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Function Theory in Exotic Clifford Algebras and its Applications

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Abstract – *In this paper, we will introduce Exotic Clifford algebras, whose structure relations are* $e_j^2+e_j+\alpha_j=0$ *and* $e_ie_j+e_je_i+e_i+e_j=0$. *From this structure, we will present Cauchy Integral formula. Moreover, boundary value problems for regular functions taking value in Exotic Clifford algebras can be solved in some low-dimensional cases.*

I. INTRODUCTION

The usual way to introduce Clifford algebras is to start from bilinear forms in linear spaces (see [5]). However, the usual Clifford algebras over $\mathbb{R}^{n+1}$ can also be constructed by equivalence classes of polynomials in n free variables $X_1, X_2, ..., X_n$, where the polynomials have relation structures X_j^2+1 and $X_iX_j+X_jX_i$, $i,j=1,...,n$, $i\neq j$ [9]. This yields the usual Clifford algebra with $e_1^2=-1$ and $e_ie_j+e_je_i=0$.

In this paper, we introduce an Exotic Clifford algebra whose structure relations are

$$\begin{cases} e_j^2+e_j+\alpha_j=0 \\ e_ie_j+e_je_i+e_i+e_j=0. \end{cases}$$

First, we derive from the algebras built into $\mathbb{R}^2$ and $\mathbb{R}^3$, and then generalized to $\mathbb{R}^{n+1}$.

In $\mathbb{R}^2$, we introduce $\mathbb{C}_{(\alpha)}$ algebra with basic elements $\{1,i\}$, where 1 is the unit element and i is the imaginary element satisfying $i^2+i+\alpha \quad \alpha\in\mathbb{R}$. An element $\alpha\in\mathbb{C}_{(\alpha)}$ can be written as $a=a_0+ia_1 \quad a_0,a_1\in\mathbb{R}$. The conjugate of $a\in\mathbb{C}_{(\alpha)}$ is defined by $\overline{a}^{(\alpha)}=a_0-(1+i)a_1$. We have $a\overline{a}^{(\alpha)}=\overline{a}^{(\alpha)}a=a_0^2-a_0a_1+\alpha a_1^2$, if $\alpha>\frac{1}{4}$ then $a\overline{a}^{(\alpha)}\geq 0$, and $a\overline{a}^{(\alpha)}=0$ if and only if $a=0$, that means $a_0=0$ and $a_1=0$. In this section, we only consider the case $\alpha>\frac{1}{4}$.

We denote $|a|_{(\alpha)}=\sqrt{a\overline{a}^{(\alpha)}}$ the norm of a. The inverse element of a nonzero element $a\in\mathbb{C}_{(\alpha)}$, we denote by a^{-1} such that:

$$a^{-1}=\frac{\overline{a}^{(\alpha)}}{|a|_{(\alpha)}^2}=\frac{a_0-a_1}{a_0^2-a_0a_1+\alpha a_1^2}-i\frac{a_1}{a_0^2-a_0a_1+\alpha a_1^2}.$$

We can prove that $\mathbb{C}_{(\alpha)}$ it is a field with the usual addition and multiplication rule. The number 0 and 1 are neutral elements under addition and multiplication, respectively (see [10]).

Let Ω be a domain in $\mathbb{R}^2$. Then a mapping

$$\begin{aligned} u:\ \ &\Omega\to\mathbb{C}_{(\alpha)} \\ &x\mapsto u(x) \end{aligned}$$

Defines a function u talking values in $\mathbb{C}_{(\alpha)}$, and u can be presented by $u(x_0,x_1)=u_0(x_0,x_1)+iu_1(x_0,x_1)$. Note that $u:\Omega\to\mathbb{C}_{(\alpha)}$, that means, its define two functions $u_0:\Omega\to\mathbb{R}$ and $u_1:\Omega\to\mathbb{R}$, respectively.

Derivative function u with respect to x_j denoted by $\frac{\partial u}{\partial x_j}, j=0,1,$ where $\frac{\partial u}{\partial x_j}=\frac{\partial u_0}{\partial x_j}+i\frac{\partial u_1}{\partial x_j}$. The function u is called to $\mathbb{C}^k(\Omega)$ if and only if all components u_j, $(j=0,1)$ belong to $\mathbb{C}^k(\Omega)$, respectively.

The Complex derivative for the function taking values $\mathbb{C}_{(\alpha)}$ is given in [1], which means: Let $u:\Omega\subseteq\mathbb{C}_{(\alpha)}\to\mathbb{C}_{(\alpha)}$ be a complex function. Then we say the function u is differentiable $z_0=\xi_0+i\xi_1$ if the following limit exists:

$$\lim_{h\to 0}(-i)\frac{u(z_0+h)-u(z_0)}{h},$$

Where $h=h_0+ih_1$ and $h=h_1-ih_0$. If this limit exists, the complex derivative z_0 is denoted by $u'(z_0)$.

