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Sum Formulas of Generalized Hexanacci Sequence: Closed Forms of the Sum Formulas $\sum_{k=0}^{n} kW_k$ and $\sum_{k=1}^{n} kW_{-k}$

Yüksel Soykan

Department of Mathematics, Art and Science Faculty,

Zonguldak Bülent Ecevit University,

67100, Zonguldak, Turkey

Abstract: *In this paper, closed forms of the sum formulas* $\sum_{k=0}^{n} kW_k$ and $\sum_{k=1}^{n} kW_{-k}$ *for generalized Hexanacci numbers are presented. As special cases, we give summation formulas of Hexanacci, Hexanacci-Lucas, and other sixth-order recurrence sequences.*

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1. Introduction

The generalized Hexanacci sequence $\{W_n(W_0, W_1, W_2, W_3, W_4, W_5; r, s, t, u, v, y)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}, \qquad (1.1)$$

$$W_0 = c_0, W_1 = c_1, W_2 = c_2, \ W_3 = c_3, W_4 = c_4, W_5 = c_5, \ n \geq 6$$

where $W_0, W_1, W_2, W_3, W_4, W_5$ are arbitrary real or complex numbers and r, s, t, u, v, y are real numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{v}{y}W_{-n+1} - \frac{u}{y}W_{-n+2} - \frac{t}{y}W_{-n+3} - \frac{s}{y}W_{-n+4} - \frac{r}{y}W_{-n+5} + \frac{1}{y}W_{-n+6}$$

for $n = 1, 2, 3, ...$ when $y \neq 0$. Therefore, recurrence (1.1) holds for all integer n.Hexanacci sequence has been studied by many authors, see for example [8,10,25] and references therein.

Table 1 A few special case of generalized Hexanacci sequences.

No	Sequences (Numbers)	Notation	References
1	Generalized Hexanacci	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 1, 1, 1, 1, 1, 1)\}$	[26]
2	Generalized Sixth order Pell	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 2, 1, 1, 1, 1, 1)\}$	[27]
3	Generalized Sixth order Jacobsthal	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 1, 1, 1, 1, 1, 2)\}$	[28]
4	Generalized 6-primes	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 2, 3, 5, 7, 11, 13)\}$	[29]

For some specific values of $W_0, W_1, W_2, W_3, W_4, W_5$ and r, s, t, u, v, y it is worth presenting these special Hexanacci numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 2) are used for the special cases of r, s, t, u, v, y and initial values.

Table 2 A few members of generalized Hexanacci sequences.

Sequences (Numbers)	Notation	OEIS [11]	Ref
Hexanacci	$\{H_n\} = \{W_n(0, 1, 1, 2, 4, 8; 1, 1, 1, 1, 1, 1)\}$	A001592	[26]
Hexanacci-Lucas	$\{E_n\} = \{W_n(6, 1, 3, 7, 15, 31; 1, 1, 1, 1, 1, 1)\}$	A074584	[26]
sixth order Pell	$\{P_n^{(6)}\} = \{W_n(0, 1, 2, 5, 13, 34; 2, 1, 1, 1, 1, 1)\}$		[27]
sixth order Pell-Lucas	$\{Q_n^{(6)}\} = \{W_n(6, 2, 6, 17, 46, 122; 2, 1, 1, 1, 1, 1)\}$		[27]
modified sixth order Pell	$\{E_n^{(6)}\} = \{W_n(0, 1, 1, 3, 8, 21; 2, 1, 1, 1, 1, 1)\}$		[27]
sixth order Jacobsthal	$\{J_n^{(6)}\} = \{W_n(0, 1, 1, 1, 1, 1; 1, 1, 1, 1, 1, 2)\}$		[28,2]
sixth order Jacobsthal-Lucas	$\{j_n^{(6)}\} = \{W_n(2, 1, 5, 10, 20, 40; 1, 1, 1, 1, 1, 2)\}$		[28,2]
modified sixth order Jacobsthal	$\{K_n^{(6)}\} = \{W_n(3, 1, 3, 10, 20, 40; 1, 1, 1, 1, 1, 2)\}$		[28]
sixth-order Jacobsthal Perrin	$\{Q_n^{(6)}\} = \{W_n(3, 0, 2, 8, 16, 32; 1, 1, 1, 1, 1, 2)\}$		[28]
adjusted sixth-order Jacobsthal	$\{S_n^{(6)}\} = \{W_n(0, 1, 1, 2, 4, 8; 1, 1, 1, 1, 1, 2)\}$		[28]
modified sixth-order Jacobsthal-Lucas	$\{R_n^{(6)}\} = \{W_n(6, 1, 3, 7, 15, 31; 1, 1, 1, 1, 1, 2)\}$		[28]
6-primes	$\{G_n\} = \{W_n(0, 0, 0, 0, 1, 2; 2, 3, 5, 7, 11, 13)\}$		[29]
Lucas 6-primes	$\{H_n\} = \{W_n(6, 2, 10, 41, 150, 542; 2, 3, 5, 7, 11, 13)\}$		[29]
modified 6-primes	$\{E_n\} = \{W_n(0, 0, 0, 0, 1, 1; 2, 3, 5, 7, 11, 13)\}$		[29]

