Banach Contraction Method and Tanh-coth Approach for the Solitary and Exact Solutions of Burger-Huxley and Kuramoto-Sivashinsky Equations

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Abstract: In this paper, the Tanh-coth and Banach contraction methods are proposed to solve the Burgers-Huxley and Kuramoto-Sivashinsky equations. The equations under study were first transformed into ordinary differential equations using specialized wave transformations as in Tanh-coth where solitary solutions were obtained, whereas the Banach contraction method gives an analytical solution after a finite number of iterations depending on the parameters. The result obtained showed, the methods are easy to implement, computationally less time consuming, accurate, reliable, promising and efficient.

Keywords: Burgers-Huxley, Kuramoto-Sivashinsky, Tanh-coth, Banach contraction method, Convergence of the Methods, Solitary, Kink, solitons

I. INTRODUCTION

Most of the nonlinear phenomena encountered in diverse fields of human endeavour like in Chemistry, Mathematics, Physics, Biology, Engineering and medicine are modelled with partial differential equations [1]. Intrigued by the inherent nonlinearity in these equations, scholars in the last decade and more, have devoted tremendous efforts to extensively study these nonlinear equations, using various advanced mathematical methods ranging from direct, exact, semi-analytical, especially in mathematics, physics, chemistry and engineering [2-4]. Some of these innovative and fascinating methods includes: Homotopy analysis method [5], Variational iteration method [6], Hirota Bilinear form [7], 8], Painleve analysis [9], Similarity transformation [10], Miura transformation [11], (G''/G) expansion method [12], Homogenous method [13], Tanh-coth method [14-19].

Of the plethora of nonlinear partial differential equations, the Burgers-Huxley equation is increasingly finding useful applications in different fields. It is used to model the interaction between reaction mechanism, convection effects, diffusion transport, motion in liquid crystals and nerve pulse propagation in nerves fibres [20]. Several methods have been used to find the exact solutions to this equation in various forms. The generalized Burgers-Huxley equation has been analysed for its exact solution using the Tanh-coth [21-22] have examined the generalized Burgers-Huxley equation using Adomian decom position method. The result showed there is agreement between those in literature and exact solution. The solitary wave solutions of the generalized Burgers-Huxley equation [23]. The spectral collocation method cum the preconditioning to solve the Burgers-Huxley equation is investigated by [24].

Equally, the Kuramoto-Shivashinsky equation was proposed by Gregory Shivashinsky and Yoshida Kuramoto. The former when he was studying laminar flame front that exhibit turbulence [25-26] and the later when he was studying diffusion-induced chaos in reaction systems [27]. They both discovered it independently and presented the equation in normalized form [28]. It is used in the modelling of plasma instabilities or turbulence in a reaction diffusion system in chemistry, propagation of flame front, viscous flow problems and spatially uniform oscillating chemical reaction in a homogenous equation has been given considerable attention [29-30]. It's a model prototype of a system which self-generates and sustain chaos in a large class of Burgers equation [31]. Many authors have used various mathematical methods to effectively solved this equation.[32] analyzed the Kuramoto-Shivashinsky equation using the Lattice Boltzmann method. The Chebyshev spectral collocation scheme was developed by [33] to study the equation for exact solutions. [34] explored the tanh function method to systematically examine the KSE equation. The result obtained showed promise and agree with literature. The local discontinuity Galerkin method was proposed by [35] to seek exact solutions. [36] investigated the KSE equation using a technique based on variational iteration

method. [37] studied the KSE equation for both solitary and exact solution using the tanh-coth method. [38-39] examined the equation in detail for the exact, solitary, kink and soliton solutions.

Our aim in this research is to implement the Banach contraction and Tanh-coth methods to the Burger-Huxley and Kuramoto-Shivashinsky equations in obtaining the solitary wave and exact solutions and show its capability and efficiency in handling highly nonlinear partial differential equations that arises in physics, Mathematics and engineering. Equally, comparative analysis is made between the methods to know which produces the most convergent solutions.