The Cauchy-Riemann operator and its adjoin are defined by

$$D=\partial_0+i\partial_1$$
$$\overline{D}^{(\alpha)}=\partial_0-(1+i)\partial_1$$

Where $\partial_j=\frac{\partial}{\partial x_j}$, $j=0,1$.

We have $\overline{D}^{(\alpha)}D=D\overline{D}^{(\alpha)}=\partial_0^2-\partial_0\partial_1+\alpha\partial_1^2$. Since $\alpha>\frac{1}{4}$ the quadratic matrix $\overline{D}^{(\alpha)}D$ is positive definite, then this operator is the elliptic operator, and it's denoted by $\Delta_2^{(\alpha)}$.

A function u is called α - regular if $Du=0$, which means:

$$\partial_0u_0-\alpha\partial_1u_1=0 \tag{1.1}$$

$$\partial_0u_1+\partial_1u_0-\partial_1u_1=0. \tag{1.2}$$

We can show that, if u is α - regular function, then $\Delta_2^{(\alpha)}u_0=0$ and $\Delta_2^{(\alpha)}u_1=0$.

Ramark 1.1. If the function u taking values $\mathbb{C}_{(\alpha)}$ is differentiable, then u it satisfies the system (1.1), (1.2) (and vice versa).

Ramark 1.1. Suppose u and v are functions taking value in $\mathbb{C}_{(\alpha)}$. Then the following product rule is true

$$D(u,v)=(Du).v+u.(Dv).$$

The integral of a function taking values *in* $\mathbb{C}_{(\alpha)}$ can be represented in [2].

The Dirichlet boundary value problem for a α - regular function in a simply connected and bounded domain as follows: Let Ω be a simply bounded domain in x_0, x_1 - plane with sufficiently smooth boundary. Find a α - regular function $u=u_0+iu_1$ in Ω whose imaginary part is prescribed on $\partial\Omega$, while its real part u_0 is given at the one point in $\overline{\Omega}$.

This problem can be solved similarly in a complex plane for a homomorphic function. Indeed, if u_0, u_1 are the solution of (1.1), (1.2), then u_0 and u_1 are a solution of the equation

$$\partial_0^2u_0-\partial_0\partial_1u_0+\alpha\partial_1^2u_0=0$$

$$\text{and } \partial_0^2u_1-\partial_0\partial_1u_1+\alpha\partial_1^2u_1=0.$$

Knowing the value of u_1 the whole boundary $\partial\Omega$, u_1 is uniquely determined in the whole domain Ω (see the book [3]).

From (1.1), (1.2), we have

$$\partial_0u_0=\alpha\partial_1u_1=p_0$$

$$\partial_1 u_0 = -\partial_0 u_1 + \partial_1 u_1 = p_1,$$

and

$$\partial_1 p_0 - \partial_0 p_1 = \alpha\partial_1^2 u_1 + \partial_0^2 u_1 - \partial_0\partial_1 u_1 = \Delta_2^{(\alpha)} u_1 = 0.$$

This implies that the compatibility condition u_0 is satisfied and, consequently, u_0 is uniquely determined up to an arbitrarily constant. This arbitrary constant is uniquely determined, too, if one prescribes the value of u_0 at one point P_0 in $\overline{\Omega}$.

II. FUNCTION THEORY IN THE EXOTIC QUATERNION ALGEBRA $\mathrm{H}_{(\alpha)}$

2.1. The Exotic Quaternion algebra $\mathrm{H}_{(\alpha)}$

In the $\mathbb{R}^3$**,** we introduce algebra $\mathrm{H}_{(\alpha)}$ which has a basic element $\{e_0 = 1, e_1, e_2\}$ with the relations

$$e_1^2 + e_1 + \alpha_1 = 0$$
$$e_2^2 + e_2 + \alpha_2 = 0$$
$$e_1 e_2 + e_2 e_1 + e_1 + e_2 = 0.$$

An element $a \in H_{(\alpha)}$ can be presented by $a = a_0 + a_1 e_1 + a_2 e_2 + a_{12} e_1 e_2$ and adjoin of a:

$$\overline{a}^{(\alpha)} = a_0 - a_1(1+e_1) - a_2(1+e_2) + a_{12}(1+e_1)(1+e_2).$$