For easy writing, from now on, we drop the superscripts from the sequences, for example we write P_n for $P_n^{(6)}$.

We present some works on summing formulas of the numbers in the following Table 3.

Table 3. A few special study of sum formulas.

Name of sequence	Papers which deal with summing formulas
Pell and Pell-Lucas	[1,4,30],[6,7]
Generalized Fibonacci	[5,12,13,14,15,16,18]
Generalized Tribonacci	[3,9,17]
Generalized Tetranacci	[19,24,31]
Generalized Pentanacci	[20,21]
Generalized Hexanacci	[22,23]

The following theorem presents some summing formulas of generalized Hexanacci numbers with positive subscripts.

THEOREM 1.1. *For $n \geq 0$ we have the following formulas:If $r+s+t+u+v+y-1 \neq 0$ then*

$$\sum_{k=0}^{n} W_k = \frac{\Theta_1}{r+s+t+u+v+y-1}$$

where

$\Theta_1 = W_{n+6} + (1-r)W_{n+5} + (1-r-s)W_{n+4} + (1-r-s-t)W_{n+3} + (1-r-s-t-u)W_{n+2} + (1-r-s-t-u-v)W_{n+1} - W_5 + (r-1)W_4 + (r+s-1)W_3 + (r+s+t-1)W_2 + (r+s+t+u-1)W_1 + (r+s+t+u+v-1)W_0.$

Proof. It is given in Soykan [23, Theorem 2.1]. □

The following theorem presents some summing formulas of generalized Hexanacci numbers with negative subscripts.

THEOREM 1.2. *For $n \geq 1$ we have the following formulas:If $r+s+t+u+v+y-1 \neq 0$, then*

$$\sum_{k=1}^{n} W_{-k} = \frac{\Theta_2}{r+s+t+u+v+y-1}$$

where

$\Theta_2 = -W_{-n+5} + (r-1)W_{-n+4} + (r+s-1)W_{-n+3} + (r+s+t-1)W_{-n+2} + (r+s+t+u-1)W_{-n+1} + (r+s+t+u+v-1)W_{-n} + W_5 + (1-r)W_4 + (1-r-s)W_3 + (1-r-s-t)W_2 + (1-r-s-t-u)W_1 + (1-r-s-t-u-v)W_0.$

Proof. It is given in Soykan [23, Theorem 3.1].

In this work, we investigate linear summation formulas of generalized Hexanacci numbers.

2. Sum Formulas of Generalized Hexanacci Numbers with Positive Subscripts

The following theorem presents some summing formulas of generalized Hexanacci numbers with positive subscripts.

THEOREM 2.1. *For $n \geq 0$ we have the following formulas:If $r+s+t+u+v+y-1 \neq 0$ then*

$$\sum_{k=0}^{n} kW_k = \frac{\Psi_1}{(r+s+t+u+v+y-1)^2}$$

where

$\Psi_1 = (-n+4r+3s+2t+u-y+nr+ns+nt+nu+nv+ny-5)W_{n+5} - (n-8r-2s-t+v+2y+nr^2-2nr-ns-nt-nu-nv+3rs+2rt-ny+ru-ry+4r^2+nrs+nrt+nru+nrv+nry+4) W_{n+4} - (n-6r-6s+u+2v+3y+nr^2+ns^2-2nr-2ns-nt-nu-nv+6rs+rt-ny+2st-rv+su-2ry-sy+3r^2+3s^2+2nrs+nrt+nru+nst+nrv+nsu+nsv+nry+nsy+3)W_{n+3} - (n-4r-4s-4t+2u+3v+4y+nr^2+ns^2+nt^2-2nr-2ns-2nt-nu-nv+4rs+4rt-ny-ru+4st-2rv-sv+tu-3ry-2sy-ty+2r^2+2s^2+2t^2+2nrs+2nrt+nru+2nst+nrv+nsu+nsv+ntu+ntv+nry+nsy+nty+2)$