II. TANH-COTH METHOD

The basics of the Tanh-coth method is outlined in the following steps

Step 1. Consider a nonlinear partial differential equation of the form

$$P(v, v_x, v_t, v_{xx}, v_{tt}, v_{xxx} \dots) = 0$$
⁽¹⁾

Let v(x, t) be the solution of Eq. (1)

Step 2. Using the special wave transformation

$$v(x,t) = f(z) = f(x - ct)$$
 (2)

Where z = x - ct and f(z) is the localized wave solution

Applying elementary laws of calculus on Eq. (2), we have the derivatives w.r.t. x and t as follows

$$\frac{dv}{dt} = -fc\frac{d}{dz}$$
$$\frac{d}{dx} = f\frac{d}{dz}$$
$$\frac{d^2}{dx^2} = f^2\frac{d^2}{dz^2}$$

Similarly, other higher derivatives include

$$\frac{d^3}{dx^2} = f^3 \frac{d^3}{dz^3}$$

The wave variable, z = x - ct transforms the partial differential equation in Eq. (1) to an ordinary differential equation of the form

Step 3.
$$Q(v, v', v'', v''', ...) = 0$$
 (3)

The ODE in step 3 is then integrated provided all terms contain derivatives, while the integration constants are taken as zero with respect to the localized solution

Step 4. We represent all derivatives and tan h by tan h itself as follows

Let
$$F = tanh(z)$$
 (4)

The successive differential coefficients of Eq. (4) become

$$F' = \operatorname{sech}^{2}(z) = 1 - \tanh^{2}(z)$$

$$F' = 1 - F^{2}$$

$$F'' = -2 \tanh(z) \operatorname{sech}^{2}(z)$$

$$F'' = -2 \tanh(1 - \tanh^{2}(z))$$

$$F'' = -2F + 2F^{3}$$

$$F''' = -2 + 8F^{2} - 6F^{4}$$

$$F^{(iv)} = 16F - 40F^{3} + 24F^{5}$$

Using step 4, we introduce a new independent variable of the form

$$Y = \tanh(\mu z), \quad z = x - ct \tag{5}$$

where μ is the wave number, then the corresponding derivatives are in the form

$$\frac{d}{dz} = \mu(1 - Y^2)\frac{d}{dY}$$
$$\frac{d^2}{dz^2} = -2\mu^2 Y(1 - Y^2)\frac{d}{dY} + \mu^2(1 - Y^2)\frac{d^2}{dz^2}$$

$$\frac{d^3}{dz^3} = 2\mu^3 (1 - Y^2)(3Y^2 - 1)\frac{d}{dY} - 6\mu^3 Y(1 - Y^2)^2 \frac{d^2}{dz^2} + \mu^3 (1 - Y^2)^3 \frac{d^3}{dz^3}$$
$$\frac{d^4}{dz^4} = -8\mu^4 Y(1 - Y^2)(3Y^2 - 2)\frac{d}{dY} + 4\mu^4 (1 - Y^2)^2 (9Y^2 - 2)\frac{d^2}{dz^2} - 12\mu^4 Y(1 - Y^2)^3 \frac{d^3}{dz^3} + \mu^4 (1 - Y^2)^4 \frac{d^4}{dY^4}$$

Step 5. Using a finite expansion of the form

$$v(x,t) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}$$
(6)

Step 6. Using exponential laws on the solution, we have the following

 $v' \rightarrow M + 1$ $v'' \rightarrow M + 2$ $v''' \rightarrow M + 3$ $v^{(r)} \rightarrow M + r$ $V \rightarrow M$ $V^{2} \rightarrow 2M$ $V^{3} \rightarrow 3M$

$$V^{(r)} \rightarrow rM$$
$$(V')^2 \rightarrow (M+1)^2$$
$$(V'')^2 \rightarrow (M+2)^2$$
$$(V^r)^2 \rightarrow (M+r)^2$$
$$(V')^r \rightarrow (M+1)^r$$

Where M is a positive integer determined from a homogenous balance method so that a closed form analytical solution can be obtained. However, for negative values of M, a transformational formula is used to overcome the inherent difficulty and avert singularities.

Step 7. By balancing the linear terms of the highest order in the resulting equation with the highest order nonlinear term, the parameter, M is determined

III. BANACH CONTRACTION METHOD (BCM)

To illustrate the basics of the Banach Contraction method, we first look at some of the underlying definitions relevant to its analysis.