We have

$$a\overline{a}^{(\alpha)} = \overline{a}^{(\alpha)} a = a_0^2 + \alpha_1 a_1^2 + \alpha_2 a_2^2 + \alpha_1\alpha_2 a_{12}^2 - a_0 x_1 - a_0 x_2 + a_0 a_{12} - \alpha_1 a_1 a_{12} - \alpha_2 a_{12} a_{12}.$$

The coefficient matrix if $a\overline{a}^{(\alpha)}$ can presented by

$$M = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \alpha_1 & 0 & -\frac{\alpha_1}{2} \\ -\frac{1}{2} & 0 & \alpha_2 & -\frac{\alpha_2}{2} \\ \frac{1}{2} & -\frac{\alpha_1}{2} & -\frac{\alpha_2}{2} & \alpha_1\alpha_2 \end{pmatrix}.$$

Let

$$M_1 = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \alpha_1 \end{pmatrix},\ M_2 = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \alpha_1 & 0 \\ -\frac{1}{2} & 0 & \alpha_2 \end{pmatrix}.$$

By simple calculation, we have: $\det(M_1) = \alpha_1 - \frac{1}{4}$, $\det(M_2) = \frac{1}{4}(4\alpha_1\alpha_2 - \alpha_1 - \alpha_2)$, $\det(M) = \frac{1}{16}(4\alpha_1\alpha_2 - \alpha_1 - \alpha_2)^2$.

Suppose

$$\alpha_1 > \frac{1}{4} \quad \text{and} \quad \alpha_1 + \alpha_2 < 4\alpha_1\alpha_2. \tag{2.1}$$

Then, we have $a\overline{a}^{(\alpha)} \geq 0$, and we denote it by $|a|^2_{(\alpha)} = a\overline{a}^{(\alpha)}$ or $|a|_{(\alpha)} = \sqrt{a\overline{a}^{(\alpha)}}$. If $a \neq 0$ then there exists uniquely inverses element of a: $a^{-1} = \dfrac{\overline{a}^{(\alpha)}}{|a|^2_{(\alpha)}}$.

Remark 2.1. It is not difficult to show that $\overline{\overline{a}^{(\alpha)}}^{(\alpha)} = a$.

Let Ω be a domain in $\mathbb{R}^3$. Then a mapping

$$\begin{aligned} u: \ \Omega &\to H_{(\alpha)} \\ x &\mapsto u(x) \end{aligned}$$

defines a function u taking values in $H_{(\alpha)}$, and u can be presented by

$$u(x_0,x_1,x_2) = u_0(x_0,x_1,x_2) + u_1(x_0,x_1,x_2)e_1 + u_2(x_0,x_1,x_2)e_2 + u_{12}(x_0,x_1,x_2)e_1e_2.$$

Derivatives of function u respect to x_j, ($j = 0,1,2$) are denoted by $\dfrac{\partial u}{\partial x_j}$ (or $\partial_j u$), where

$$\frac{\partial u}{\partial x_j} = \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial x_j}e_1 + \frac{\partial u_2}{\partial x_j}e_2 + \frac{\partial u_{12}}{\partial x_j}e_1e_2.$$

The function u to be called belongs to $C^k(\Omega)$ if and only if components u_j ($j = 0,1,2,12$) belong to $C^k(\Omega)$, respectively.

The Cauchy - Riemann operator and its adjoin can be presented by

$$\begin{aligned} D &= \partial_0 + e_1\partial_1 + e_2\partial_2 \\ \overline{D}^{(\alpha)} &= \partial_0 - (1+e_1)\partial_1 - (1+e_2)\partial_2. \end{aligned}$$

And, we have

$$\overline{D}^{(\alpha)}D = \partial_0^2 + \alpha_1\partial_1^2 - \partial_0\partial_1 - \partial_0\partial_2.$$

Suppose that α_1 α_2 satisfies (2.1). Then $\overline{D}^{(\alpha)}D$ is the eliptic operator, we denote by $\Delta_3^{(\alpha)}$.

2.2. Fundamental solution, Cauchy – pompeiu Integral Formula

From the above definition, the coefficient matrix $\overline{D}^{(\alpha)}D$ can be presented by

$$M_2 = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \alpha_1 & 0 \\ -\frac{1}{2} & 0 & \alpha_2 \end{pmatrix}.$$

In this section, we assume that condition (2.1) is satisfied. Therefore the matrix M_2 has an inverse matrix in the form

$$A = \frac{1}{4\alpha_1\alpha_2 - \alpha_1 - \alpha_2}\begin{pmatrix} 4\alpha_1\alpha_2 & 2\alpha_2 & 2\alpha_1 \\ 2\alpha_2 & 4\alpha_2 - 1 & 1 \\ 2\alpha_1 & 1 & 4\alpha_1 - 1 \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}$$

