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Stability And Hopf Bifurcation of A Double Quality Level Supply Chain With Double Delay

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Abstract — *In this paper, the bifurcation theory is used to study the mass adjustment dynamic time-delay model with double time delay. Firstly, the delay decision is selected as the delay parameter to obtain the stability point and the critical value of maintaining stability, and the conditions for Hopf bifurcation are discussed. In addition, the Hopf bifurcation direction and the stability of periodic solutions are studied by using the central manifold theorem and the gauge theory, and the calculation formula is obtained. Finally, the validity of the conclusion is verified by numerical simulation with mathematical software.*

Keywords — *Center manifold theorem, Double mass level, Hopf bifurcation, Normal form theory, Stability*

I. INTRODUCTION

With the increasing improvement of national life and the gradual improvement of people's quality of life, consumers have a wider range of choices for consumption content. In addition to the price of products, consumers pay more attention to the product quality of manufacturers and the service quality of retailers.Therefore, the manufacturer's choice of product quality level and the retailer's adjustment of service quality level are of great practical significance. In terms of research content, many scholars mainly conducted unilateral static research on product quality level and service quality level, and few researches on dynamic aspects. Literature [4] introduces the dual quality level into the supply chain. As manufacturers or retailers make decisions on the next phase price with reference to the historical price, price risks and uncertainties can be effectively avoided. Therefore, manufacturers and retailers usually adopt a delay strategy when making decisions on product quality and service level, and make future development direction according to consumers' past demands. Therefore, the author studies the system stability of their dynamic time-delay model and the existence of Hopf branch. In this paper, the system stability and the existence of Hopf bifurcation of the dynamic two-delay model are studied by considering the differences between manufacturers and retailers.

II. MODEL BUILDING

According to literature [1], the dynamic process of product quality level Q and service quality level S is

$$\begin{cases} \dot{Q} = v_m Q[-k_m Q + \eta(\omega - c)], \\ \dot{S} = v_r S[-k_r S + \theta(P - \omega)]. \end{cases}$$

Where Q is the quality level of the product and S is the service level of the retailer, v_m and v_r are the adjustment speed of product quality level and service quality level respectively, k_m and k_r are cost coefficients of product quality level and service quality level, the rest of the retailer's costs are negligible, ω is the wholesale price a manufacturer pays to sell

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goods to a retailer, P is the retail price that the retailer sells, C is the production cost per unit product, η is the sensitivity coefficient of the quality level, θ is the sensitivity coefficient of service quality level.

Based on the above basis, Literature [4] considered the quality adjustment dynamic model with time delay for retailers and manufacturers to obtain information

$$\begin{cases}\dot{Q} = v_m Q[-k_m Q(t-\tau) + \eta(\omega - c)], \\ \dot{S} = v_r S[-k_r S(t-\tau) + \theta(P-\omega)].\end{cases}$$

Considering that the market information obtained by the manufacturer and the retailer respectively not only has time delay, but also has different time delay, this paper presents a quality adjustment model with two time delays

$$\begin{cases}\dot{Q} = v_m Q[-k_m Q(t-\tau_1) + \eta(\omega - c)], \\ \dot{S} = v_r S[-k_r S(t-\tau_2) + \theta(P-\omega)],\end{cases} \tag{1}$$

Where τ_1, τ_2 are respectively the pricing delay period of the manufacturer and the retailer.

III. STABPLPTY AND LOCAL HOPF BIFURCATION ANALYSIS

The stability and *Hopf* bifurcation of the positive equilibrium point (Q_0, S_0) of system (1) are discussed below.

First, let the equilibrium point of system (1) be E (Q_0, S_0), then it satisfies the following equation:

$$\begin{cases}v_m Q_0[-k_m Q_0 + \eta(\omega - c)] = 0, \\ v_r S_0[-k_r S_0 + \theta(P-\omega)] = 0,\end{cases} \tag{2}$$

solve the above equation, the positive equilibrium point is obtained,

$$\begin{cases}Q_0 = \dfrac{\eta(\omega - c)}{k_m}, \\ S_0 = \dfrac{\theta(P-\omega)}{k_r}.\end{cases} \tag{3}$$

Let $x(t) = Q(t) - Q_0$, and $y(t) = S(t) - S_0$. After linearizing the controlled system (1) at the equilibrium point, the linearization equation is

$$\begin{cases}\dot{x}(t) = v_m(x(t) + Q_0)[-k_m(x(t-\tau_1) + Q_0) + \eta(\omega - c)], \\ \dot{y}(t) = v_r(y(t) + S_0)[-k_r(y(t-\tau_2) + S_0) + \theta(P-\omega)],\end{cases} \tag{4}$$

reduction to

$$\begin{cases}\dot{x}(t) = a_1 x(t) + a_2 x(t-\tau_1), \\ \dot{y}(t) = b_1 y(t) + b_2 y(t-\tau_2),\end{cases} \tag{5}$$

among them

$$a_1 = -k_m v_m Q_0 + v_m \eta(\omega - c)\,,\ a_2 = -k_m v_m Q_0\,,$$

$$b_1 = -k_r v_r S_0 + v_r \theta(P-\omega)\,,\ b_2 = -k_r v_r S_0\,.$$

The characteristic equation of the system (5) is

$$\lambda^2 - \lambda(a_2 e^{-\lambda\tau_1} + b_2 e^{-\lambda\tau_2}) + a_2 b_2 e^{-\lambda(\tau_1+\tau_2)} = 0 \tag{6}$$

In order to study the stability and *Hopf* bifurcation of the equilibrium *E* of the system, we only need to discuss the distribution of the roots of the characteristic equation (6). If all the roots of the equation (6) have a negative real part, then the equilibrium *E* is asymptotically stable; if one root of the equation has a positive real part, then the equilibrium *E* is unstable. Since the dynamic properties of differential equations with multiple delays are very complex, the two time delays τ_1 and τ_2 of system (1) are discussed in the following cases.

Case 1: $\tau_1 = \tau_2 = 0$

Theorem 1. For system (5), $\tau_1 = \tau_2 = 0$, if $a_2 + b_2 < 0, a_2 b_2 > 0$, the equilibrium *E* is stable.

Proof. When $\tau_1 = \tau_2 = 0$, the characteristic equation of system (6) becomes

$$\lambda^2 - (a_2 + b_2)\lambda + a_2 b_2 = 0. \tag{7}$$

Therefore, it is clear from the characteristic equation (7) that only when $a_2 + b_2 < 0, a_2 b_2 > 0$, both roots of equation (7) have negative real parts, and at $\tau_1 = \tau_2 = 0$, the equilibrium point of the system is asymptotically stable.