Definition 3.1 (Contraction Mapping)

Let (X, d) be a metric space. A mapping $f: X \to X$ is said to be contraction mapping if there exist a positive number k > 1, such that the following hold

 $d(f(x), f(y)) \le kd(x, y) \quad \forall x, y \in X$

Definition 3.2 (Banach Contraction Principle)

Let (X, d) be a complete metric space and $f: X \to X$ be a contraction mapping. Then f has a unique fixed point x_0 and for each $x \in X$, we have the

- (i) $\lim_{n \to \infty} f^n(x) = x_0$
- (ii) For each $x \in X$, we have $d(f^n(x), x_0) \le \frac{k^n}{1-k} d(f(x), x)$

Definition 3.3 (Banach Fixed Point Theorem)

Let (X, d) be a non-empty complete metric space with a contraction mapping $f: X \to X$, then f admits a unique fixed-point $x \in X$, if f(x) = x

Definition 3.4 Let *F* be a mapping of a complete metric space (X, d) into itself such that F^k is a contraction mapping of *X* for some positive integer *k*. Then *F* has a unique fixed point in *X*

Following [40-42], we consider a general functional equation as

$$v(x) = N(v) + f(x) \tag{7}$$

Where N(v) is a nonlinear operator from a Banach space $B \rightarrow B$, f(x) is a known integrable function of x and v(x) is an unknown function

We seek a solution of v(x) of Eq. (7) in series form as

$$v = \sum_{m=0}^{\infty} y_m \tag{8}$$

Decomposing the nonlinear operator, N as an infinite series

$$N(\sum_{m=0}^{\infty} y_m) = N(v_0) + \sum_{m=1}^{\infty} \{N(\sum_{m=0}^{m} y_m) - N(\sum_{m=0}^{m-1} y_m)\}$$
(9)

Combining Eqs. (8) and (9), Eq. (7) is rewritten in the form

$$\sum_{m=0}^{\infty} y_m = f(x) + N(v_0) + \sum_{m=1}^{\infty} \{N(\sum_{m=0}^m y_m) - N(\sum_{m=0}^{m-1} y_m)\}$$
(10)

Next, we define the recursive sequence of approximations as

$$v'_{0}(x) = f(x) \Longrightarrow v_{0}(x) = \int_{0}^{x} f(x) dx$$
$$v_{1}(x) = v_{0}(x) + N(v_{0})$$
$$v_{2}(x) = v_{0}(x) + N(v_{1})$$
$$v_{3}(x) = v_{0}(x) + N(v_{2})$$
$$v_{4}(x) = v_{0}(x) + N(v_{3})$$
$$\vdots$$
$$v_{n}(x) = v_{0}(x) + N(v_{n-1}), \qquad n \ge 1$$

From the above recursive scheme, if N^k is a contraction operator for some positive integer k, then N(v) has a unique fixed-point and hence the sequence above is convergent in view of theorem 3.4 see [43]

Thus, the solution of Eq. (7) is given by

$$V(x) = \lim_{n \to \infty} V_n(x) \tag{11}$$

IV. NUMERICAL EXAMPLE

In this section, we solve the Kuramoto-Shivashinsky and Burgers-Huxley equations using the two methods and make comparative analysis between them to show which is more efficient and gives the solution at a lesser time.

4.1 Burger-Huxley Equation: Tanh-coth solution

Given the Burger-Huxley equation as

$$u_t - u_{xx} = uu_x + u(k - u)(u - 1)$$
(12)

Let u(x,t) = f(z), z = x - ct be solution of Eq. (12) (13)

Putting Eq. (13) into Eq. (12), the PDE transforms into an ODE of the form

$$cu' + uu' + u'' + u(k - u)(u - 1)$$
(14)

Balancing the nonlinear term u^3 , that has exponent 3M, with the highest order derivative u'', that has the exponent M + 2, we have

$$3M = M + 2$$

So that

M = 1

The tanh-coth admits the use of the finite expression of the form

$$U(\mu z) = S(Y) = \sum_{k=0}^{m} a_k Y^k + \sum_{k=1}^{m} b_k Y^{-k}$$
$$= a_0 + a_1 Y + b_1 Y^{-1}$$
(15)

