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Basic Concept of Lie Groups

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Abstract - *Lie groups is an intersection of two fundamental fields of mathematics: algebra and geometry. Lie groups is a first of all the group and secondly it is a smooth manifold which is a specific kind of geometric objects. The circle and the sphere are example of smooth manifolds. A circle has a continuous group of symmetries. You can rotate the circle an arbitrarily small amount and it looks the same. Finally, we can say that A Lie group is a group of symmetries where the symmetries are continuous.*

Keywords - *History of lie group, Definition of Lie group, Classification of Lie group, Importance of Lie groups*

Research Objectives - In mathematics and abstract algebra, group theory studies the algebraic structures known as groups. Thus, group theory and the closely related representation theory have many important applications in physics, chemistry and materials science etc. Group theory is also control to public key cryptography. So, it is most important thing to know about the group theory and basic concept of group theory.

Research Designed - A research design is the set of methods and procedures used in collecting and analyzing measures of variables specified in the problem research. A Research design is a frame work that has been created to find answer to research questions. There are numerous types of research design that are appropriate for the different types of research projects. The choice of which design to apply depends on the nature of the problems posed by the research aim. Each type of research design has a range of research methods that are commonly used to collect and analyses the type of data that is generated by the investigations. Here is a list of some of the more common research design, which are used to find expected outcome of my proposed work.

1. **Historical-** This aim at a systematic and objective evaluation and synthesis of evidence in order of establish facts and draw conclusion about past events. It uses primary historical data, such as archaeological remains as well as documentary sources of the past. It is usually to carry out tests in order to check the authenticity of these sources.
2. **Descriptive-** This design relies an observation as a means of collecting data. It attempts to examine situations, observation can take many forms depending on the type of information sought, people can be interviewed questionnaires distributed. Visual records mode, even sounds and smells records.
3. **Correlation-** This design is used to examine a relationship between two concepts. There are two broad classifications of relational statements an association between two concepts.
4. **Comparative-** This design is used to compare past and present or different parallel situations. It can look at situations at different scales, macro (international, national) or micro (community, individual).
5. **Experimental-** Experimental research attempts to isolate and control every relevant condition which determines the events investigated and the observes the effects when the conditions are manipulated.

I. Introduction

In mathematics a group is a set equipped with a binary operation combines any two elements to from a third element in such a way that four condition called group axioms are satisfied, namely closure, associativity, identity and invertibility.

One of the most familiar examples of group is the set of integers together with the addition operation, but groups are encountered in numerous areas within and outside mathematics, and help focusing on essential structural aspects, by detaching them from the concrete nature of the subject of the study. Group share a fundamental kinship with the notion of symmetry. For example, a symmetry group encodes symmetry features of a geometrical object. The group consist of the set of transformation that leave the object unchanged and the operation of combining two such transformations by performing one after the other. Lie groups are the symmetry groups, used in the standard model of particle physics, Poincare groups, which are also Lie groups, can express the physical symmetry underlying special relativity, and point groups are used to help understand symmetry phenomena in molecular chemistry.

The concept of a group arose from the study of polynomial equations, starting with Evariste Galois in the 1830’s, who introduced the team of group for the symmetry group of the roots of an equation, now called a Galois group. After contribution from other fields such as number theory and geometry, the group notion was generalized and firmly established around 1870.

In the 19th century we have further development in the group theory. group theory is spreads in basically three parts first is Finite groups, second is Infinite discrete groups and third is continuous groups which are known as Lie groups.

II. History of Lie groups

As Lie group theory has developed it has also become more and more pervasive in its influence on other mathematical disciplines. The original founder of this theory was a Norwegian, Marius Sophus Lie, who was born in Nonreformed, 1842.

on the early history of Lie groups , Sophus Lie himself considered the winter of 1873–1874 as the birth date of his theory of continuous groups. Hawkins, however, suggests that it was "Lie's prodigious research activity during the four-year period from the fall of 1869 to the fall of 1873" that led to the theory's creation. Some of Lie's early ideas were developed in close collaboration with Felix Klein. Lie met with Klein every day from October 1869 through 1872: in Berlin from the end of October 1869 to the end of February 1870, and in Paris, Göttingen and Erlangen in the subsequent two years. Lie stated that all of the principal results were obtained by 1884. But during the 1870s all his papers (except the very first note) were published in Norwegian journals, which impeded recognition of the work throughout the rest of Europe. In 1884 a young German mathematician, Friedrich Engel, came to work with Lie on a systematic treatise to expose his theory of continuous groups. From this effort resulted the three-volume Theories der Transformations gruppen, published in 1888, 1890, and 1893. The term groups de Lie first appeared in French in 1893 in the thesis of Lie's student Arthur Tresses’.