Where $a_{ij} = a_{ji}$, $i,j = 1,2$. Using these coefficients, define for two points $x = (x_0,x_1,x_2)$ and $\xi = (\xi_0,\xi_1,\xi_2)$ of $\mathbb{R}^3$ a (non-Euclidean) distance ρ by

$$\rho^2=\sum_{i,j=0}^{2}a_{ij}(x_i-\xi_i)(x_j-\xi_j). \tag{2.2}$$

Let

$$H=\sum_{i,j=0}^{2}a_{0j}(x_j-\xi_j)-\sum_{i=1}^{2}(1+e_i)\sum_{k=0}^{2}a_{ik}(x_k-\xi_k). \tag{2.3}$$

Now we are going to show that

$$E(x,\xi)=\frac{1}{4\pi^2}.\frac{H}{\rho^3} \tag{2.4}$$

is a fundamental solution of $Du=0$ with singularity at ξ.

Indeed, to simplify the calculation, choose $\xi=(0,0,0)$, we get

$$\begin{aligned}DE&=\frac{1}{4\pi^2}\sum_{j=0}^{3}e_j\partial_jE_0\\&=\frac{1}{4\pi^2}\left(\frac{1}{\rho^3}\sum_{j=0}^{3}e_j\partial_jH-\frac{3}{\rho^4}\sum_{j=0}^{3}e_jH\partial_j\rho\right)\\&=\frac{1}{4\pi^2}\big((I)-(II)\big).\end{aligned}$$

We have ($k=0,1,2$)

$$\partial_kH=a_{0k}-(1+e_1)a_{1k}-(1+e_2)a_{2k}$$

$$\partial_k\rho=\frac{1}{\rho}\sum_{j=0}^{2}a_{kj}x_j.$$

Since $M_2A=I$ and $a_{ij}=a_{ji}$, we have

$$\begin{aligned}(I)&=\frac{a_{00}-a_{10}-a_{20}+\alpha_1a_{11}+\alpha_2a_{22}}{\rho^3}\\&=\frac{(a_{00}-\frac{1}{2}a_{10}-\frac{1}{2}a_{20})+(-\frac{1}{2}a_{01}+\alpha_1a_{11})+(-\frac{1}{2}a_{02}+\alpha_2a_{22})}{\rho^3}\\&=\frac{3}{\rho^3}\ ,\end{aligned}$$

and

$$\begin{aligned}(II)&=\frac{3}{\rho^5}\left(H\sum_{j=0}^{2}a_{0j}x_j+e_1H\sum_{j=0}^{2}a_{1j}x_j+e_2H\sum_{j=0}^{2}a_{2j}x_j\right)\\&=\frac{3}{\rho^5}\left[\left(\sum_{j=0}^{2}a_{0j}x_j\right)^2+\alpha_1\left(\sum_{j=0}^{2}a_{1j}x_j\right)^2+\alpha_2\left(\sum_{j=0}^{2}a_{2j}x_j\right)^2-\left(\sum_{j=0}^{2}a_{0j}x_j\right)\left(\sum_{k=0}^{2}a_{1k}x_k\right)-\left(\sum_{j=0}^{2}a_{0j}x_j\right)\left(\sum_{k=0}^{2}a_{2k}x_k\right)\right].\end{aligned}$$

In this equation using $M_2A=I$, the coefficient for x_0^2 can be written in the following form

$$\begin{aligned}&a_{00}^2+\alpha_1 a_{10}^2+\alpha_2 a_{20}^2-a_{00}a_{01}-a_{00}a_{01}\\&=a_{00}\left(a_{00}-\frac{1}{2}a_{01}-\frac{1}{2}a_{02}\right)+a_{01}\left(-\frac{1}{2}a_{00}+\alpha_1 a_{01}\right)+a_{02}\left(-\frac{1}{2}a_{00}+\alpha_2 a_{02}\right)\\&=a_{00}.\end{aligned}$$

Similarly, we can calculate the coefficients of $x_1^2, x_2^2, x_0x_1, x_0x_2$ and x_1x_2 are $a_{11}, a_{22}, 2a_{02}$ and a_{12}, respectively. And so

$$\begin{aligned}(II)&=\frac{3}{\rho^5}\left(a_{00}x_0^2+a_{11}x_1^2+a_{22}x_2^2+2a_{01}x_0x_1+2a_{02}x_0x_2+2a_{12}x_1x_2\right)\\&=\frac{3}{\rho^5}.\rho^2=\frac{3}{\rho^3}.\end{aligned}$$

Therefore, $DE=0$. By analogy, we can prove that $ED=0$.

