence* $\{y_n\}$ *of points in* TX *such that* $y_n = Sx_n = Tx_{n+1}$, *for all* $n \geq 0$.

Now, $Ty_{n+1} = TSx_{n+1} = STx_{n+1} = Sy_n = z_n, n \geq 1$.

Now, we prove that $\{z_n\}$ *is a non-Newtonian Cauchy sequence and then non-Newtonian convergent to some* $z \in X$. *Further we shall show that* $T^2z = STz$.

Since $lim_{n\to\infty} Ty_n = lim_{n\to\infty} TSx_n = lim_{n\to\infty} STx_n = lim_{n\to\infty} Sy_n = lim_{n\to\infty} z_n = z$, *it follows that* $lim_{n\to\infty} T^4x_n = lim_{n\to\infty} T^3Sx_n = lim_{n\to\infty} ST^3x_n = T^2z$, *since* T^2 *is continuous. Now, we obtain*

$$d_N(T^2z, STz) \dot{\leq} d_N(T^2z, T^3Sx_n) \dot{+} d_N(T^3Sx_n, STz)$$
$$\dot{\leq} d_N(T^2z, T^3Sx_n) \dot{+} t \dot{\times} \nu_n,$$

where

$$\nu_n \in \{d_N(T^4x_n, T^2z), d_N(T^4x_n, ST^3x_n), d_N(T^2z, STz), d_N(T^4x_n, STz), d_N(T^2z, ST^3x_n)\}.$$

Choose a natural number n_0 *such that for all* $n \geq n_0$, *because* $T^3Sx_n \overset{N}{\to} T^2z$ *and* $T^4x_n \overset{N}{\to} T^2z$, *then we have* $d_N(T^2z, T^3Sx_n) \dot{\leq} a_n$ *and* $d_N(T^4x_n, T^2z) \dot{\leq} b_n$ *as* $a_n \overset{N}{\to} \beta(0)$ *and* $b_n \overset{N}{\to} \beta(0)$.

Again we have the following cases;

Case 1: $d_N(T^2z, STz) \dot{\leq} d_N(T^2z, T^3Sx_n) \dot{+} t \dot{\times} d_N(T^4Sx_n, T^2z) \dot{\leq} a_n \dot{+} t \dot{\times} b_n$,

Case 2: $d_N(T^2z, STz) \dot{\leq} d_N(T^2z, T^3Sx_n) \dot{+} t \dot{\times} d_N(T^4Sx_n, ST^3z)$

$$\dot{\leq} d_N(T^2z, T^3Sx_n) \dot{+} t \dot{\times} (d_N(T^4x_n, T^2z) \dot{+} t \dot{\times} d_N(T^2z, ST^3x_n))$$
$$\dot{\leq} a_n \dot{+} t \dot{\times} (b_n \dot{+} a_n)$$

Case 3: $d_N(T^2z, STz) \dot{\leq} d_N(T^2z, T^3Sx_n) \dot{+} t \dot{\times} d_N(T^4Sx_n, ST^3z) \dot{\leq} \frac{a_n}{\beta(1) \dot{-} t}$

Case 4: $d_N(T^2z, STz) \dot{\leq} d_N(T^2z, T^3Sx_n) \dot{+} t \dot{\times} d_N(T^4x_n, STz)$

$$\dot{\leq} d_N(T^2z, T^3Sx_n) \dot{+} t \dot{\times} (d_N(T^4x_n, T^2z) \dot{+} d_N(T^2z, STz))$$
$$\dot{\leq} \frac{a_n \dot{+} t \dot{\times} b_n}{\beta(1) \dot{-} t}$$

Case 4: $d_N(T^2z, STz) \dot{\leq} d_N(T^2z, T^3Sx_n) \dot{+} t \dot{\times} d_N(T^4x_n, STz)$

$$\dot{\leq} d_N(T^2z, T^3Sx_n) \dot{+} t \dot{\times} (d_N(T^4x_n, T^2z) \dot{+} d_N(T^2z, STz))$$
$$\dot{\leq} \frac{a_n \dot{+} t \dot{\times} b_n}{\beta(1) \dot{-} t}$$

Case 5: $d_N(T^2z, STz) \dot{\leq} d_N(T^2z, T^3Sx_n) \dot{+} t \dot{\times} d_N(T^2z, ST^3x_n)$

$$\dot{\leq} a_n \dot{+} t \dot{\times} a_n$$
$$\dot{\leq} (\beta(1) \dot{+} t) \dot{\times} a_n.$$

Therefore, since the infimum of sequences on the right side of last inequality are zero, then $d_N(T^2z, STz) = \beta(0)$ *that is* $T^2z = STz$ *and so* STz *is a common fixed point for* T *and* S. *Really, putting in* $d_N(Sx, Sy) \dot{\leq} t \dot{\times} \nu(x, y)$, $x = STz$.

Example 2.3. Consider the non-Newtonian metric space $X = \mathrm{R}(N)$,
$d_N(x, y) = (\alpha \dot{\times} |x \dot{-} y|_N, \gamma \dot{\times} |x \dot{-} y|_N)$, $\alpha, \gamma \in \mathrm{R}^+(N)$, $Sx = x^{2N} \dot{+} \beta(2)$,
$Tx = \beta(2) \dot{\times} x^{2N}$. Then, for all $x, y \in X$ we have
$d_N(Sx, Sy) = (\alpha \dot{\times} |x^{2N} \dot{-} y^{2N}|_N, \gamma \dot{\times} |x^{2N} \dot{-} y^{2N}|_N) = \beta(\frac{1}{3}) \dot{\times} d_N(Tx, Ty) \dot{\leq} t \dot{\times} d_N(Tx, Ty)$ for
$t \in [\beta(\frac{1}{3}), \beta(1))$, $S(X) \subset T(X)$ and $S(X)$ is non-Newtonian complete subspace of X and T is a continuous self map on X.
Therefore all conditions of Corollary are satisfied. Thus S and T have an unique common fixed point.

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