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Titchmarsh Theorem and its Generalization for the Bessel type transform

Balasaheb Bhagaji Waphare[1],Yashodha Sanjay Sindhe[2]
[1]*Professor,*[2]*Research Scholar, Mathematics Department, MAEER's MIT Arts, Commerce and Science College, Alandi, Pune-412105,Maharashtra, India*

Abstract-In this paper we obtain a generalization of Titchmarsh's Theorem for the Bessel type transform for functions satisfying the $\psi-$ Bessel type Lipschitz condition in $L_{z,a,b}(\mathbb{R})$ by using a generalized translation operator.

Keywords-Bessel type operator, Bessel type transform, generalized translation operator, Bessel type function.
Mathematics subject classification:42A38,42B10.

I Introduction and Preliminaries

In past and recent years Bessel transform is used in engineering, mechanics, Physics, Computational Mathematics etc.
Inspired by Hamma & Daher[3], we obtain generalization of Titchmarsh's theorem for the Bessel type transform. In this paper Titchmarsh[7, Theorem 85] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipshitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms. We have

Theorem1.1: Let $\alpha \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalent.
(i) $\|f(t+h) - f(t)\|_{L^2(\mathbb{R})} = O(h^\alpha)$ as $h \to 0$
(ii)$\int_{|\lambda|\geq r} |g(\lambda)|^2 d\lambda = O(r^{-2\alpha})$ as $r \to \infty$,
where g stands for the Fourier transform of f. Our main objective in this paper is to obtain a generalization of Theorem 1.1 for the Bessel type operator. Let $B_{a,b} = D^2{}_x + \frac{a-b}{x} D_x, D_x \equiv \frac{d}{dx}$, be the Bessel type differential operator.
Now, for $(a-b) \geq 0$, we introduce the Bessel type normalized function of the first kind $j_{\frac{a-b-1}{2}}$ defined by

$$j_{\frac{a-b-1}{2}} = \Gamma(\frac{a-b+1}{2}) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\frac{2n+a-b+1}{2})} (\frac{x}{2})^{2n} \tag{1.1}$$

Where $\Gamma(x)$ is the Gamma function (see[5])
From (1.1), it is easily deduced that

$$\lim_{x\to 0} \frac{j_{\frac{a-b-1}{2}}(x) - 1}{x^2} \neq 0$$

by consequence, there exist $c > 0$ and $\eta > 0$ satisfying

$$|x| \leq \eta \Rightarrow |j_{\frac{a-b-1}{2}}(x) - 1| \geq c|x|^2 \tag{1.2}$$

The function $y = j_{\frac{a-b-1}{2}(x)}$ satisfies the differential equation

$$B_{a,b}(y) + y = 0$$

with the initial conditions that $y(0) = 1$ and $y'(0) = 0, j_{\frac{a-b-1}{2}}(x)$ is function infinitely differentiable, even and moreover entire analytic.

Lemma1.1: The following inequalities are valid for the Bessel type function $j_{\frac{a-b-1}{2}}$:

(i)$|j_{\frac{a-b-1}{2}}(x)| \leq C$, for all $x \in \mathbb{R}^+$, where C is positive constant.

(ii)$1 - j_{\frac{a-b-1}{2}}(x) = O(x^2), 0 \leq x \leq 1$

Proof. Proof is clear from [1]

Let $L_{2,a,b}(\mathbb{R}^+), (a, b) \geq 0$ be the Hilbert space of measurable functions f(x)on $\mathbb{R}^+$ with the finite norm

$$\|f\|_{2,a,b} = (\int_0^\infty |f(x)|^2 x^{a-b} dx)^{1/2}$$

. The generalized Bessel type translation T_h defined by

$$T_h f(t) = c_{a,b} \int_0^\pi f(\sqrt{t^2 + h^2 - 2th\cos\varphi}) \sin^{a-b-1}\varphi \ \ d\varphi.$$

where

$$c_{a,b} = (\int_0^\pi \sin^{a-b-1}\varphi \ \ d\varphi)^{-1} = \frac{\Gamma(\frac{a-b+1}{2})}{\sqrt{\pi}\Gamma(\frac{a-b}{2})}.$$

The Bessel type transform is defined by (see[4,5,6])

$$\hat{f}(\lambda) = \int_0^\infty f(t) j_{\frac{a-b-1}{2}}(\lambda t) t^{a-b} dt, \lambda \in \mathbb{R}^+.$$

The inverse Bessel type transform is given by the formula

$$f(t) = (2^{\frac{a-b+1}{2}}\Gamma(\frac{a-b+1}{2}))^{-2} \int_0^\infty \hat{f}(\lambda) j_{\frac{a-b-1}{2}}(\lambda t) \lambda^{a-b} d\lambda,$$

that is the direct and inverse Bessel type transform differ by the factor$(2^{\frac{a-b+1}{2}}\Gamma(\frac{a-b+1}{2}))^{-2}$
The connection between the Bessel type generalized translation and the Bessel type transform in [2] is given by

$$\hat{T_h} f(\lambda) = j_{\frac{a-b-1}{2}}(\lambda h) \hat{f}(\lambda). \tag{1.3}$$

□

II Main result

In this section we prove the main result of this paper. First we need to define $\psi-$ Bessel type Lipschitz class.