Case 2: $\tau_1 = 0, \tau_2 > 0$

Lemma 1. For system (5), if $\tau_1 = 0, \tau_2 > 0$ is satisfied, then equation (7) has pure imaginary root $\pm i\omega_{2\pm}$, where

$$\omega_{2\pm} = \sqrt{\frac{-({a_2}^2 - {b_2}^2) \pm \sqrt{({a_2}^2 - {b_2}^2)^2 + 4{a_2}^2{b_2}^2}}{2}} \tag{8}$$

$$\tau_{2k}^{\pm} = \frac{1}{\omega_{2\pm}}[\arcsin(\frac{{\omega_2}^3 + {a_2}^2\omega_2}{-b_2{\omega_2}^2 - {a_2}^2 b_2}) + 2k\pi], k = 0,1,2,\cdots \tag{9}$$

Proof. When $\tau_1 = 0,\ \ \tau_2 > 0$,the characteristic equation of system (5) becomes

$$\lambda^2 - \lambda(a_2 + b_2 e^{-\lambda\tau_2}) + a_2 b_2 e^{-\lambda\tau_2} = 0. \tag{10}$$

First, we assume that $\lambda = i\omega_2 (\omega_2 > 0)$ is a root of the characteristic equation (10), then it satisfies the following equation

$$(i\omega_2)^2 - i\omega_2(a_2 + b_2 e^{-i\omega_2\tau_2}) + a_2 b_2 e^{-i\omega_2\tau_2} = 0. \tag{11}$$

That is

$$-{\omega_2}^2 - ia_2\omega_2 - ib_2\omega_2(\cos\omega_2\tau_2 - i\sin\omega_2\tau_2) + a_2 b_2(\cos\omega_2\tau_2 - i\sin\omega_2\tau_2) = 0. \tag{12}$$

The separation of the real and imaginary parts, it follows

$$\begin{cases} -{\omega_2}^2 - b_2\omega_2\sin\omega_2\tau_2 + a_2 b_2\cos\omega_2\tau_2 = 0, \\ -a_2\omega_2 - b_2\omega_2\cos\omega_2\tau_2 - a_2 b_2\sin\omega_2\tau_2 = 0. \end{cases} \tag{13}$$

From (13) we obtain

$${\omega_2}^4 + ({a_2}^2 - {b_2}^2){\omega_2}^2 - {b_2}^2{a_2}^2 = 0. \tag{14}$$

Because of $({a_2}^2 - {b_2}^2)^2 + 4{b_2}^2{a_2}^2 \geq 0$, equation (14) has positive roots ω_{2+} and ω_{2-}

$$\omega_{2+}=\sqrt{\frac{-(a_2{}^2-b_2{}^2)+\sqrt{(a_2{}^2-b_2{}^2)^2+4a_2{}^2b_2{}^2}}{2}}$$

$$\omega_{2-}=-\sqrt{\frac{-(a_2{}^2-b_2{}^2)-\sqrt{(a_2{}^2-b_2{}^2)^2+4a_2{}^2b_2{}^2}}{2}}$$

Therefore, there is a pure imaginary root $\pm i\omega_{2\pm}$, which can be calculated from Equation (13)

$$\tau_{2k}^{\pm}=\frac{1}{\omega_{2\pm}}[\arcsin(\frac{\omega_2{}^3+a_2{}^2\omega_2}{-b_2\omega_2{}^2-a_2{}^2b_2})+2k\pi],k=0,1,2,\cdots$$

This completes the proof.

Lemma 2. Let $\tau_{20}=min\{\tau_{2k}^{\pm}|k=0,1,2,\cdots\}=\tau_{20}^{+}$, and let the corresponding ω_2 be ω_{20}.

Let $\lambda(\tau_2)=\alpha(\tau_2)+i\omega(\tau_2)$ is the root of the characteristic equation (10) and the conditions $\alpha(\tau_{20})=0$ and $\omega(\tau_{20})=\omega_{20}$ are satisfied, then the transversely condition $\mathrm{Re}(\frac{d\lambda}{d\tau_2})^{-1}|_{\tau=\tau_2}>0$ is true.

Proof. By differentiating both sides of equation (10) with regard to τ_2 and applying the implicit function theorem, we have :

$$\frac{d\lambda}{d\tau_2}|_{\tau=\tau_2}=\frac{a_2b_2\lambda e^{-\lambda\tau_2}-\lambda^2b_1e^{-\lambda\tau_2}}{2\lambda+b_2\tau_2\lambda e^{-\lambda\tau_2}-a_2-b_2e^{-\lambda\tau_2}-a_2b_2\tau_2e^{-\lambda\tau_2}}|_{\tau=\tau_2}$$

so

$$[\frac{d\lambda}{d\tau_2}]^{-1}=\frac{i2\omega_{20}\cos\omega_{20}\tau_{20}+ib_2\omega_{20}\tau_{20}-ia_2\sin\omega_{20}\tau_{20}-2\omega_{20}\sin\omega_{20}\tau_{20}-a_2\cos\omega_{20}\tau_{20}-b_2-a_2b_2\tau_{20}}{ia_2b_2\omega_{20}+b_2\omega_{20}{}^2}$$

$$\mathrm{Re}(\frac{d\lambda}{d\tau_2})^{-1}|_{\tau=\tau_2}=\frac{-2b_2\omega_{20}{}^3\sin\omega_{20}\tau_{20}-b_2{}^2\omega_{20}{}^2+a_2b_2\omega_{20}{}^2\cos\omega_{20}\tau_{20}-a_2{}^2b_2\omega_{20}\sin\omega_{20}\tau_{20}}{b_2{}^2\omega_{20}{}^4+a_2{}^2b_2{}^2\omega_{20}{}^2}$$

because $\sin\omega_{20}\tau_{20}=-\frac{\omega_{20}{}^3+a_2{}^2\omega_{20}}{b_2\omega_{20}{}^2-a_2{}^2b_2}$, we know

$$\mathrm{Re}(\frac{d\lambda}{d\tau_2})^{-1}|_{\tau=\tau_2}=\frac{2\omega_{20}{}^6+3\omega_{20}{}^4(a_2{}^2-b_2{}^2)+2b_2{}^2\omega_{20}{}^4+a_2{}^2\omega_{20}{}^2(a_2{}^2-b_2{}^2)}{(b_2{}^2\omega_{20}{}^4+a_2{}^2b_2{}^2\omega_{20}{}^2)(a_2{}^2+\omega_{20}{}^2)}.$$

When (H_1): $a_2>b_2$, the transversal condition $\mathrm{Re}(\frac{d\lambda}{d\tau_2})^{-1}|_{\tau=\tau_2}>0$ holds.

The proof is completed.

According to the above analysis and *Hopf* bifurcation theory, we get the following theorem.

Theorem 2. For system (1), suppose that (H_1) is true.When $\tau_1=0$, $\tau_2\in[0,\tau_{20})$, its positive equilibrium point E is asymptotically stable; at $\tau_2>\tau_{20}$, the positive equilibrium point is unstable;when $\tau_2=\tau_{20}$, system (1) shows a *Hopf* branch at the equilibrium point.

Case 3: $\tau_1 = \tau_2 = \tau$

Theorem 3. According to literature [4], under corresponding conditions, when $\tau \in [0, \tau_0)$, the positive equilibrium point of system (5) is locally asymptotically stable; When $\tau = \tau_0$, system (5) has a Hopf branch at the equilibrium point; When $\tau > \tau_0$;The positive equilibrium point of system (5) is unstable.