Substituting Eq. (15) into Eq. (14) and rearranging gives

$$c\frac{dU(z)}{dY} + U(z)\frac{dU(z)}{dY} + \frac{d^{2}U(z)}{dY^{2}} + U(z)(k - U(z))(U(z) - 1)$$

Expressing the above in terms of the change in derivative, we obtain

$$c\mu(1-Y^2)\frac{dS(Y)}{dY} + \mu U(z)(1-Y^2)\frac{dS(Y)}{dY} - 2\mu^2 Y(1-Y^2)\frac{dS(Y)}{dY} + \mu^2(1-Y^2)\frac{d^2S(Y)}{dY^2}$$
(16)

Collecting the coefficients of Y^k , $k \ge 0$, and setting each coefficient to zero leads to a system of algebraic equations in a_0, a_1, b_1, μ and c.

Solving the resulting system in Eq. (16) using Mathematica, we obtain the following twelve set of solutions.

Case 1. Putting $b_1 = 0$, we have the following results for the unknowns

$$a_{0} = \frac{1}{2}, a_{1} = -\frac{1}{2}, \mu = \frac{1}{4}, c = \frac{1}{2}(1 - 4k)$$

$$a_{0} = \frac{k}{2}, a_{1} = -\frac{k}{2}, \mu = \frac{k}{4}, c = \frac{1}{2}(k - 4)$$

$$a_{0} = \frac{k + 1}{2}, a_{1} = -\frac{k - 1}{2}, \mu = \frac{k - 1}{4}, c = \frac{1}{2}(k + 1)$$

$$a_{0} = \frac{1}{2}, a_{1} = \frac{1}{2}, \mu = \frac{1}{2}, c = k - 1$$

$$a_{0} = \frac{k}{2}, a_{1} = \frac{k}{2}, \mu = \frac{k}{2}, c = 1 - k$$

$$a_{0} = \frac{k + 1}{2}, a_{1} = \frac{k - 1}{2}, \mu = \frac{k - 1}{2}, c = -(1 + k)$$

Case 2. When $a_1 = 0$, we get the following results

$$a_0 = \frac{1}{2}, b_1 = -\frac{1}{2}, \mu = \frac{1}{4}, c = \frac{1}{2}(1 - 4k)$$

$$a_{0} = \frac{k}{2}, b_{1} = -\frac{k}{2}, \mu = \frac{k}{4}, c = \frac{1}{2}(k-4)$$

$$a_{0} = \frac{k+1}{2}, b_{1} = -\frac{k-1}{2}, \mu = \frac{k-1}{4}, c = \frac{1}{2}(k+1)$$

$$a_{0} = \frac{1}{2}, b_{1} = \frac{1}{2}, \mu = \frac{1}{2}, c = k-1$$

$$a_{0} = \frac{k}{2}, b_{1} = \frac{k}{2}, \mu = \frac{k}{2}, c = 1-k$$

$$a_{0} = \frac{k+1}{2}, b_{1} = \frac{k-1}{2}, \mu = \frac{k-1}{2}, c = -(1+k)$$

The first case gives the kink solution of Eq. (1) as follows

$$u_{1}(x,t) = \frac{1}{2} \left(1 - \tanh\left[\frac{1}{4}\left(x - \frac{1-4k}{2}t\right)\right] \right)$$
$$u_{2}(x,t) = \frac{k}{2} \left(1 - \tanh\left[\frac{k}{4}\left(x - \frac{k-4}{2}t\right)\right] \right)$$
$$u_{3}(x,t) = \frac{k+1}{2} - \frac{k-1}{2} \tanh\left[\frac{k-1}{4}\left(x - \frac{k+1}{2}t\right)\right]$$
$$u_{4}(x,t) = \frac{1}{2} \left(1 + \tanh\left[\frac{1}{2}\left(x - (k-1)t\right)\right] \right)$$
$$u_{5}(x,t) = \frac{k}{2} \left(1 + \tanh\left[\frac{k}{2}\left(x - (1-k)t\right)\right] \right)$$
$$u_{6}(x,t) = \frac{k+1}{2} - \frac{k-1}{2} \tanh\left[\frac{k-1}{2}\left(x + (1+k)t\right)\right]$$