III. Definition and example of Lie group-

A real Lie group is a group that is also a finite-dimensional real smooth manifold, in which the group operations of multiplication and inversion are smooth maps. Smoothness of the group multiplication

$\mu: G\times G \to G \qquad \mu(x,y) = xy$

means that μ is a smooth mapping of the product manifold G × G into G. These two requirements can be combined to the single requirement that the mapping

$(x , y) \mapsto x^{-1} y$

be a smooth mapping of the product manifold into G.

example

The 2×2 real invertible matrices form a group under multiplication, denoted by GL(2, **R**) or by $GL_2(\mathbf{R})$:: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$: det A =

ad – b c ≠ 0

This is a four-dimensional noncompact real Lie group; it is an open subset of R^4. This group is disconnected; it has two connected components corresponding to the positive and negative values of the determinant.

IV. Classification of Lie groups

Lie groups are classified according to their algebraic properties (simple, semi simple, solvable , nilpotent, abelian),their connectedness (connected or simply Connected) and their compactness.

A first key result is the Levi decomposition, which says that every simply connected Lie group is the semidirect product of a solvable normal subgroup and a semi simple subgroup.

- Connected compact Lie groups are all known: they are finite central quotients of a product of copies of the circle group S1 and simple compact Lie groups (which correspond to connected Dynkin diagrams).
- Any simply connected solvable Lie group is isomorphic to a closed subgroup of the group of invertible upper triangular matrices of some rank, and any finite-dimensional irreducible representation of such a group is 1-dimensional. Solvable groups are too messy to classify except in a few small dimensions.
- Any simply connected nilpotent Lie group is isomorphic to a closed subgroup of the group of invertible upper triangular matrices with 1's on the diagonal of some rank, and any finite-dimensional irreducible representation of such a group is 1-dimensional. Like solvable groups, nilpotent groups are too messy to classify except in a few small dimensions.
- Simple Lie groups are sometimes defined to be those that are simple as abstract groups, and sometimes defined to be connected Lie groups with a simple Lie algebra. For example, SL(2, R) is simple according to the second definition but not according to the first. They have all been classified (for either definition).
- Semi simple Lie groups are Lie groups whose Lie algebra is a product of simple Lie algebras. They are central extensions of products of simple Lie groups.

The identity component of any Lie group is an open normal subgroup, and the quotient group is a discrete group. The universal cover of any connected Lie group is a simply connected Lie group, and conversely any connected Lie group is a quotient of a simply connected Lie group by a discrete normal subgroup of the center. Any Lie group *G* can be decomposed into discrete, simple, and abelian groups in a canonical way as follows. Write

G_{con} for the connected component of the identity
G_{sol} for the largest connected normal solvable subgroup
G_{nil} for the largest connected normal nilpotent subgroup

so that we have a sequence of normal subgroups
$1 \subseteq G_{nil} \subseteq G_{sol} \subseteq G_{con} \subseteq G$.

Then
G/G_{con} is discrete
G_{con}/G_{sol} is a central extension of a product of simple connected Lie groups.
G_{sol}/G_{nil} is abelian. A connected abelian Lie group is isomorphic to a product of copies of **R** and the circle group S^1.
$G_{nil}/1$ is nilpotent, and therefore its ascending central series has all quotients abelian.

This can be used to reduce some problems about Lie groups (such as finding their unitary representations) to the same problems for connected simple groups and nilpotent and solvable subgroups of smaller dimension.

- The diffeomorphism group of a Lie group acts transitively on the Lie group

- Every Lie group is parallelizable, and hence an orientable manifold (there is a bundle isomorphism between its tangent bundle and the product of itself with the tangent space at the identity)

V. Importance of Lie groups

Other than the fact that Lie groups have rich structures and that the mathematical theories of Lie groups are very beautiful, Lie groups also have applications to many different areas:

(1) (Lie's original work) Lie groups are important in studying differential equations
(2) Special functions such as Bessel functions, spherical harmonics etc can be unified by viewing them as coefficient of representation of Lie groups
(3) According to Klein's Erlanger program, the essence of geometry is to study the invariance of groups acting on homogeneous spaces
(4) Lie groups appears naturally in modern geometric theories, e.g. as the isometry group of a compact Riemannian manifold
(5) Lie groups describe symmetries in many physics' theories, includes classical mechanics (conservative quantities), relativity theories (Lorenz group), quantum mechanics (Heisenberg group) etc.

To illustrate the power of symmetry in solving mathematical problems, let's look

VI. Conclusion

Among all groups, Lie groups are of particular importance. They were first studied by the Norwegian mathematician Sophus Lie at the end of nineteenth century. Roughly speaking, a Lie group is a group of symmetries where the symmetries varies smoothly. More precisely, a Lie group admits three structures

- an algebraic structure as a group,
- a geometric structure as a topological manifold,
- a smooth structure so that one can do analysis,

and moreover, these three structures are compatible, i.e. they are naturally related to each other. This make Lie groups into important objects as well as tools in mathematics.

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