$W_{n+2} - (n - 2r - 2s - 2t - 2u + 4v + 5y + nr^2 + ns^2 + nt^2 + nu^2 - 2nr - 2ns - 2nt - 2nu - nv + 2rs + 2rt - ny + 2ru + 2st - 3rv + 2su - 2sv + 2tu - tv - 4ry - 3sy - 2ty - uy + r^2 + s^2 + t^2 + u^2 + 2nrs + 2nrt + 2nru + 2nst + nrv + 2nsu + nsv + 2ntu + ntv + nry + nuv + nsy + nty + nuy + 1)W_{n+1} + y(-n+5r+4s+3t+2u+v+nr+ns+nt+nu+nv+ny-6)W_n - (4r+3s+2t+u-y-5)W_5 + (-8r-2s-t+v+2y+3rs+2rt+ru-ry+4r^2+4)W_4 + (-6r-6s+u+2v+3y+6rs+rt+2st-rv+su-2ry-sy+3r^2+3s^2+3) W_3 + (-4r-4s-4t+2u+3v+4y+4rs+4rt-ru+4st-2rv-sv+tu-3ry-2sy-ty+2r^2+2s^2+2t^2+2) W_2 + (-2r-2s-2t-2u+4v+5y+2rs+2rt+2ru+2st-3rv+2su-2sv+2tu-tv-4ry-3sy-2ty-uy+r^2+s^2+t^2+u^2+1)W_1 - y(5r+4s+3t+2u+v-6)W_0.$

Proof. Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}$$

i.e.

$$yW_{n-6} = W_n - rW_{n-1} - sW_{n-2} - tW_{n-3} - uW_{n-4} - vW_{n-5}$$

we obtain

$$\begin{aligned}
y \times (-1) \times W_{-1} &= (-1) \times W_5 - r \times (-1) \times W_4 - s \times (-1) \times W_3 - t \times (-1) \times W_2 \\
&\quad -u \times (-1) \times W_1 - v \times (-1) \times W_0 \\
y \times 0 \times W_0 &= 0 \times W_6 - r \times 0 \times W_5 - s \times 0 \times W_4 - t \times 0 \times W_3 - u \times 0 \times W_2 - v \times 0 \times W_1 \\
y \times 1 \times W_1 &= 1 \times W_7 - r \times 1 \times W_6 - s \times 1 \times W_5 - t \times 1 \times W_4 - u \times 1 \times W_3 - v \times 1 \times W_2 \\
y \times 2 \times W_2 &= 2 \times W_8 - r \times 2 \times W_7 - s \times 2 \times W_6 - t \times 2 \times W_5 - u \times 2 \times W_4 - v \times 2 \times W_3 \\
y \times 3 \times W_3 &= 3 \times W_9 - r \times 3 \times W_8 - s \times 3 \times W_7 - t \times 3 \times W_6 - u \times 3 \times W_5 - v \times 3 \times W_4
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
y(n-3)W_{n-3} &= (n-3)W_{n+3} - r(n-3)W_{n+2} - s(n-3)W_{n+1} - t(n-3)W_n \\
&\quad -u(n-3)W_{n-1} - v(n-3)W_{n-2} \\
y(n-2)W_{n-2} &= (n-2)W_{n+4} - r(n-2)W_{n+3} - s(n-2)W_{n+2} - t(n-2)W_{n+1} \\
&\quad -u(n-2)W_n - v(n-2)W_{n-1} \\
y(n-1)W_{n-1} &= (n-1)W_{n+5} - r(n-1)W_{n+4} - s(n-1)W_{n+3} - t(n-1)W_{n+2} \\
&\quad -u(n-1)W_{n+1} - v(n-1)W_n \\
y \times n \times W_n &= n \times W_{n+6} - r \times n \times W_{n+5} - s \times n \times W_{n+4} - t \times n \times W_{n+3} \\
&\quad -u \times n \times W_{n+2} - v \times n \times W_{n+1}
\end{aligned}$$