The travelling wave solutions for the second case are as follows

$$u_{7}(x,t) = \frac{1}{2} \left(1 - \coth\left[\frac{1}{4}\left(x - \frac{1-4k}{2}t\right)\right] \right)$$
$$u_{8}(x,t) = \frac{k}{2} \left(1 - \coth\left[\frac{k}{4}\left(x - \frac{k-4}{2}t\right)\right] \right)$$
$$u_{9}(x,t) = \frac{k+1}{2} - \frac{k-1}{2} \coth\left[\frac{k-1}{4}\left(x - \frac{k+1}{2}t\right)\right]$$
$$u_{10}(x,t) = \frac{1}{2} \left(1 + \coth\left[\frac{1}{2}\left(x - (k-1)t\right)\right] \right)$$
$$u_{11}(x,t) = \frac{k}{2} \left(1 + \coth\left[\frac{k}{2}\left(x - (1-k)t\right)\right] \right)$$
$$u_{12}(x,t) = \frac{k+1}{2} + \frac{k-1}{2} \coth\left[\frac{k-1}{2}\left(x + (1+k)t\right)\right]$$



Figure 1. Kink solution of $u_1(x,t)$ for $c = 0.5, -10 \le x \le 10, -0.4 \le t \le 0.4$



Figure 2. kink solution of $u_3(x, t)$ for $c = 0.5, -10 \le x \le 10, -0.4 \le t \le 0.4$



Figure 3. kink solution of $u_4(x,t)$ for $c=0.5,-10\leq x\leq 10,-0.4\leq t\leq 0.4$



Figure 4. Kink solution of $u_6(x, t)$ for $c = 0.5, -10 \le x \le 10, -0.4 \le t \le 0.4$



Figure 5. Travelling wave solution of $u_7(x, t)$ for $c = 0.5, -10 \le x \le 10, -0.4 \le t \le 0.4$



Figure 6. Travelling wave solution of $u_9(x, t)$ for $c = 0.5, -10 \le x \le 10, -0.4 \le t \le 0.4$



Figure 7. Travelling wave solution of $u_{10}(x, t)$ for $c = 0.5, -10 \le x \le 10, -0.4 \le t \le 0.4$



Figure 8. Travelling wave solution of $u_{12}(x,t)$ for $c = 0.5, -10 \le x \le 10, -0.4 \le t \le 0.4$

4.2. Kuramoto-Shivashinsky Equation: Tanh-coth solution

$$u_t + a u u_x + b u_{2x} + k u_{4x} = 0 (17)$$

Using the wave transformation,

$$u(x,t) = f(z), \ z = x - ct$$

The PDE transforms into an ODE and on integration and setting the constants of integration as zero, we get

$$-cu + \frac{a}{2}u^2 + bu'' + ku''' = 0 \tag{18}$$

Now by balancing the highest order derivative term, u''' with the power of the nonlinear term, u^2

We have,
$$M + 3 = 2M \Longrightarrow M = 3$$

Using the Tanh-coth ansatz of the form

$$u(z) = S(Y) = \sum_{k=0}^{3} a_k Y^k + \sum_{k=1}^{3} b_k Y^{-k}, \quad Y = \tanh(\mu z)$$
(19)

Putting Eq. (19) into Eq. (18), we obtain an equation in the form

$$u(x,t) = S(Y) = \sum_{k=0}^{3} a_{k}Y^{k} + \sum_{k=1}^{3} b_{k}Y^{-k}$$

$$S(Y) = a_{0} + a_{1}Y + a_{2}Y^{2} + a_{3}Y^{3} + b_{1}Y^{-1} + b_{2}Y^{-2} + b_{3}Y^{-3}$$

$$S(Y) = a_{0} + a_{1}Y + a_{2}Y^{2} + a_{3}Y^{3} + \frac{b_{1}}{Y} + \frac{b_{2}}{Y^{2}} + \frac{b_{3}}{Y^{3}}$$