Remark 2.2. In (2.3), we denote

$$\begin{aligned}&H_i=\sum_{k=0}^{2}a_{ik}(x_k-\xi_k),\ i=0,1,2,\\&H_{12}=0.\end{aligned}$$

Therefore

$$\begin{aligned}&H=H_0-H_1(1+e_1)-H_2(1+e_2)+H_{12}(1+e_1)(1+e_2),\\&\overline{H}^{(\alpha)}=H_0+H_1e_1+H_2e_2+H_{12}e_1e_2.\end{aligned}$$

And so (see calculate with (II)), we have $\overline{H}^{(\alpha)}H=\rho^2$. From this, we obtain

$$|E(x,\xi)|_{(\alpha)}=\frac{|H|_{(\alpha)}}{\rho^3}=\frac{\sqrt{\overline{H}^{(\alpha)}H}}{\rho^3}=\frac{1}{\rho^2}. \tag{2.5}$$

Now we consider the usual Euclidean distance $r=|x-\xi|$ for $x\neq\xi$, we can write $x-\xi$ in the form rx^*, where $|x^*|=1$. For such a point, the non- Euclidean distance ρ_0 between the point x^* and $(0,0,...,0)$ has a positive infimum c_0, i.e., $\rho_0\geq c_0$. And, from the definition of ρ shows that $\rho=r\rho_0$. Hence we have $\frac{1}{\rho^2}\leq\frac{\text{Constant}}{r^2}$. This implies that $E(x,\xi)$ it has a weak singularity $x=\xi$.

Let Ω be a domain in $\mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$, $N=(N_0,N_1,N_2)$ the outer unit normal of $\partial\Omega$, f_0, f_1, f_2 real-valued functions continuously differentiable in $\overline{\Omega}$. Then we have

$$\int_{\Omega}\sum_{j=0}^{2}\partial_j f_j dx=\int_{\partial\Omega}\sum_{j=0}^{2}\partial_j N_j d\mu, \tag{2.6}$$

where $d\mu=$measure element of $\partial\Omega$.

Let $u=\sum_A u_A e_A,\ u=\sum_B u_B e_B,\ A,B\in\{0,1,2,12\}$ are $H_{(\alpha)}$ - valued functions defined $\overline{\Omega}$, and suppose that $u_A,\ v_B$ are continuously differentiable functions in $\overline{\Omega}$. We consider (2.6) for $f_j=v_B.u_A$, $f_k=0$ for all $k\neq j$

$$\int_{\Omega}\partial_j(v_Bu_A)dx=\int_{\partial\Omega}v_Bu_AN_jd\mu=\int_{\partial\Omega}v_BN_jd\mu u_A.$$

Apply the product rule for real-valued functions u_A and v_B, we have

$$\int_{\Omega}(\partial_j v_B u_A + v_B \partial_j u_A)dx = \int_{\partial\Omega} v_B N_j d\mu u_A \tag{2.7}$$

From this, we get

$$\int_{\Omega}\left(\partial_j(v_B e_B)e_j u_A e_A + v_B e_B e_j \partial_j(u_A e_A)\right)dx = \int_{\partial\Omega}(v_B e_B)e_j N_j d\mu(u_A e_A).$$

Sum up over A, B and $j = 0,1,2$, then we have following formula, to be called Green Integral Formula

$$\int_{\Omega}\left(vD.u + v.Du\right)dx = \int_{\partial\Omega} vd\sigma u, \tag{2.8}$$

where $\partial\sigma = \sum_{j=0}^{2} e_j N_j d\mu$.

Now, we consider $u_\varepsilon(\xi) = |x-\xi| < \varepsilon$ (neighborhood of ξ), $\Omega_\varepsilon = \Omega \setminus \overline{u_\varepsilon(\xi)}$. Apply (2.8) in Ω_ε with functions v and $u = E(x,\xi)$ assume that v is continuously differentiable in $\overline{\Omega}$. Then we have

$$\int_{\Omega_\varepsilon} vD.E(x,\xi)dx = \int_{\partial\Omega} vd\sigma.E(x,\xi) - \int_{|x-\xi|=\varepsilon} vd\sigma E(x,\xi). \tag{2.9}$$

Limiting process $\varepsilon \to 0$ in this formula. We have, in the left-hand side of (2.9), because $E(x,\xi)$ it is weakly singular at $x = \xi$, then

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} vD.E(x,\xi)dx = \int_{\Omega} vD.E(x,\xi)dx.$$