If we add the equations side by side, we obtain

$$\begin{aligned} y\sum_{k=0}^{n}kW_k &= (nW_{n+6}+(n-1)W_{n+5}+(n-2)W_{n+4}+(n-3)W_{n+3}+(n-4)W_{n+2}+(n-5)W_{n+1} \\ &\quad -(-1)W_5-(-2)W_4-(-3)W_3-(-4)W_2-(-5)W_1-(-6)W_0+\sum_{k=0}^{n}(k-6)W_k) \\ &\quad -r(nW_{n+5}+(n-1)W_{n+4}+(n-2)W_{n+3}+(n-3)W_{n+2}+(n-4)W_{n+1} \\ &\quad -(-1)W_4-(-2)W_3-(-3)W_2-(-4)W_1-(-5)W_0+\sum_{k=0}^{n}(k-5)W_k) \\ &\quad -s(nW_{n+4}+(n-1)W_{n+3}+(n-2)W_{n+2}+(n-3)W_{n+1}-(-1)W_3-(-2)W_2 \\ &\quad -(-3)W_1-(-4)W_0+\sum_{k=0}^{n}(k-4)W_k)-t(nW_{n+3}+(n-1)W_{n+2} \\ &\quad +(n-2)W_{n+1}-(-1)W_2-(-2)W_1-(-3)W_0+\sum_{k=0}^{n}(k-3)W_k) \\ &\quad -u(nW_{n+2}+(n-1)W_{n+1}-(-1)W_1-(-2)W_0 \\ &\quad +\sum_{k=0}^{n}(k-2)W_k)-v(nW_{n+1}-(-1)W_0+\sum_{k=0}^{n}(k-1)W_k) \end{aligned}$$

Then using Theorem 1.1, we get the result. □

Taking $r=s=t=u=v=y=1$ in Theorem 2.1, we obtain the following Proposition.

PROPOSITION 2.2. *If $r=s=t=u=v=y=1$ then for $n\geq 0$ we have the following formula:*

$\sum_{k=0}^{n}kW_k=\frac{1}{25}((5n+4)W_{n+5}-5W_{n+4}-(5n+9)W_{n+3}-(10n+8)W_{n+2}-(15n+2)W_{n+1}+(5n+9)W_n-4W_5+5W_4+9W_3+8W_2+2W_1-9W_0)$.

From the above Proposition, we have the following Corollary.

For Hexanacci numbers, take $W_n=H_n$ with $H_0=0,H_1=1,H_2=1,H_3=2,H_4=4,H_5=8$ in the last Proposition. For Hexanacci-Lucas numbers, take $W_n=E_n$ with $E_0=6,E_1=1,E_2=3,E_3=7,E_4=15,E_5=31$ in the last Proposition.

COROLLARY 2.3. *For $n\geq 0$, we have the following properties:*

(a): $\sum_{k=0}^{n}kH_k=\frac{1}{25}((5n+4)H_{n+5}-5H_{n+4}-(5n+9)H_{n+3}-(10n+8)H_{n+2}-(15n+2)H_{n+1}+(5n+9)H_n+16)$.

(b): $\sum_{k=0}^{n}kE_k=\frac{1}{25}((5n+4)E_{n+5}-5E_{n+4}-(5n+9)E_{n+3}-(10n+8)E_{n+2}-(15n+2)E_{n+1}+(5n+9)E_n-14)$.

Taking $r=2,s=t=u=v=y=1$ in Theorem 2.1, we obtain the following Proposition.

PROPOSITION 2.4. *If $r=2,s=t=u=v=y=1$ then for $n\geq 0$ we have the following formula:*

$\sum_{k=0}^{n}kW_k=\frac{1}{18}((3n+4)W_{n+5}-(3n+7)W_{n+4}-2(3n+4)W_{n+3}-3(3n+2)W_{n+2}-(12n+1)W_{n+1}+(3n+7)W_n-4W_5+7W_4+8W_3+6W_2+W_1-7W_0)$.

From the last proposition, we have the following corollary.

For sixth-order Pell numbers, take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, P_4 = 13, P_5 = 34$ in the last proposition. For sixth-order Pell-Lucas numbers, take $W_n = Q_n$ with $Q_0 = 6, Q_1 = 2, Q_2 = 6, Q_3 = 17, Q_4 = 46, Q_5 = 122$ in the last proposition.

COROLLARY 2.5. *For $n \geq 0$, we have the following properties:*

- **(a):** $\sum_{k=0}^{n} kP_k = \frac{1}{18}((3n+4)P_{n+5} - (3n+7)P_{n+4} - 2(3n+4)P_{n+3} - 3(3n+2)P_{n+2} - (12n+1)P_{n+1} + (3n+7)P_n + 8)$.
- **(b):** $\sum_{k=0}^{n} kQ_k = \frac{1}{18}((3n+4)Q_{n+5} - (3n+7)Q_{n+4} - 2(3n+4)Q_{n+3} - 3(3n+2)Q_{n+2} - (12n+1)Q_{n+1} + (3n+7)Q_n - 34)$.