Case 4: $\tau_1 > 0,\ \ \tau_2 > 0$

Lemma 3. For system (5), if $\tau_1 > 0,\ \ \tau_2 > 0$ is satisfied, then the characteristic equation (6) has a pair of pure imaginary roots $\pm i\omega_{11}$, where

$$\omega_{11} = \pm a_2,\ \tau_1 = \tau_1^j = -\frac{1}{a_2}\left(2j\pi + \frac{3\pi}{2}\right), (j = 0,1,2,\cdots).$$

Proof. When $\tau_1 > 0,\ \ \tau_2 > 0$, The characteristic equation (6) can be reduced to

$$(\lambda - a_2 e^{-\lambda\tau_1})(\lambda - b_2 e^{-\lambda\tau_2}) = 0,$$

let $\lambda = i\omega_1 (\omega_1 > 0)$ be the root of characteristic equation (6), plug it into the equation

$$(i\omega_1 - a_2 e^{-i\omega_1\tau_1})(i\omega_1 - b_2 e^{-i\omega_1\tau_2}) = 0,$$

then

$$i\omega_1 - a_2(\cos\omega_1\tau_1 - i\sin\omega_1\tau_1) = 0 \quad \text{or} \quad i\omega_1 - b_2(\cos\omega_1\tau_2 - i\sin\omega_1\tau_2) = 0$$

the separation of the real and imaginary parts, it follows

$$\begin{cases} \omega_1 + a_2\sin\omega_1\tau_1 = 0 \\ a_2\cos\omega_1\tau_1 = 0 \end{cases} \text{or} \begin{cases} \omega_1 + b_2\sin\omega_1\tau_2 = 0 \\ b_2\cos\omega_1\tau_2 = 0 \end{cases}$$

we obtain

$$\omega_{11} = \pm a_2, \tau_1 = \tau_1^j = -\frac{1}{a_2}\left(2j\pi + \frac{3\pi}{2}\right), (j = 0,1,2,\cdots),$$

$$\omega_{12} = \pm b_2, \tau_2 = \tau_2^j = -\frac{1}{b_2}\left(2j\pi + \frac{3\pi}{2}\right), (j = 0,1,2,\cdots),$$

So when $\tau_1 = \tau_1^j$, equation (6) has a pair of pure imaginary roots $\pm i\omega_{11}$. The proof is completed.

Lemma 4. Let $\tau_{11} = \min\{\tau_1^j \,|\, j = 0,1,2,\cdots\}$, and let the corresponding ω_1 be ω_{11}.

Let $\lambda(\tau_1) = \alpha(\tau_1) + i\omega(\tau_1)$ is the root of the characteristic equation (6) at $\tau_1 = \tau_{11}$ and the conditions $\alpha(\tau_{11}) = 0$ and $\omega(\tau_{11}) = \omega_{11}$ are satisfied, then the transversely condition $\mathrm{Re}(\frac{d\lambda}{d\tau_1})^{-1}|_{\tau=\tau_1} > 0$ is true.

Proof. By differentiating both sides of equation (6) with regard to τ_1 and applying the implicit function theorem, we have :

$$\frac{d\lambda}{d\tau_1}|_{\tau=\tau_1}=\frac{-a_2\lambda^2e^{-\lambda\tau_1}+\lambda a_2b_2e^{-\lambda(\tau_1+\tau_2)}}{2\lambda+a_2\tau_1\lambda e^{-\lambda\tau_1}-b_2\tau_2e^{-\lambda\tau_2}-a_2e^{-\lambda\tau_1}-b_2e^{-\lambda\tau_2}-(\tau_1+\tau_2)a_2b_2e^{-\lambda(\tau_1+\tau_2)}}$$

then

$$\mathrm{Re}(\frac{d\lambda}{d\tau_1})^{-1}|_{\tau=\tau_1}=\frac{R_1I_2+R_2I_1}{I_1^{\ 2}+I_2^{\ 2}}$$

Among in

$$R_2=\omega_{11}a_2\tau_{11}\sin\omega_{11}\tau_{11}-b_2\tau_2\cos\omega_{11}\tau_2-a_2\cos\omega_{11}\tau_{11}-b_2\cos\omega_{11}\tau_2-(\tau_{11}+\tau_2)a_2b_2\cos\omega_{11}(\tau_{11}+\tau_2)$$

$$R_1=2\omega_{11}+\omega_{11}a_2\tau_{11}\cos\omega_{11}\tau_{11}+b_2\tau_2\sin\omega_{11}\tau_2+a_2\sin\omega_{11}\tau_{11}+b_2\sin\omega_{11}\tau_2+(\tau_{11}+\tau_2)\sin\omega_{11}(\tau_{11}+\tau_2)$$

$$I_1=\omega_{11}a_2\cos\omega_{11}\tau_{11}+\omega_{11}a_2b_2\sin\omega_{11}(\tau_{11}+\tau_2)\,,\ I_2=\omega_{11}a_2b_2\cos\omega_{11}(\tau_{11}+\tau_2)-\omega_{11}a_2\sin\omega_{11}\tau_{11}\,.$$

Thus, when $(H_2): R_1I_2+R_2I_1\neq 0$ is satisfied, $\mathrm{Re}(\frac{d\lambda}{d\tau_1})^{-1}|_{\tau=\tau_1}\neq 0$ is true, that is, the transversal condition is true. Done.

According to the above analysis and *Hopf* bifurcation theory, we get the following theorem.

Theorem 3. For system (1), when $\tau_1>0,\ \ \tau_2\in[0,\tau_{20})$,assume that (H_2) are true.the following conclusions are true: when $\tau_1\in[0,\tau_{11})$, the equilibrium point E is asymptotically uniformly stable; when $\tau_1=\tau_{11}$, its positive equilibrium point is unstable; When $\tau_1=\tau_{11}$, it has *Hopf* branch at the positive equilibrium point.

IV. DIRECTION AND STABILITY OF THE HOPF BIFURCATION

In the analysis in the previous section, we have obtained the conditions for the system to generate *Hopf* bifurcation. In this section, we will use the normative theory and the central manifold theorem in literature [5-6] to give the calculation formula for the direction of Hopf bifurcation generated by the system (1) and the stability of the periodic solution of the bifurcation.

First we consider the Taylor expansion of model (1) at equilibrium, let $x(t)=Q(t)-Q_0, y(t)=S(t)-S_0$, so system (1) becomes:

$$\begin{cases}\dot{x}(t)=a_1x(t)+a_2x(t-\tau_1)+a_3x(t)x(t-\tau_1)\\ \dot{y}(t)=b_1y(t)+b_2y(t-\tau_2)+b_3y(t)y(t-\tau_2)\end{cases}$$

that is

$$\begin{cases}\dot{y}(t)=b_1y(t)+b_2y(t-\tau_2)+b_3y(t)y(t-\tau_2)\\ \dot{x}(t)=a_1x(t)+a_2x(t-\tau_1)+a_3x(t)x(t-\tau_1)\end{cases}\tag{15}$$

among them

$$a_1=-k_mv_mQ_0+v_m\eta(\omega-c)\,,\ a_2=-k_mv_mQ_0\,,\ a_3=-k_mv_m\,,$$

$$b_1=-k_rv_rS_0+v_r\theta(P-\omega)\,,\ b_2=-k_rv_rS_0\,,\ b_3=-k_rv_r\ .$$

Assuming $\tau_{11}>\tau_2^*\ (\tau_2^*\in[0,\tau_{20}))$, $\tau_1=\tau_{11}+\mu$, then $\mu=0$ represents that system (1) generates *Hopf* bifurcations at τ_{11}. Let $u_1(t)=x(t)-0,\ \ u_2(t)=y(t)-0$, linearize the time delay with scale $t\to\frac{t}{\tau_{11}}$, then system (1) is equivalent to the following functional differential equation form

$$\dot{u}(t) = L_\mu + F(u_t, \mu), \tag{16}$$

there

$$L_\mu \varphi = (\tau_{11} + \mu)(B_1\varphi(0) + B_2\varphi(-\frac{\tau_2^*}{\tau_{11}}) + B_3\varphi(-1)) \tag{17}$$

and

$$F(\varphi, \mu) = (\tau_{11} + \mu)\begin{pmatrix} b_3\varphi_2(0)\varphi_1(-\frac{\tau_2^*}{\tau_{11}}) \\ a_3\varphi_1(0)\varphi_1(-1) \end{pmatrix} \tag{18}$$

where L_μ is a bounded operator of $C'([-1,0], \tau^2) \to \tau^2$ and $\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta))^T \in C'[-1,0]$,

$$B_1 = \begin{pmatrix} 0 & b_1 \\ a_1 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 \\ a_2 & 0 \end{pmatrix}.$$