In the right-hand side of (2.9), we consider

$$K = \int_{|x-\xi|=\varepsilon} vd\sigma E(x,\xi).$$

First, we have

$$d\sigma E(x,\xi) = \sum_{j=0}^{2} e_j N_j d\mu.\frac{1}{4\pi^2}.\frac{H}{\rho^3}.$$

Observe that, on the ε-sphere: $|x-\xi| = \varepsilon$, a point $x = (x_0, x_1, x_2)$ can be represented on the form $x - \xi = \varepsilon.X$, where X is a point on the unit sphere, and $(N_0, N_1, N_2) = (X_0, X_1, X_2)$. From (2.4), and furthermore one gets $d\mu = e^2 d\mu_1$, where $d\mu_1$ is the surface element of the unit sphere, we have

$$d\sigma.E(x,\xi)=\sum_{i=0}^{2}e_iX_i\varepsilon^2d\mu_1.\frac{1}{4\pi^2}.\frac{-\varepsilon\left(\sum_{j=0}^{2}a_{0j}X_j-\sum_{i=0}^{2}(1+e_i)\sum_{k=0}^{2}a_{ik}X_k\right)}{\varepsilon^3\left(\sum_{i,j=0}^{2}a_{ij}X_iX_j\right)^{3/2}}$$

$$=-\frac{1}{4\pi^2}.\frac{\sum_{i=0}^{2}e_iX_i\left(\sum_{j=0}^{2}a_{0j}X_j-\sum_{i=0}^{2}(1+e_i)\sum_{k=0}^{2}a_{ik}X_k\right)}{\left(\sum_{i,j=0}^{2}a_{ij}X_iX_j\right)^{3/2}}.$$

Therefore, the integral K does not depend on ε. We denote by $-c(\alpha_1,\alpha_2)$. Therefore

$$\int_{|x-\xi|=\varepsilon} vd\sigma E(x,\xi)=\int_{|x-\xi|=\varepsilon}(v(x)-v(\xi))d\sigma E(x,\xi)+v(\xi)\int_{|x-\xi|=\varepsilon}d\sigma E(x,\xi).$$

In the right-hand side of the above equation, the first integral tends to zero as $\varepsilon\to 0$ (because v it is continuous), and the second integral has the value: $-v(\xi).c(\alpha_1,\alpha_2)$.

We obtain

$$\lim_{\varepsilon\to 0}\int_{|x-\xi|=\varepsilon} v(x)d\sigma E(x,\xi)=-v(\xi).c(\alpha_1,\alpha_2).$$

To sum up, the following theorems have been proven.

Theorem 2.3. *The fundamental solution of the equation* $Du=0$ *can be represented by*

$$E(x,\xi)=\frac{1}{4\pi^2}.\frac{H}{\rho^3}.$$

Theorem 2.4 (Cauchy-Pompeiu Integral Formula). *Let* $E(x,\xi)$ *be a fundamental solution to the equation* $Du=0$. *Then in any interior point* ξ *of* Ω *each function,* v *twice continuously differentiable with values in the* $H_{(\alpha)}$ *algebra can be represented by*

$$v(\xi).c(\alpha_1,\alpha_2)=\int_{\partial\Omega}vd\sigma E(x,\xi)-\int_{\Omega}vD.E(x,\xi)dx.$$

III. FUNCTION THEORY IN THE EXOTIC CLIFFORD ALGEBRAS $A_{n,(\alpha)}$

In $\mathbb{R}^{n+1}$, we introduce algebras $A_{n,(\alpha)}$ whose have basis elements $\{e_0=1,e_1,e_2,...,e_n\}$ with the relation

$$e_j^2+e_j+\alpha_j=0,\ \ j=1,...,n$$
$$e_ie_j+e_je_i+e_i+e_j=0,\ \ i\neq j.$$

The Cauchy-Riemann operator and its adjoin:

$$D = \partial_0 + \sum_{j=1}^{n} e_j \partial_j,$$

$$\overline{D}^{(\alpha)} = \partial_0 - \sum_{j=1}^{n} (1+e_j)\partial_j.$$

We have

$$\overline{D}^{(\alpha)} D = \partial_0^2 + \sum_{j=1}^{n} \left(\alpha_j \partial_j^2 - \partial_0 \partial_j \right). \tag{3.1}$$

The coefficient matrix if $\overline{D}^{(\alpha)} D$ can presented by

$$M = \begin{pmatrix} 1 & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & \alpha_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -\frac{1}{2} & 0 & \cdots & \alpha_n \end{pmatrix}. \tag{3.2}$$