Taking $r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ in Theorem 2.1, we obtain the following Proposition.

PROPOSITION 2.6. *If $r = 1, s = 1, t = 1, u = 1, v = 2, y = 2$ then for $n \geq 0$ we have the following formulas:*

$\sum_{k=0}^{n} kW_k = \frac{1}{12}((2n+1)W_{n+5} - 2W_{n+4} - (2n+3)W_{n+3} - (4n+2)W_{n+2} - (6n-1)W_{n+1} + 2(2n+3)W_n - W_5 + 2W_4 + 3W_3 + 2W_2 - W_1 - 6W_0)$.

From the last proposition, we have the following corollary.

For sixth-order Jacobsthal numbers, take $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1, J_5 = 1$ in the last Proposition. For sixth order Jacobsthal-Lucas numbers, take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20, j_5 = 40$. For modified sixth order Jacobsthal numbers, take $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20, K_5 = 40$. For sixth-order Jacobsthal Perrin numbers, take $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16, Q_5 = 32$. For adjusted sixth-order Jacobsthal numbers, take $W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4, S_5 = 8$. For modified sixth-order Jacobsthal-Lucas numbers, take $W_n = R_n$ with $R_0 = 6, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15, R_5 = 31$.

COROLLARY 2.7. *For $n \geq 0$, we have the following properties:*

- **(a):** $\sum_{k=0}^{n} kJ_k = \frac{1}{12}((2n+1)J_{n+5} - 2J_{n+4} - (2n+3)J_{n+3} - (4n+2)J_{n+2} - (6n-1)J_{n+1} + 2(2n+3)J_n + 5)$.
- **(b):** $\sum_{k=0}^{n} kj_k = \frac{1}{12}((2n+1)j_{n+5} - 2j_{n+4} - (2n+3)j_{n+3} - (4n+2)j_{n+2} - (6n-1)j_{n+1} + 2(2n+3)j_n + 27)$.
- **(c):** $\sum_{k=0}^{n} kK_k = \frac{1}{12}((2n+1)K_{n+5} - 2K_{n+4} - (2n+3)K_{n+3} - (4n+2)K_{n+2} - (6n-1)K_{n+1} + 2(2n+3)K_n + 17)$.
- **(d):** $\sum_{k=0}^{n} kQ_k = \frac{1}{12}((2n+1)Q_{n+5} - 2Q_{n+4} - (2n+3)Q_{n+3} - (4n+2)Q_{n+2} - (6n-1)Q_{n+1} + 2(2n+3)Q_n + 10)$.
- **(e):** $\sum_{k=0}^{n} kS_k = \frac{1}{12}((2n+1)S_{n+5} - 2S_{n+4} - (2n+3)S_{n+3} - (4n+2)S_{n+2} - (6n-1)S_{n+1} + 2(2n+3)S_n + 7)$.
- **(f):** $\sum_{k=0}^{n} kR_k = \frac{1}{12}((2n+1)R_{n+5} - 2R_{n+4} - (2n+3)R_{n+3} - (4n+2)R_{n+2} - (6n-1)R_{n+1} + 2(2n+3)R_n - 11)$.

Taking $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$ in Theorem 2.1, we obtain the following proposition.

PROPOSITION 2.8. *If $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$ then for $n \geq 0$ we have the following formula:*

$\sum_{k=0}^{n} kW_k = \frac{1}{200}((5n+2)W_{n+5} - (5n+7)W_{n+4} - (20n+8)W_{n+3} - (45n-2)W_{n+2} - (80n-33)W_{n+1} + 13(5n+7)W_n - 2W_5 + 7W_4 + 8W_3 - 33W_1 - 2W_2 - 91W_0).$

From the last proposition, we have the following corollary.

For 6-primes numbers, take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 0, G_4 = 1, G_5 = 2$ in the last proposition. For Lucas 6-primes numbers, take $W_n = H_n$ with $H_0 = 6, H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150, H_5 = 542$. For modified 6-primes numbers, take $W_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 0, E_4 = 1, E_5 = 1$.