By the Riesz representation theorem, there exists a bounded variation function $\eta(\theta, \mu)$, $\theta \in [-1,0]$, such that

$$L_\mu \varphi = \int_{-1}^{0} d\eta(\theta, \mu)\varphi(\theta), \varphi \in C. \tag{19}$$

In fact, we can choose

$$\eta(\theta, \mu) = \begin{cases} (\tau_{11} + \mu)(B_1 + B_2 + B_3), & \theta = 0, \\ (\tau_{11} + \mu)(B_2 + B_3), & \theta \in [-\frac{\tau_2^*}{\tau_{11}}, 0), \\ (\tau_{11} + \mu)B_3, & \theta \in (-1, -\frac{\tau_2^*}{\tau_{11}}), \\ 0, & \theta = -1. \end{cases} \tag{20}$$

Here $\delta(\theta)$ is a Delta function. The operators A and R are defined as follows:

$$A(\mu)\varphi(\theta) = \begin{cases} \frac{d(\varphi(\theta))}{d\theta}, & \theta \in [-1,0), \\ \int_{-1}^{0} d(\eta(\theta, \mu)\varphi(\theta)), & \theta = 0. \end{cases} \tag{21}$$

$$R(\mu)\varphi(\theta) = \begin{cases} 0, & \theta \in [-1,0), \\ F(\mu, \varphi), & \theta = 0. \end{cases} \tag{22}$$

Then equation (16) can be rewritten into the following form:

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t. \tag{23}$$

Where $u_t = u(t + \theta)$, $\theta \in [-1,0)$.

For $\psi \in C'[0,1]$, we define the adjoint operator $A^*(0)$ of $A(0)$ as

$$A^*(\mu)\psi(s)=\begin{cases}-\dfrac{d\psi(s)}{ds}, & s\in(0,1],\\ \int_{-1}^{0}d(\eta^T(s,0)\psi(-s)), & s=0.\end{cases} \tag{24}$$

For $\varphi(\theta)\in C'[-1,0)$ and $\psi\in C'[0,1]$, we define the bilinear inner product

$$<\psi,\varphi>=\overline{\psi}^T(0)\varphi(0)-\int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}\overline{\psi}(\xi-\theta)[d\eta(\theta)]\varphi(\xi)d\xi. \tag{25}$$

where $\eta(\theta)=\eta(\theta,0)$.

Lemma 5. Let $\mu=0$ The eigenvectors $q(\theta)=Ve^{i\omega_{11}\tau_{11}\theta}$ and $q^*(s)=DV^*e^{i\omega_{11}\tau_{11}s}$ are respectively the eigenvectors corresponding to the eigenvalues $i\omega_{11}\tau_{11}$ and $-i\omega_{11}\tau_{11}$ of $A(0)$ and $A^*(0)$, and

$$<q^*,q>=1,\ <q^*,\overline{q}>=0,$$

where

$$V=(\rho_1,1)^T=(\frac{i\omega_{11}\tau_{11}}{a_1+a_2e^{-i\omega_{11}\tau_{11}}},1)^T,\quad V^*=(\rho_1^*,1)^T=(-\frac{i\omega_{11}\tau_{11}}{b_1+b_2e^{-i\omega_{11}\tau_2^*}},1)^T$$

$$\overline{D}=[1+\rho_1\overline{\rho}_1^*+\tau_{11}(a_1\rho_1+b_1\overline{\rho}_1^*)+\tau_{11}b_2\overline{\rho}_1^*e^{-i\omega_{11}\tau_2^*}+\tau_{11}a_2\rho_1e^{-i\omega_{11}\tau_{11}}]^{-1}$$

Proof. Since $i\omega_{11}\tau_{11}$ is the eigen value of $A(0)$, they are also eigen values of $A^*(0)$. In order to determine the standard form of the operator $A(0)$, let's assume that $q(\theta)$ and $q^*(s)$ are eigen vectors corresponding to $A(0)$ and $A^*(0)$'s eigen values $i\omega_{11}\tau_{11}$ and $-i\omega_{11}\tau_{11}$, respectively.

$$\begin{cases}A(0)q(\theta)=i\omega_{11}\tau_{11}q(\theta)\\ A^*(0)q^*(s)=-i\omega_{11}\tau_{11}q^*(s)\end{cases} \tag{26}$$

from (19) and (21), (26) can be written as

$$\begin{aligned}&\frac{dq(\theta)}{d\theta}=i\omega_{11}\tau_{11}q(\theta), \qquad \theta\in[-1,0).\\ &L(0)q(0)=-i\omega_{11}\tau_{11}q(0), \qquad \theta=0.\end{aligned} \tag{27}$$

Therefore,

$$q(\theta)=q(0)e^{i\omega_{11}\tau_{11}\theta}, \qquad \theta\in[-1,0].$$

$q(0)=(q_1(0),q_2(0))^T\in C^2$ is a constant vector, which can be obtained from (17) and (26)

$$(B_1+B_2e^{-i\omega_{11}\tau_2^*}+B_3e^{-i\omega_{11}\tau_{11}}]q(0)=i\omega_{11}\tau_{11}Iq(0)$$

we get

$$q(0)=\begin{pmatrix}\rho_1\\1\end{pmatrix}=\begin{pmatrix}\dfrac{i\omega_{11}\tau_{11}}{a_1+a_2e^{-i\omega_{11}\tau_{11}}}\\1\end{pmatrix}$$

we make

$$V=(\rho_1,1)^T=(\frac{i\omega_{11}\tau_{11}}{a_1+a_2e^{-i\omega_{11}\tau_{11}}},1)^T$$

then

$$q(\theta)=Ve^{i\omega_{11}\tau_{11}\theta}$$

for non-zero vectors $q^*(s), s\in[0,1]$, we have

$$(B_1{}^T+B_2{}^Te^{-i\omega_{11}\tau_2^*}+B_3{}^Te^{-i\omega_{11}\tau_{11}})q^*(0)=-i\omega_{11}\tau_{11}Iq^*(0).$$

Similarly

$$q^*(0)=\begin{pmatrix}\rho_1^*\\1\end{pmatrix}=\begin{pmatrix}-\dfrac{i\omega_{11}\tau_{11}}{b_1+b_2e^{-i\omega_{11}\tau_2^*}}\\1\end{pmatrix}.$$

We make

$$V^*=(\rho_1^*,1)^T=(-\frac{i\omega_{11}\tau_{11}}{b_1+b_2e^{-i\omega_{11}\tau_2^*}},1)^T.$$

So $q^*(s)=DV^*e^{-i\omega_0 s}$.