$$-cS(Y) + \frac{a}{2}S^{2}(Y) - \mu^{2}b\left[2Y(1-Y^{2})\frac{dS(Y)}{dY} - (1-Y^{2})\frac{d^{2}S(Y)}{dY^{2}}\right] + k\left[-2Y(1-Y^{2})\left(-2Y\frac{d^{3}S(Y)}{dY^{3}}\right)\right] + (1-Y^{2})\frac{d^{2}S(Y)}{dY^{2}} + (1-Y^{2})^{2}\left(-2\frac{dS(Y)}{dY} - 2Y\frac{d^{2}S(Y)}{dY^{3}}\right) + (1-Y^{2})\frac{d^{3}S(Y)}{dY^{3}}$$

$$(20)$$

Collecting the coefficients of Y, and solving the resulting algebraic system of equations in Eq. (20) for $a_0, a_1, a_2, a_3, b_1, b_2, b_3$ and μ , we obtain the constants with the help of Mathematica as follows

$$a_{0} = \frac{30b}{19a} \sqrt{\frac{-b}{19k}}, a_{1} = \frac{135b}{152a} \sqrt{\frac{-b}{19k}}, a_{2} = 0, a_{3} = -\frac{15b}{152a} \sqrt{\frac{-b}{19k}}$$
$$b_{1} = \frac{135b}{152a} \sqrt{\frac{-b}{19k}}, b_{2} = 0, b_{3} = -\frac{15b}{152a} \sqrt{\frac{-b}{19k}}, \mu = \frac{1}{4} \sqrt{\frac{-b}{19k}}, c = \frac{30b}{19a} \sqrt{\frac{-b}{19k}}, \frac{b}{k} < 0$$

Similarly, for $\frac{b}{k} > 0$, we have the following results for the coefficients

$$a_0 = \frac{30b}{19a} \sqrt{\frac{11b}{19k}}, a_1 = -\frac{45b}{152a} \sqrt{\frac{11b}{19k}}, a_2 = 0, a_3 = \frac{165b}{152a} \sqrt{\frac{11b}{19k}}$$

$$b_1 = -\frac{45b}{152a} \sqrt{\frac{11b}{19k}}, b_2 = 0, b_3 = \frac{165b}{152a} \sqrt{\frac{11b}{19k}}, \mu = \frac{1}{4} \sqrt{\frac{11b}{19k}}, c = \frac{30b}{19} \sqrt{\frac{11b}{19k}}$$

If $\frac{b}{k} < 0$, the soliton solution for the first set is given as

$$u(x,t) = \frac{15b}{152a} \sqrt{\frac{-b}{19k}} (16 + 9Y - Y^3 + 9Y^{-1} - Y^{-3})$$
(21)

Similarly, for $\frac{b}{k} > 0$ the soliton solution become

$$u(x,t) = \frac{15b}{152a} \sqrt{\frac{11b}{19k}} (16 - 3Y + 11Y^3 - 3Y^{-1} + 11Y^{-3})$$

4.3. Solution of Burger-Huxley Equation by Banach Contraction Method

The general Burgers-Huxley equation is a nonlinear PDE of the form

$$u_t + \alpha u^{\delta} u_x - u_{xx} = \beta u (1 - u^{\delta}) (u^{\delta} - \gamma), 0 \le x \le 1, t \ge 0$$

$$\tag{22}$$

Where α , β , γ and δ are parameters, $\beta \ge 0$, γ , $\delta > 0$

When $\alpha = 0, \delta = 1$, Eq. (22) reduces to the Huxley Equation expressible of the form

$$u_t - u_{xx} = u(k - u)(u - 1), k \neq 0$$
(23)

Now putting $\delta = \beta = \gamma = 1$, $\alpha = -1$ in Eq. (22) we have the Burger-Huxley equation as

$$u_t - u_{xx} = uu_x - u(u - 1)^2$$
(24)

Integrating both sides of Eq. (24) from 0 to t and using the initial condition at t = 0, we obtain

$$u(x,t) = 2x + \int_0^t [u_{xx} + uu_x - u(u-1)^2] dt$$
⁽²⁵⁾

Applying Banach Contraction method to Eq. (25), we get

$$u(x,t) = f(x,t) + N(v)$$

Where $u_0(x,t) = f(x,t) = 2x$

$$N(v) = \int_0^t [u_{xx} + uu_x - u(u-1)^2] dt$$

Then the recursive scheme for Eq. (25) become

$$u_{n}(x,t) = u_{0}(x,t) + \int_{0}^{t} \left[\left(u_{n-1}(x,t) \right)_{xx} + u_{n-1}(x,t) \left(u_{n-1}(x,t) \right)_{x} - u_{n-1}(x,t) (u_{n-1}(x,t) - 1)^{2} \right] dt, n \ge 1$$
(26)