COROLLARY 2.9. *For $n \geq 0$, we have the following properties:*

- **(a):** $\sum_{k=0}^{n} kG_k = \frac{1}{200}((5n+2)G_{n+5} - (5n+7)G_{n+4} - (20n+8)G_{n+3} - (45n-2)G_{n+2} - (80n-33)G_{n+1} + 13(5n+7)G_n + 3)$.
- **(b):** $\sum_{k=0}^{n} kH_k = \frac{1}{200}((5n+2)H_{n+5} - (5n+7)H_{n+4} - (20n+8)H_{n+3} - (45n-2)H_{n+2} - (80n-33)H_{n+1} + 13(5n+7)H_n - 338)$.
- **(c):** $\sum_{k=0}^{n} kE_k = \frac{1}{200}((5n+2)E_{n+5} - (5n+7)E_{n+4} - (20n+8)E_{n+3} - (45n-2)E_{n+2} - (80n-33)E_{n+1} + 13(5n+7)E_n + 5)$.

3. Sum Formulas of Generalized Hexanacci Numbers with Negative Subscripts

The following Theorem presents some linear summing formulas of generalized Hexanacci numbers with negative subscripts.

THEOREM 3.1. *For $n \geq 1$ we have the following formula: If $r + s + t + u + v + y - 1 \neq 0$, then*

$$\sum_{k=1}^{n} kW_{-k} = \frac{\Psi_2}{(r+s+t+u+v+y-1)^2}$$

where

$\Psi_2 = (n + 4r + 3s + 2t + u - y - nr - ns - nt - nu - nv - ny - 5)W_{-n+5} + (n + 8r + 2s + t - v - 2y + nr^2 - 2nr - ns - nt - nu - nv - 3rs - 2rt - ny - ru + ry - 4r^2 + nrs + nrt + nru + nrv + nry - 4) W_{-n+4} + (n + 6r + 6s - u - 2v - 3y + nr^2 + ns^2 - 2nr - 2ns - nt - nu - nv - 6rs - rt - ny - 2st + rv - su + 2ry + sy - 3r^2 - 3s^2 + 2nrs + nrt + nru + nst + nrv + nsu + nsv + nry + nsy - 3)W_{-n+3} + (n + 4r + 4s + 4t - 2u - 3v - 4y + nr^2 + ns^2 + nt^2 - 2nr - 2ns - 2nt - nu - nv - 4rs - 4rt - ny + ru - 4st + 2rv + sv - tu + 3ry + 2sy + ty - 2r^2 - 2s^2 - 2t^2 + 2nrs + 2nrt + nru + 2nst + nrv + nsu + nsv + ntu + ntv + nry + nsy + nty - 2)W_{-n+2} + (n + 2r + 2s + 2t + 2u - 4v - 5y + nr^2 + ns^2 + nt^2 + nu^2 - 2nr - 2ns - 2nt - 2nu - nv - 2rs - 2rt - ny - 2ru - 2st + 3rv - 2su + 2sv - 2tu + tv + 4ry + 3sy + 2ty + uy - r^2 - s^2 - t^2 - u^2 + 2nrs + 2nrt + 2nru + 2nst + nrv + 2nsu + nsv + 2ntu + ntv + nry + nuv + nsy + nty + nuy - 1)W_{-n+1} + (n - 6y + nr^2 + ns^2 + nt^2 + nu^2 + nv^2 - 2nr - 2ns - 2nt - 2nu - 2nv - ny + 5ry + 4sy + 3ty + 2uy + vy + 2nrs + 2nrt + 2nru + 2nst + 2nrv + 2nsu + 2nsv + 2ntu + 2ntv + nry + 2nuv + nsy + nty + nuy + nvy)$

$W_{-n}-(4r+3s+2t+u-y-5)W_5+(-8r-2s-t+v+2y+3rs+2rt+ru-ry+4r^2+4)W_4+(-6s+u+2v+3y+6rs+rt+2st-rv+su-2ry-sy+3r^2+3s^2+3-6r)W_3+(-4r-4s-4t+2u+3v+4y+4rs+4rt-ru+4st-2rv-sv+tu-3ry-2sy-ty+2r^2+2s^2+2t^2+2)W_2+(-2r-2s-2t-2u+4v+5y+2rs+2rt+2ru+2st-3rv+2su-2sv+2tu-tv-4ry-3sy-2ty-uy+r^2+s^2+t^2+u^2+1)W_1-y(5r+4s+3t+2u+v-6)W_0.$

Proof. Using the recurrence relation

$$W_{-n}=\frac{1}{y}W_{-n+6}-\frac{v}{y}W_{-n+1}-\frac{u}{y}W_{-n+2}-\frac{t}{y}W_{-n+3}-\frac{s}{y}W_{-n+4}-\frac{r}{y}W_{-n+5}$$

i.e.