In this section, we assume that the determinant of the matrix M is different from zero. In this case, for instance, if (3.1) is elliptic, then M it has an inverses matrix in the following form

$$A = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{pmatrix} \tag{3.3}$$

where $a_{ij} = a_{ji,}\ i, j = 1, \ldots, n$. Using these coefficients, define for two points $x = (x_0, x_1, \ldots, x_n)$ and $\xi = (\xi_0, \xi_1, \ldots, \xi_n)$ of $\mathbb{R}^{n+1}$ a distance ρ by $\rho^2 = \sum_{i,j=0}^{n} a_{ij}(x_i - \xi_i)(x_j - \xi_j).$

By an analogy method in Section II, we can prove the following result. Let

$$H = \sum_{j=0}^{n} a_{0j}(x_j - \xi_j) - \sum_{i=1}^{n} (1+e_i) \sum_{k=0}^{n} a_{ik}(x_k - \xi_k).$$

Then

$$E(x,\xi) = \frac{1}{\omega_{n+1}} . \frac{H}{\rho^{n+1}}, \tag{3.4}$$

where ω_{n+1} - the surface measure of the unit sphere is a fundamental solution of $Du = 0$ with singularity at ξ.

IV. BOUNDARY VALUE PROBLEM FOR A α - REGULAR FUNCTION TAKING VALUES IN $H_{(\alpha)}$ ALGEBRA

A function $u = u_0 + u_1 e_1 + u_2 e_2 + u_{12} e_1 e_2$ taking values in $H_{(\alpha)}$ is called α **-**regular if $Du = 0$, that means:

$$\partial_0 u_0 - \alpha_1 \partial_1 u_1 - \alpha_2 \partial_2 (u_2 - u_{12}) = 0 \tag{4.1}$$

$$\partial_0 u_1 + \partial_1 (u_0 + u_1) - \partial_2 (u_1 - \alpha_2 u_{12}) = 0 \tag{4.2}$$

$$\partial_0 u_2 - \alpha_1 \partial_1 u_{12} + \partial_2 (u_0 - u_1 - u_2 + u_{12}) = 0 \tag{4.3}$$

$$\partial_0 u_{12} + \partial_1 (u_2 - u_{12}) - \partial_2 u_1 = 0. \tag{4.4}$$

Note that, if u is α -regular function, then all components of following satisfying equations

$$\Delta_3^{(\alpha)} u_j = 0, \; (j = 0,1,2,12). \tag{4.5}$$

Now we consider a boundary value problem $H_{(\alpha)}$. Let Ω be the unit ball in $\mathbb{R}^3$ and Ω_{01} the unit ball in $x_0 \;\; x_1$ - space. Then we have the following theorem:

Theorem 4.1. *The* α *-regular function* u *talking values in* $H_{(\alpha)}$ *is uniquely determined by*

- *The values of components* $u_2,\ u_{12}$ *on the whole boundary* $\partial\Omega$,
- *The value of component* u_1 *on the distinguishing part* $\partial\Omega_{01} \subset \partial\Omega$,
- *The value of component* u_0 *at one point* $P_0 \subset \partial\Omega$.

Proof

Since all components of α the -regular function u satisfy equation (4.5). Let $\varphi_2,\ \varphi_{12}$ are boundary values of $u_2,\ u_{12}$, *resp.* Then corresponding u_2 and u_{12} are uniquely determined in the whole domain Ω.

From (4.4), we have

$$\partial_2 u_1 = \partial_0 u_{12} + \partial_1 u_2 - \partial_1 u_{12} = F(x_0, x_1, x_2). \tag{4.6}$$

Then u_1 can be represented by integration in x_2 - direction and the values g_1 on the lower covering surface $x_2 = \psi_1(x_0, x_1)$:

$$u_1(x_0,x_1,x_2) = g_1(x_0,x_1) + \int_{\psi_1(x_0,x_1)}^{x_2} F(x_0,x_1,\xi)\,d\xi.$$

In order to simplify the calculations, we represent u_1 with the values g_1 on the middle surface Ω_{01}

$$u_1(x_0,x_1,x_2) = g_1(x_0,x_1) + \int_0^{x_2} F(x_0,x_1,\xi)\,d\xi. \tag{4.7}$$

We see that $\Delta_3^{(\alpha)} u_1 = 0$ is not true for all g_1. In order to find the necessary condition g_1, we calculate $\Delta_3^{(\alpha)} u_1$.
From (4.6), we have ($j = 0,1$)

$$\partial_j^2 u_1 = \partial_j^2 g_1 + \int_0^{x_2} \partial_j^2 F(x_0,x_1,\xi)\,d\xi$$

and

$$\partial_0\partial_1 u_1 = \partial_0\partial_1 g_1 + \int_0^{x_2} \partial_0\partial_1 F(x_0,x_1,\xi)\,d\xi.$$