Now let's evaluate $\langle q^*,q\rangle$ and $\langle q^*,\bar{q}\rangle$, from equation (25), we get

$$\begin{aligned}\langle q^*,q\rangle&=\overline{q^*}^Tq(0)-\int_{\theta=-\tau_0}^{0}\int_{\xi=0}^{\theta}\overline{q^*}^T(\xi-\theta)d\eta(\theta)q(\xi)d\xi\\&=\bar{D}[\overline{V^*}^TV-\int_{\theta=-\tau_0}^{0}\int_{\xi=0}^{\theta}\overline{V^*}^Te^{-i\omega_0(\xi-\theta)}d\eta(\theta)Ve^{i\omega_0\xi}d\xi]\\&=\bar{D}[\overline{V^*}^TV-\int_{\theta=-\tau_0}^{0}\overline{V^*}^T[d\eta(\theta)]\theta e^{i\omega_0\theta}V]\\&=\bar{D}[\overline{V^*}^TV-\tau_0e^{-i\omega_0\tau_0}\overline{V^*}^TB_2V].\end{aligned}\tag{28}$$

So, let $\bar{D}=[1+\rho_1\overline{\rho_1^*}+\tau_{11}(a_1\rho_1+b_1\overline{\rho_1^*})+\tau_{11}b_2\overline{\rho_1^*}e^{-i\omega_{11}\tau_2^*}+\tau_{11}a_2\rho_1e^{-i\omega_{11}\tau_{11}}]^{-1}$, we can obtain $\langle q^*,q\rangle=1$.

Since $\langle\psi,A\varphi\rangle=\langle A^*\psi,\varphi\rangle$, we have

$$-i\omega_{11}\tau_{11}\langle q^*,\bar{q}\rangle=\langle q^*,A\bar{q}\rangle=\langle A^*q^*,\bar{q}\rangle=\langle -i\omega_{11}\tau_{11}q^*,\bar{q}\rangle=i\omega_{11}\tau_{11}\langle q^*,\bar{q}\rangle\tag{29}$$

therefore $\langle q^*,\bar{q}\rangle=0$, this completes the proof.

Next, we will use the method proposed by Hassard *et al.* to construct coordinates on the central epidemic C_0 at $\mu=0$. Define

$$z(t)=<q^*,u_t>,\tag{30}$$

and

$$W(t,\theta)=u_t(\theta)-2\operatorname{Re}\{z(t)q(\theta)\}. \tag{31}$$

On the center manifold C_0, we have

$$W(t,\theta)=W(z(t),\bar{z}(t),\theta). \tag{32}$$

Where $W(z,\bar{z},\theta)=W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\bar{z}+W_{02}(\theta)\frac{\bar{z}^2}{2}+\cdots$

For central manifold C_0, z and $\bar{z}$ represent the local coordinates of the central epidemic in the directions of q and $\overline{q^*}$ respectively. If u_t is real, then W is real, here we are dealing only with the real solution case, Since $\mu=0$, it is easy to see that

$$\begin{aligned}\dot{z}(t)&=\langle q^*,\dot{\mu}_t\rangle=\langle q^*,(A(0)+R(0))\mu_t\rangle\\&=<q^*,A\mu_t>+<q^*,R\mu_t>\\&=i\omega_{11}\tau_{11}z+\bar{q}^{*T}f_0(z,\bar{z})\end{aligned} \tag{33}$$

abbreviate (33) as follows

$$\dot{z}(t)=i\omega_{11}\tau_{11}z+g(z,\bar{z}) \tag{34}$$

where

$$g(z,\bar{z})=g_{20}\frac{z^2}{2}+g_{11}z\bar{z}+g_{02}\frac{\bar{z}^2}{2}+\cdots, \tag{35}$$

from (23) and (35), we have

$$\dot{W}=\dot{u}_t-\dot{z}q-\dot{\bar{z}}\dot{q}=\begin{cases}AW-2\operatorname{Re}\overline{q^*}^T(0)f_0(z,\bar{z})q(\theta), & \theta\in[-1,0)\\ AW-2\operatorname{Re}\{\overline{q^*}^T(0)f_0(z,\bar{z})q(\theta)\}+f_0(z,\bar{z}), & \theta=0\end{cases} \tag{36}$$

the above equation can be rewritten as

$$\dot{W}=AW+H(z,\bar{z},\theta). \tag{37}$$

where

$$H(z,\bar{z},\theta)=H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\bar{z}+H_{02}(\theta)\frac{\bar{z}^2}{2}+\cdots. \tag{38}$$

On the other hand, on the central manifold C_0, there is

$$\dot{W}=W_z\dot{z}+W_{\bar{z}}\dot{\bar{z}}. \tag{39}$$

Substitute Equations (33) and (35) for W_z and $\dot{z}$ into (39), respectively, we can get another expression of $\dot{W}$

$$\dot{W}=i\omega_{11}\tau_{11}W_{20}(\theta)z^2-i\omega_{11}\tau_{11}W_{02}(\theta)\bar{z}^2+\cdots \tag{40}$$

comparing the coefficients of the above equation with those of (38) and (41), we get

$$\begin{cases}(A-2i\omega_{11}\tau_{11}I)W_{20}(\theta)=-H_{20}(\theta)\\ AW_{11}(\theta)=-H_{11}(\theta)\\ (A+2i\omega_{11}\tau_{11}I)W_{02}(\theta)=-H_{02}(\theta)\end{cases} \tag{41}$$

notice that $u_t(\theta)=W(z(t),\bar{z}(t),\theta)+zq+\bar{z}\bar{q}$ and $q(\theta)=(1,\rho_1)^T e^{i\omega_{11}\tau_{11}\theta}$, we have

$$u_t(\theta)=\begin{pmatrix} W^{(1)}(z,\bar{z},\theta) \\ W^{(2)}(z,\bar{z},\theta) \end{pmatrix}+z\begin{pmatrix} \rho_1 \\ 1 \end{pmatrix}e^{i\omega_{11}\tau_{11}\theta}+\bar{z}\begin{pmatrix} \bar{\rho}_1 \\ 1 \end{pmatrix}e^{-i\omega_{11}\tau_{11}\theta} \tag{42}$$

therefore, we can obtain

$$y(t+\theta)=z\rho_1 e^{i\omega_{11}\tau_{11}\theta}+\bar{z}\overline{\rho_1}e^{-i\omega_{11}\tau_{11}\theta}+W_{20}{}^{(1)}(\theta)\frac{z^2}{2}+W_{11}{}^{(1)}(\theta)z\bar{z}+W_{02}{}^{(1)}(\theta)\frac{\bar{z}^2}{2}+\cdots$$

$$x(t+\theta)=ze^{i\omega_{11}\tau_{11}\theta}+\bar{z}e^{-i\omega_{11}\tau_{11}\theta}+W_{20}{}^{(2)}(\theta)\frac{z^2}{2}+W_{11}{}^{(2)}(\theta)z\bar{z}+W_{02}{}^{(2)}(\theta)\frac{\bar{z}^2}{2}+\cdots$$