$$yW_{-n}=W_{-n+6}-rW_{-n+5}-sW_{-n+4}-tW_{-n+3}-uW_{-n+2}-vW_{-n+1}$$

we obtain

$$\begin{aligned}
y\times n\times W_{-n} &= \times n\times W_{-n+6}-r\times n\times W_{-n+5}-s\times n\times W_{-n+4}-t\times n\times W_{-n+3}\\
&\quad -u\times n\times W_{-n+2}-v\times n\times W_{-n+1}\\
y(n-1)W_{-n+1} &= (n-1)W_{-n+7}-r(n-1)W_{-n+6}-s(n-1)W_{-n+5}-t(n-1)W_{-n+4}\\
&\quad -u(n-1)W_{-n+3}-v(n-1)W_{-n+2}\\
y(n-2)W_{-n+2} &= (n-2)W_{-n+8}-r(n-2)W_{-n+7}-s(n-2)W_{-n+6}-t(n-2)W_{-n+5}\\
&\quad -u(n-2)W_{-n+4}-v(n-2)W_{-n+3}
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
y\times 4\times W_{-4} &= 4\times W_2-r\times 4\times W_1-s\times 4\times W_0-t\times 4\times W_{-1}\\
&\quad -u\times 4\times W_{-2}-v\times 4\times W_{-3}\\
y\times 3\times W_{-3} &= 3\times W_3-r\times 3\times W_2-s\times 3\times W_1-t\times 3\times W_0\\
&\quad -u\times 3\times W_{-1}-v\times 3\times W_{-2}\\
y\times 2\times W_{-2} &= 2\times W_4-r\times 2\times W_3-s\times 2\times W_2-t\times 2\times W_1\\
&\quad -u\times 2\times W_0-v\times 2\times W_{-1}\\
y\times 1\times W_{-1} &= 1\times W_5-r\times 1\times W_4-s\times 1\times W_3-t\times 1\times W_2\\
&\quad -u\times 1\times W_1-v\times 1\times W_0.
\end{aligned}$$

If we add the above equations side by side, we get

$$\begin{aligned} y\sum_{k=1}^{n}kW_{-k} &= (-(n+1)W_{-n+5}-(n+2)W_{-n+4}-(n+3)W_{-n+3}-(n+4)W_{-n+2}-(n+5)W_{-n+1} \\ &\quad -(n+6)W_{-n}+1\times W_5+2W_4+3W_3+4W_2+5W_1+6W_0+\sum_{k=1}^{n}(k+6)W_{-k}) \\ &\quad -r(-(n+1)W_{-n+4}-(n+2)W_{-n+3}-(n+3)W_{-n+2}-(n+4)W_{-n+1} \\ &\quad -(n+5)W_{-n}+1\times W_4+2W_3+3W_2+4W_1+5W_0+\sum_{k=1}^{n}(k+5)W_{-k}) \\ &\quad -s(-(n+1)W_{-n+3}-(n+2)W_{-n+2}-(n+3)W_{-n+1}-(n+4)W_{-n} \\ &\quad +1\times W_3+2W_2+3W_1+4W_0+\sum_{k=1}^{n}(k+4)W_{-k})-t(-(n+1)W_{-n+2} \\ &\quad -(n+2)W_{-n+1}-(n+3)W_{-n}+1\times W_2+2W_1+3W_0+\sum_{k=1}^{n}(k+3)W_{-k}) \\ &\quad -u(-(n+1)W_{-n+1}-(n+2)W_{-n}+1\times W_1+2W_0 \\ &\quad +\sum_{k=1}^{n}(k+2)W_{-k})-v(-(n+1)W_{-n}+1\times W_0+\sum_{k=1}^{n}(k+1)W_{-k}) \end{aligned}$$

Then, using Theorem 1.2, we get the result. □

Taking $r=s=t=u=v=y=1$ in Theorem 3.1, we obtain the following Proposition.

PROPOSITION 3.2. *If $r=s=t=u=v=y=1$ then for $n\geq 1$ we have the following formula:*
$\sum_{k=1}^{n}kW_{-k}=\frac{1}{25}(-(5n-4)W_{-n+5}-5W_{-n+4}+(5n-9)W_{-n+3}+(10n-8)W_{-n+2}+(15n-2)W_{-n+1}+(20n+9)W_{-n}-4W_5+5W_4+9W_3+8W_2+2W_1-9W_0)$.

From the above Proposition, we have the following Corollary.