Then we have the iterate of the problem as follows using Eq. (26)

 $u_0(x,t) = 2x$

$$u_{1}(x,t) = u_{0}(x,t) + \int_{0}^{t} \left[\left(u_{0}(x,t) \right)_{xx} + u_{0}(x,t) \left(u_{0}(x,t) \right)_{x} - u_{0}(x,t) (u_{0}(x,t) - 1)^{2} \right] dt$$
(27)

 $u_1(x,t) = 2x(1+t+4xt-4x^2t)$

$$u_{2}(x,t) = u_{0}(x,t) + \int_{0}^{t} \left[\left(u_{1}(x,t) \right)_{xx} + u_{1}(x,t) \left(u_{1}(x,t) \right)_{x} - u_{1}(x,t) (u_{1}(x,t) - 1)^{2} \right] dt$$

4.4. Kuramoto-Shivashinsky Equation by Banach Contraction Method

A nonlinear PDE of the form below where a, b and k are constants is called Kuramoto-Shivashinsky equation

$$u_t + auu_x + bu_{2x} + ku_{4x} = 0 (28)$$

Putting a = b = k = 1, we get the simplified form of Eq. (28) as

$$u_t + uu_x + u_{2x} + u_{4x} = 0$$

Rearranging the above gives the form

$$u_t = -uu_x - u_{2x} - u_{4x} \tag{29}$$

Subject to the initial condition

$$u(x,0) = 2x \tag{30}$$

Integrating both sides of Eq. (29) from 0 to t subject to Eq. (30), we obtain

$$u(x,t) = 2x + \int_0^t (-uu_x - u_{2x} - u_{4x})dt$$
(31)

Applying Banach Contraction method (BCM) to Eq. (30), we get

$$u(x,t) = f(x,t) + N(v)$$

Where f(x, t) = 2x

$$N(v) = \int_0^t (-uu_x - u_{2x} - u_{4x})dt$$

The recursive relation for Eq. (28) becomes

$$u_{n}(x,t) = u_{0}(x,t) - \int_{0}^{t} \left[u_{n-1}(x,t) \left(u_{n-1}(x,t) \right)_{x} + \left(u_{n-1}(x,t) \right)_{2x} + \left(u_{n-1}(x,t) \right)_{4x} \right] dt, \ n = 1,2,3.$$
(32)

Then we have the iterates of the problem as follows

$$u_{0}(x,t) = f(x,t) = 2x$$

$$u_{1}(x,t) = u_{0}(x,t) - \int_{0}^{t} \left[u_{0}(x,t) (u_{0}(x,t))_{x} + (u_{0}(x,t))_{2x} + (u_{0}(x,t))_{4x} \right] dt \qquad (33)$$

$$u_{1}(x,t) = 2x - 4xt$$

Similarly, for the next iterate we get

$$u_{2}(x,t) = u_{0}(x,t) - \int_{0}^{t} \left[u_{1}(x,t) (u_{1}(x,t))_{x} + (u_{1}(x,t))_{2x} + (u_{1}(x,t))_{4x} \right] dt$$
(34)
$$u_{2}(x,t) = 2x - 4xt + 8xt^{2} - \frac{16}{3}xt^{3}$$

Continuing in the same way, the problem converges to the exact solution using the identity

$$u(x,t) = \lim_{n \to \infty} v_n(x,t)$$

 $u(x,t) = \frac{2x}{(1+t)^2}$

(35)

V. CONCLUDING REMARKS

In this research article, two powerful semi-analytical methods in Tanh-coth and Banach Contraction method is introduced to handle the nonlinear Kuramoto-Shivashinsky and Burger-Huxley equations. The efficiency of the methods is shown by applying the procedure to successfully solve the aforementioned equations to produce analytical solutions. The study reveals the performance of the methods are reliable and efficient in handling nonlinear problems to obtain a variety of exact as well as solitary solutions.

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