obviously

$$\varphi_1(0)=z\rho_1+\bar{z}\overline{\rho_1}+W_{20}{}^{(1)}(0)\frac{z^2}{2}+W_{11}{}^{(1)}(0)z\bar{z}+W_{02}{}^{(1)}(0)\frac{\bar{z}^2}{2}+\cdots,$$

$$\varphi_1(-1)=z\rho_1 e^{-i\omega_{11}\tau_{11}}+\bar{z}\overline{\rho_1}e^{-i\omega_{11}\tau_{11}}+W_{20}{}^{(1)}(-1)\frac{z^2}{2}+W_{11}{}^{(1)}(-1)z\bar{z}+W_{02}{}^{(1)}(-1)\frac{\bar{z}^2}{2}+\cdots,$$

$$\varphi_2(0)=z+\bar{z}+W_{20}{}^{(2)}(0)\frac{z^2}{2}+W_{11}{}^{(2)}(0)z\bar{z}+W_{02}{}^{(2)}(0)\frac{\bar{z}^2}{2}+\cdots,$$

$$\varphi_2(-\frac{\tau_2^*}{\tau_{11}})=ze^{-i\omega_{11}\tau_2^*}+\bar{z}e^{-i\omega_{11}\tau_2^*}+W_{20}{}^{(2)}(-\frac{\tau_2^*}{\tau_{11}})\frac{z^2}{2}+W_{11}{}^{(2)}(-\frac{\tau_2^*}{\tau_{11}})z\bar{z}+W_{02}{}^{(2)}(-\frac{\tau_2^*}{\tau_{11}})\frac{\bar{z}^2}{2}+\cdots.$$

From (18), we obtain

$$f_0(z,\bar{z})=\tau_{11}\begin{pmatrix} K_1z^2+K_2z\bar{z}+K_3\bar{z}^2+K_4z^2\bar{z} \\ K_5z^2+K_6z\bar{z}+K_7\bar{z}^2+K_8z^2\bar{z} \end{pmatrix}$$

where

$K_1=a_3\rho_1{}^2e^{-i\omega_{11}\tau_{11}}$, $K_2=a_3\rho_1\bar{\rho}_1(e^{i\omega_{11}\tau_{11}}+e^{-i\omega_{11}\tau_{11}})$, $K_3=a_3\bar{\rho}_1{}^2e^{i\omega_{11}\tau_{11}}$,

$K_4=a_3[\rho_1W_{11}{}^{(1)}(-1)+\frac{1}{2}\bar{\rho}_1W_{20}{}^{(1)}(-1)+\frac{1}{2}\bar{\rho}_1e^{i\omega_{11}\tau_{11}}W_{20}{}^{(1)}(0)+\rho_1e^{-i\omega_{11}\tau_{11}}W_{11}{}^{(1)}(0)]$,

$K_5=b_3e^{-i\omega_{11}\tau_2^*}$, $K_6=b_3(e^{-i\omega_{11}\tau_2^*}+e^{-i\omega_{11}\tau_2^*})$, $K_7=b_3e^{i\omega_{11}\tau_2^*}$,

$K_8=b_3[W_{11}{}^{(2)}(-\frac{\tau_2^*}{\tau_{11}})+\frac{1}{2}W_{20}{}^{(2)}(--\frac{\tau_2^*}{\tau_{11}})+\frac{1}{2}e^{i\omega_{11}\tau_2^*}W_{20}{}^{(2)}(0)+e^{-i\omega_{11}\tau_2^*}W_{11}{}^{(2)}(0)]$.

Since $\bar{q}^{*T}(0)=\bar{D}(\bar{\rho}_1{}^*,1)$, we obtain

$$\begin{aligned} g(z,\bar{z})&=\bar{q}^{*T}(0)f_0(z,\bar{z}) \\ &=\tau_{11}\bar{D}(\bar{\rho}_1{}^*,1)\begin{pmatrix} K_1z^2+K_2z\bar{z}+K_3\bar{z}^2+K_4z^2\bar{z} \\ K_5z^2+K_6z\bar{z}+K_7\bar{z}^2+K_8z^2\bar{z} \end{pmatrix} \\ &=\tau_{11}\bar{D}[(\bar{\rho}_1{}^*K_1+K_5)z^2+(\bar{\rho}_1{}^*K_2+K_6)z\bar{z}+(\bar{\rho}_1{}^*K_3+K_7)\bar{z}^2+(\bar{\rho}_1{}^*K_4+K_8)z^2\bar{z}] \end{aligned}$$

Comparing the coefficients of the above equation with those in (35), we have

$$\begin{aligned} &g_{20}=2\tau_{11}\bar{D}(\bar{\rho}_1{}^*K_1+K_5),\quad g_{11}=\tau_{11}\bar{D}(\bar{\rho}_1{}^*K_2+K_6), \\ &g_{02}=2\tau_{11}\bar{D}(\bar{\rho}_1{}^*K_3+K_7),\quad g_{21}=2\tau_{11}\bar{D}(\bar{\rho}_1{}^*K_4+K_8). \end{aligned} \tag{43}$$

In order to get the value of g_{21}, we also need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. For $\theta \in [-1,0)$, we have

$$\begin{aligned} H(z,\bar{z},\theta) &= -2\,\mathrm{Re}[\bar{q}^{*T}(0) f_0(z,\bar{z}) q(\theta)] \\ &= -(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots) q(\theta) \\ &\quad -(\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{z^2}{2} + \cdots)\bar{q}(\theta). \end{aligned} \tag{44}$$

Comparing the coefficients of the above equation with those in (38), we have

$$\begin{aligned} H_{20}(\theta) &= -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \\ H_{11}(\theta) &= -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \end{aligned} \tag{45}$$

When $\theta = 0$, we have

$$\begin{aligned} H(z,\bar{z},0) &= -2\,\mathrm{Re}[\bar{q}^{*T}(0) f_0(z,\bar{z}) q(0)] + f_0(z,\bar{z}) \\ &= -(g_{20}(0)\frac{z^2}{2} + g_{11}(0)z\bar{z} + g_{02}(0)\frac{\bar{z}^2}{2} + \cdots) q(0) \\ &\quad -(\bar{g}_{20}(0)\frac{\bar{z}^2}{2} + \bar{g}_{11}(0)z\bar{z} + \bar{g}_{02}(0)\frac{z^2}{2} + \cdots)\bar{q}(0) + \tau_{11}\begin{pmatrix} K_1 z^2 + K_2 z\bar{z} + K_3\bar{z}^2 + K_4 z^2\bar{z} \\ K_5 z^2 + K_6 z\bar{z} + K_7\bar{z}^2 + K_8 z^2\bar{z} \end{pmatrix}. \end{aligned}$$

Comparing the coefficients with (38), we have

$$\begin{aligned} H_{20}(0) &= -g_{20}q(0) - \bar{g}_{20}\bar{q}(0) + 2\tau_{11}\begin{pmatrix} K_1 \\ K_5 \end{pmatrix} \\ H_{11}(0) &= -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \tau_{11}\begin{pmatrix} K_2 \\ K_6 \end{pmatrix} \end{aligned} \tag{46}$$

using (41) and (45), we obtain

$$\begin{aligned} W_{20}(\theta) &= \frac{ig_{20}}{\omega_{11}\tau_{11}} q(0) e^{i\omega_{11}\tau_{11}\theta} + \frac{i\bar{g}_{02}}{3\omega_{11}\tau_{11}} \bar{q}(0) e^{-i\omega_{11}\tau_{11}\theta} + E_1 e^{2i\omega_{11}\tau_{11}\theta}, \\ W_{11}(\theta) &= -\frac{ig_{11}}{\omega_{11}\tau_{11}} q(0) e^{i\omega_{11}\tau_{11}\theta} + \frac{i\bar{g}_{11}}{\omega_{11}\tau_{11}} \bar{q}(0) e^{-i\omega_{11}\tau_{11}\theta} + E_2, \end{aligned} \tag{47}$$

where $E_1 = (E_1^{(1)}, E_1^2) \in R^2, E_2 = (E_2^{(1)}, E_2^2) \in R^2$ are two two-dimensional vectors.