Definition 2.1: A function $f \in L_{2,a,b}(\mathbb{R}^+)$ is said to be in the $\psi-$ Bessel type Lipschitz class, denoted by Lip$(\psi, a, b, 2)$, if

$$\|T_h f(t) - f(t)\|_{2,a,b} = O(\psi(h)), as \quad h \to 0,$$

Where $\psi(t)$ is a continuous increasing function on $[0,\infty), \psi(0) = 0$ and $\psi(ts) = \psi(t)\psi(s)$ for all $t, s \in [0,\infty)$and this function verify $\int_0^{1/h} s\psi(s^{-2})ds = O(\frac{1}{h^2}\psi(h^2))$ as $h \to 0$
Theorem 2.1: Let $f \in L_{2,a,b}(\mathbb{R}^+)$ then the following are equivalents:
(i)$f \in Lip(\psi, a, b, 2)$
(ii)$\int_r^\infty |\hat{f}(\lambda)|^2\lambda^{a-b}\, d\lambda = O(\psi(r^{-2}))$ as $r \to \infty$.

Proof. (i)⇒(ii): Suppose that $f \in Lip(\psi, a, b, 2)$. Then we obtain

$$\|T_h f(t) - f(t)\|_{2,a,b} = O(\psi(h)), ash \to 0$$

By using (1.3) and Parseval's identity, we obtain

$$\|T_h f(t) - f(t)\|^2{}_{2,a,b} = \frac{1}{(2^{\frac{a-b-1}{2}}\Gamma(\frac{a-b+1}{2}))^2}\int_0^\infty |1 - j_{\frac{a-b-1}{2}}(\lambda h)|^2|\hat{f}(\lambda)|^2\lambda^{a-b}d\lambda$$

From (1.2), we have

$$\int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} |1 - j_{\frac{a-b-1}{2}}(\lambda h)|^2|\hat{f}(\lambda)|^2\lambda^{a-b}d\lambda \geq \frac{c^2\eta^4}{16}\int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} |\hat{f}(\lambda)|^2\lambda^{a-b}d\lambda$$

We can deduce that

$$\int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} |1 - j_{\frac{a-b-1}{2}}(\lambda h)|^2|\hat{f}(\lambda)|^2\lambda^{a-b}d\lambda \leq \int_0^\infty |1 - j_{\frac{a-b-1}{2}}(\lambda h)|^2|\hat{f}(\lambda)|^2\lambda^{a-b}d\lambda$$

There exists a positive C_2 such that

$$\int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} |\hat{f}(\lambda)|^2\lambda^{a-b}d\lambda \leq C_2\psi(h^2)$$

Now we obtain

$$\int_r^{2r} |\hat{f}(\lambda)|^2\lambda^{a-b}d\lambda \leq C_2\psi(2^{-2}\eta^2 r^{-2})$$

Now there exists a positive constant K such that

$$\int_r^{2r} |\hat{f}(\lambda)|^2\lambda^{a-b}d\lambda \leq K\psi(r^{-2}),\ for\ all\ r > 0$$

Thus

$$\begin{aligned}\int_r^\infty |\hat{f}(\lambda)|^2\lambda^{a-b}d\lambda &= [\int_r^{2r} + \int_{2r}^{4r} + \int_{4r}^{8r} + \cdots]|\hat{f}(\lambda)|^2\lambda^{a-b}d\lambda \\ &= O(\psi(r^{-2}) + \psi(2^{-2}r^{-2}) + \cdots) \\ &= O(\psi(r^{-2}) + \psi(r^{-2}) + \cdots) \\ &= O(\psi(r^{-2}))\end{aligned}$$

This proves that

$$\int_r^{\infty} |\hat{f}(\lambda)|^2 \lambda^{a-b} d\lambda == O(\psi(r^{-2})), as\ r \to \infty.$$

now we prove (ii)⇒(i)

$$Let \int_r^{\infty} |\hat{f}(\lambda)|^2 \lambda^{a-b} d\lambda = O(\psi(r^{-2})), as\ r \to \infty.$$

we write

$$\int_0^{\infty} |1 - j_{\frac{a-b-1}{2}}(\lambda h)|^2 |\hat{f}(\lambda)|^2 \lambda^{a-b} d\lambda = I_1 + I_2,$$

$$I_1 = \int_0^{1/h} |1 - j_{\frac{a-b-1}{2}}(\lambda h)|^2 |\hat{f}(\lambda)|^2 \lambda^{a-b} d\lambda,$$

and

$$I_2 = \int_{1/h}^{\infty} |1 - j_{\frac{a-b-1}{2}}(\lambda h)|^2 |\hat{f}(\lambda)|^2 \lambda^{a-b} d\lambda.$$

Estimate the summands I_1 and I_2
Firstly we have from (1.1)in Lemma 1.2

$$I_2 \leq (1+c)^2 \int_{1/h}^{\infty} |\hat{f}(\lambda)|^2 \lambda^{a-b} d\lambda = O(\psi(h^2))$$

Now set

$$\phi(x) = \int_x^{\infty} |\hat{f}(\lambda)|^2 \lambda^{a-b} d\lambda.$$

From Lemma 1.2 we have that

$$|1 - j_{\frac{a-b-1}{2}}(\lambda h)| \leq C_1 \lambda^2 h^2 for\ \lambda\ h \leq 1.$$

Then $I_1 \leq -C_1 h^2 \int_0^{1/h} x^2 \phi'(x) dx$
Integration by parts gives

$$\begin{aligned} I_1 &\leq -C_1 h^2 \int_0^{1/h} x^2 \varphi'(x) dx \\ &\leq C_1 \phi\left(\frac{1}{h}\right) + 2C_1 h^2 \int_0^{1/h} x\phi(x) dx \\ &\leq C_3 h^2 \int_0^{1/h} x\psi(x^{-2}) dx \\ &\leq C_3 h^2 \frac{1}{h^2} \psi(h^2) \\ &\leq C_3 \psi(h^2) \end{aligned}$$

where C_3 is a positive constant and thus proof is completed □

Remarks:(i)If we take $a = p + \frac{3}{4}, b = -p - \frac{1}{4}$ throughout this paper, we obtain the results studied in [3].
(ii)Author claims that the results studied in this paper are general than that of [3].

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