From (4.6), we have

$$\partial_2^2 u_1 = \partial_2 F$$
$$\partial_0\partial_2 u_1 = \partial_0 F.$$

And so $\Delta_3^{(\alpha)} u_1 = 0$ if and only if

$$\Delta_2^{(\alpha)} g_1(x_0,x_1) + \alpha_2 \partial_2 F(x_0,x_1,x_2) - \partial_0 F(x_0,x_1,x_2) + \int_0^{x_2} \left(\partial_0^2 F + \alpha_1 \partial_1^2 F\right) d\xi = 0,$$

Where the variables in the integrand are (x_0, x_1, ξ) and $\Delta_2^{\alpha_2} = \partial_0^2 - \partial_0\partial_1 + \alpha_2\partial_1^2$. Since u_2, u_{12} satisfy (4.5), derivatives of these functions also satisfy (4.5), therefore, $\Delta_3^{(\alpha)}F = 0$, and the integral can be written in the form

$$-\int_0^{x_2}\left(\alpha_2\partial_2 F - \partial_0\partial_2 F\right)d\xi = -\int_0^{x_2}\partial_2\left(\alpha_2\partial_2 F - \partial_0 F\right)d\xi,$$

so that, the integral the value

$$-\alpha_2\partial_2 F(x_0,x_1,x_2) + \partial_0 F(x_0,x_1,x_2) + \alpha_2\partial_2 F(x_0,x_1,0) - \partial_0 F(x_0,x_1,0).$$

And so, $\Delta_3^{(\alpha)}u_1 = 0$ in Ω, if the initial function g_1 satisfies the Poisson equation in Ω_{01}:

$$\Delta_2^{(\alpha_2)}g_1 = -\alpha_2\partial_2 F(x_0,x_1,0) + \partial_0 F(x_0,x_1,0) \text{ in } \Omega_{01} \tag{4.8}$$

$$g_1 = \varphi_1 \text{ on } \partial\Omega_{01}. \tag{4.9}$$

Here φ_1 are the values of the function u_1 $\partial\Omega_{01}$.

Finally, the component u_0 can be constructed from (4.1)-(4.3)

$$\begin{aligned}
\partial_0 u_0 &= \alpha_1\partial_1 u_1 + \alpha_2\partial_2 u_2 - \alpha_2\partial_2 u_{12} = p_0\\
\partial_1 u_0 &= -\partial_0 u_1 + \partial_1 u_1 + \partial_2 u_1 - \alpha_2\partial_2 u_{12} = p_1\\
\partial_2 u_0 &= -\partial_0 u_2 + \alpha_1\partial_1 u_{12} + \partial_2 u_1 + \partial_2 u_2 - \partial_2 u_{12} = p_2.
\end{aligned}$$

We can prove that $\partial_j p_k = \partial_k p_j$ for all $0 \le j \ne k \le 2$.

Indeed, we have

$$\begin{aligned}
\partial_1 p_0 - \partial_0 p_1 &= \alpha_1\partial_1^2 u_1 + \alpha_2\partial_1\partial_2 u_2 - \alpha_2\partial_1\partial_2 u_{12} + \partial_0^2 u_1 - \partial_0\partial_1 u_1 - \partial_0\partial_2 u_1 - \alpha_2\partial_0\partial_2 u_{12}\\
&= \partial_0^2 u_1 + \alpha_1\partial_1^2 u_1 - \partial_0\partial_1 u_1 - \partial_0\partial_2 u_1 + \alpha_2\partial_2\left(\partial_1 u_2 - \partial_1 u_{12} - \partial_0 u_{12}\right)\\
&= \partial_0^2 u_1 + \alpha_1\partial_0^2 u_1 - \partial_0\partial_1 u_1 - \partial_0\partial_2 u_1 + \alpha_2\partial_2^2 u_1\\
&= \Delta_3^{(\alpha)}u_1 = 0.
\end{aligned}$$

Applying $\Delta_3^{(\alpha)}u_2 = 0$, $\Delta_3^{(\alpha)}u_{12} = 0$ and from (4.4) we can prove $\partial_2 p_0 - \partial_0 p_2 = 0$ and $\partial_2 p_1 - \partial_1 p_2 = 0$.

Therefore, the system turns out to be completely integrable. Since the domain Ω is homotopically simple connected, u_0 it is uniquely determined by

$$u_0(P) = u_0(P_0) + \int_\gamma\left(p_0 d\xi_0 + p_1 d\xi_1 + p_2 d\xi_2\right),$$

Where $P_0 \in \partial\Omega$ γ is an arbitrarily curve in Ω starting from P_0 to $P \in \Omega$? □

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