According to the definition of $A(0)$ and formula (41), we have

$$\begin{aligned} \int_{-1}^{0} d\eta(\theta) W_{20}(\theta) &= 2i\omega_{11}\tau_{11} W_{20}(0) - H_{20}(0), \\ \int_{-1}^{0} d\eta(\theta) W_{11}(\theta) &= -H_{11}(0), \end{aligned}$$

and

$$\begin{aligned} (i\omega_{11}\tau_{11} I - \int_{-1}^{0} e^{i\omega_{11}\tau_{11}\theta} d\eta(\theta)) q(0) &= 0, \\ (-i\omega_{11}\tau_{11} I - \int_{-1}^{0} e^{-i\omega_{11}\tau_{11}\theta} d\eta(\theta)) \bar{q}(0) &= 0, \end{aligned}$$

hence, we can get

$$(2i\omega_{11}\tau_{11}I-\int_{-1}^{0}e^{2i\omega_{11}\tau_{11}\theta}d\eta(\theta))E_1=2\tau_{11}\begin{pmatrix}K_1\\K_5\end{pmatrix},$$

$$(\int_{-1}^{0}d\eta(\theta))E_2=-\tau_{11}\begin{pmatrix}K_2\\K_6\end{pmatrix},$$

therefore, we have

$$\begin{cases}\begin{pmatrix}i2\omega_{11} & -b_1-b_2e^{-2i\omega_{11}\tau_{11}}\\ -a_1-a_2e^{-2i\omega_{11}\tau_{11}} & 2i\omega_{11}\end{pmatrix}E_1=2\tau_{11}\begin{pmatrix}K_1\\K_5\end{pmatrix}\\ \begin{pmatrix}0 & b_1+b_2\\ a_1+a_1 & 0\end{pmatrix}E_2=-\tau_{11}\begin{pmatrix}K_2\\0\end{pmatrix}\end{cases} \tag{48}$$

by calculation, we have

$$E_1=2\tau_{11}\begin{pmatrix}i2\omega_{11} & -b_1-b_2e^{-2i\omega_{11}\tau_{11}}\\ -a_1-a_2e^{-2i\omega_{11}\tau_{11}} & 2i\omega_{11}\end{pmatrix}^{-1}\begin{pmatrix}K_1\\K_5\end{pmatrix} \tag{49}$$

and

$$E_2=-\tau_{11}\begin{pmatrix}0 & b_1+b_2\\ a_1+a_1 & 0\end{pmatrix}^{-1}\begin{pmatrix}K_2\\K_6\end{pmatrix} \tag{50}$$

Based on the above analysis, we next determine several important values of the properties of Hopf periodic solutions at the critical value τ_{11}:

$$\begin{aligned}&C_1(0)=\frac{i}{2\omega_{11}\tau_{11}}(g_{20}g_{11}-2|g_{11}|^2-\frac{1}{3}|g_{02}|^2)+\frac{g_{21}}{2},\\&\mu_2=-\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_{11})\}},\\&\beta_2=2\,\mathrm{Re}\{C_1(0)\},\\&T_2=-\frac{\mathrm{Im}\{C_1(0)\}+\mu_2(\mathrm{Im}\{\lambda'(\tau_{11})\})}{\omega_{11}\tau_{11}}\end{aligned} \tag{51}$$

Theorem 4. In the case of system (1), the conclusion holds:

(1) The direction of the Hopf bifurcation is determined by the parameter μ_2. If $\mu_2>0$ ($\mu_2<0$), the Hopf bifurcation is supercritical (subcritical) .

(2) The value of β_2 determines the stability of Hopf bifurcation periodic solution. If $\beta_2<0$ ($\beta_2>0$), then the branching periodic solution is asymptotically stable (unstable).

(3) The value of T_2 determines the period of the Hopf bifurcation periodic solution. If $T_2>0$ $(T_2<0)$, then the period of the periodic solution increases (decreases).

V. NUMERICAL SIMULATION

In this section, we verified the validity of the above theoretical analysis results by using mathematica, a mathematical software, for numerical simulation.

When $\tau_1=0$, $\tau_2>0$, first we select the parameter: $\eta=0.5$, $\theta=0.8$, $k_m=0.7$, $k_r=0.8$, $P=3$, $\omega=2$, $c=1$, $v_x=v_r=0.5$. Through calculation: $Q_0\approx0.714$, $S_0=1$, $\tau_{20}=4.90874$, $\omega_{20}=0.32$, and these coefficients satisfy (H_1). If $\tau_2=4<\tau_{20}$ is taken, system (1) is asymptotically stable at the equilibrium point, as shown in Fig. 1.

Take $\tau_2 = 4.90874 = \tau_{20}$, system (1) generates *Hopf* bifurcation at the equilibrium point, as shown in Fig. 2; As the value of τ_2 increases, system (1) is unstable at the equilibrium point.

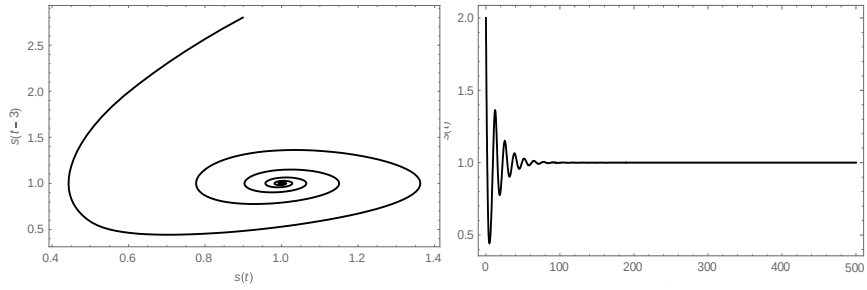

Fig. 1 the equilibrium point is asymptotically stable with $\tau_2 = 4$

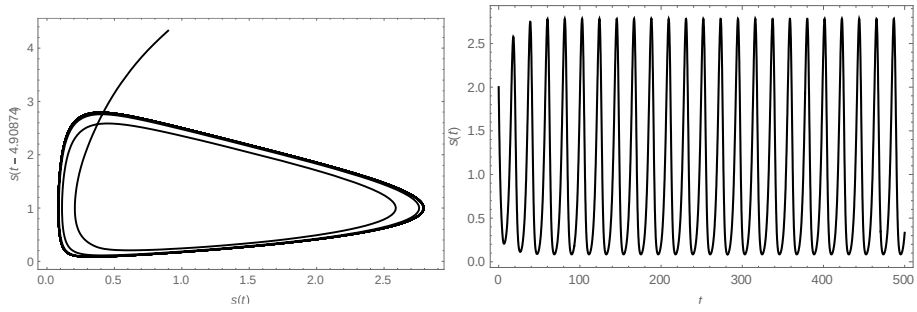

Fig. 2 an unstable periodic solution appears at $\tau_2 = 4.90874$

When $\tau_1 = \tau_2 = \tau$, first we select the parameter: $\eta = 0.4$, $\theta = 0.4$, $k_m = 2$, $k_r = 2$, $P = 3$, $\omega = 2$, $c = 1$, $v_x = v_r = 0.5$. Through calculation: $Q_0 \approx 0.2$, $S_0 = 0.2$, $\tau_0 = 7.314$, $\omega_0 = 0.04$. If $\tau = 7 < \tau_0$ is taken, system (1) is asymptotically stable at the equilibrium point, as shown in Fig. 3.Take $\tau = 9 > \tau_0$, system (1) generates *Hopf* bifurcation at the equilibrium point, as shown in Fig. 4; As the value of τ increases, system (1) is unstable at the equilibrium point.

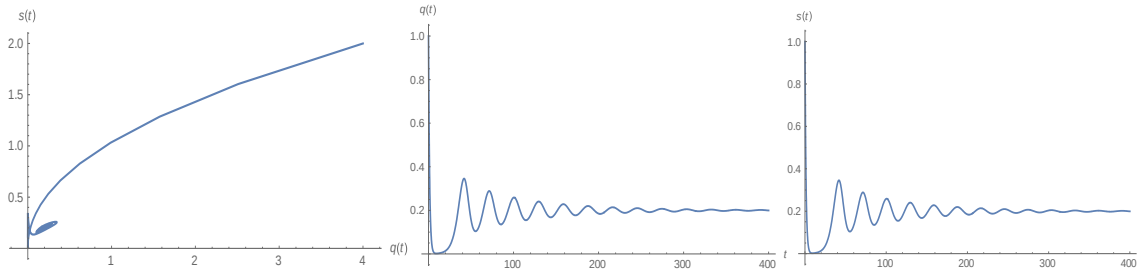

Fig. 3 the equilibrium point is asymptotically stable with $\tau = 7$

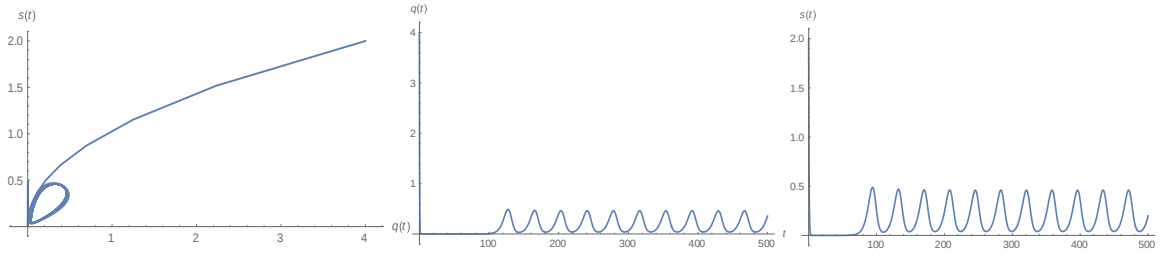

Fig. 4 an unstable periodic solution appears at $\tau = 9$

When $\tau_1 > 0$, $\tau_2 > 0$, first we select the parameter: $\eta = 0.5$, $\theta = 0.8$, $k_m = 0.7$, $k_r = 0.8$, $P = 3$, $\omega = 2$, $c = 1$, $v_x = v_r = 0.5$. Through calculation: $Q_0 \approx 0.714$, $S_0 = 1$, $\tau_{11} \approx 6.5$, and these coefficients satisfy (H_2). If $\tau_1 = 5 < \tau_{11}$ is taken, system (1) is asymptotically stable at the equilibrium point, as shown in Fig. 5.Take $\tau_1 = 7 > \tau_{11}$, system (1) generates *Hopf* bifurcation at the equilibrium point, as shown in Fig. 6; As the value of τ_1 increases, system (1) is unstable at the equilibrium point.

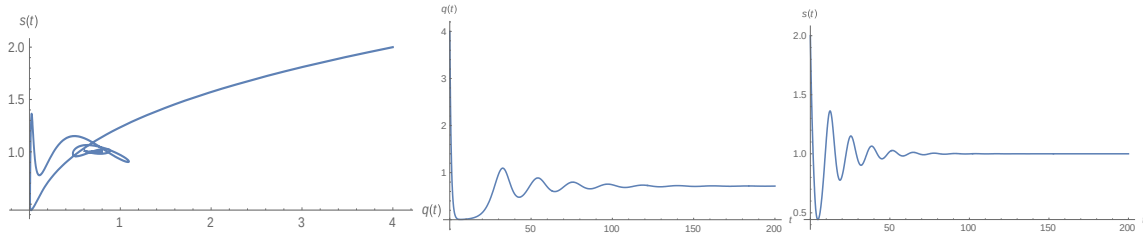

Fig. 5 the equilibrium point is asymptotically stable with $\tau_1 = 5$

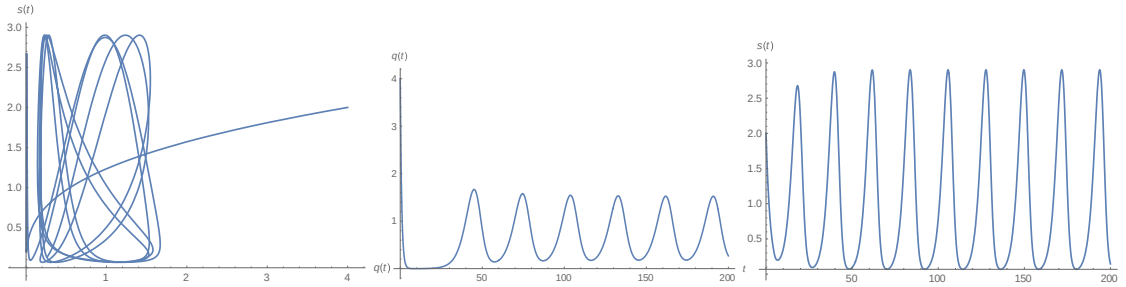

Fig. 6 an unstable periodic solution appears at $\tau_1 = 7$

VI. CONCLUSIONS

Based on the dynamic time-delay model of double quality level, this paper studies the dynamic model of quality adjustment of supply chain with double time-delay, analyzes the stability of the system and the existence of *Hopf* branch by the eigenvalue method, and obtains the conditions of system stability. The research shows that, when $\tau_1 = 0$, $\tau_1 = \tau_2$, $\tau_1 \neq 0$, respectively, and each condition is satisfied, the relevant conclusions about the existence of unique positive equilibrium point and the stability of the system and the existence conditions of Hopf bifurcation are obtained. Then the stability of bifurcated periodic solutions and the bifurcated direction of *Hopf* bifurcated solutions are analyzed by using the central manifold theorem and the gauge method. Finally, mathematical software is used to verify the correctness of the results. Therefore, it can be known that the time delay parameters in the model should not be too large, otherwise the stability will be lost. That is to say, when referring to past information, although there is no necessary connection between the manufacturer and retailer in the selection of the long time of delay information, neither manufacturer nor retailer should choose the information that is too long, otherwise both parties will have adverse effects.

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