For Hexanacci numbers, take $W_n=H_n$ with $H_0=0,H_1=1,H_2=1,H_3=2,H_4=4,H_5=8$ in the last Proposition. For Hexanacci-Lucas numbers, take $W_n=E_n$ with $E_0=6,E_1=1,E_2=3,E_3=7,E_4=15,E_5=31$ in the last Proposition.

COROLLARY 3.3. *For $n\geq 1$, we have the following properties:*

- **(a):** $\sum_{k=1}^{n}kH_{-k}=\frac{1}{25}(-(5n-4)H_{-n+5}-5H_{-n+4}+(5n-9)H_{-n+3}+(10n-8)H_{-n+2}+(15n-2)H_{-n+1}+(20n+9)H_{-n}+16)$.
- **(b):** $\sum_{k=1}^{n}kE_{-k}=\frac{1}{25}(-(5n-4)E_{-n+5}-5E_{-n+4}+(5n-9)E_{-n+3}+(10n-8)E_{-n+2}+(15n-2)E_{-n+1}+(20n+9)E_{-n}-14)$.

Taking $r=2,s=t=u=v=y=1$ in Theorem 3.1, we obtain the following Proposition.

PROPOSITION 3.4. *If $r=2,s=t=u=v=y=1$ then for $n\geq 1$ we have the following formula:*
$\sum_{k=1}^{n}kW_{-k}=\frac{1}{15}(-(3n-4)W_{-n+5}+(3n-7)W_{-n+4}+(6n-8)W_{-n+3}+(9n-6)W_{-n+2}+(12n-1)W_{-n+1}+(15n+7)W_{-n}-4W_5+7W_4+8W_3+6W_2+W_1-7W_0)$.

From the last proposition, we have the following corollary.

For sixth-order Pell numbers, take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, P_4 = 13, P_5 = 34$ in the last proposition. For sixth-order Pell-Lucas numbers, take $W_n = Q_n$ with $Q_0 = 6, Q_1 = 2, Q_2 = 6, Q_3 = 17, Q_4 = 46, Q_5 = 122$ in the last proposition.

COROLLARY 3.5. *For $n \geq 1$, we have the following properties:*

- **(a):** $\sum_{k=1}^{n} kP_{-k} = \frac{1}{18}(-(3n-4)P_{-n+5} + (3n-7)P_{-n+4} + (6n-8)P_{-n+3} + (9n-6)P_{-n+2} + (12n-1)P_{-n+1} + (15n+7)P_{-n} + 8)$.
- **(b):** $\sum_{k=1}^{n} kQ_{-k} = \frac{1}{18}(-(3n-4)Q_{-n+5} + (3n-7)Q_{-n+4} + (6n-8)Q_{-n+3} + (9n-6)Q_{-n+2} + (12n-1)Q_{-n+1} + (15n+7)Q_{-n} - 34)$.

Taking $r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ in Theorem 3.1, we obtain the following Proposition.

PROPOSITION 3.6. *If $r = 1, s = 1, t = 1, u = 1, v = 2, y = 2$ then for $n \geq 1$ we have the following formulas:*

$\sum_{k=1}^{n} kW_{-k} = \frac{1}{12}(-(2n-1)W_{-n+5} - 2W_{-n+4} + (2n-3)W_{-n+3} + (4n-2)W_{-n+2} + (6n+1)W_{-n+1} + (8n+6)W_{-n} - W_5 + 2W_4 + 3W_3 + 2W_2 - W_1 - 6W_0)$.

From the last proposition, we have the following corollary.

For sixth-order Jacobsthal numbers, take $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1, J_5 = 1$ in the last Proposition. For sixth order Jacobsthal-Lucas numbers, take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20, j_5 = 40$. For modified sixth order Jacobsthal numbers, take $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20, K_5 = 40$. For sixth-order Jacobsthal Perrin numbers, take $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16, Q_5 = 32$. For adjusted sixth-order Jacobsthal numbers, take $W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4, S_5 = 8$. For modified sixth-order Jacobsthal-Lucas numbers, take $W_n = R_n$ with $R_0 = 6, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15, R_5 = 31$.

COROLLARY 3.7. *For $n \geq 1$, we have the following properties:*

- **(a):** $\sum_{k=1}^{n} kJ_{-k} = \frac{1}{12}(-(2n-1)J_{-n+5} - 2J_{-n+4} + (2n-3)J_{-n+3} + (4n-2)J_{-n+2} + (6n+1)J_{-n+1} + (8n+6)J_{-n} + 5)$.
- **(b):** $\sum_{k=1}^{n} kj_{-k} = \frac{1}{12}(-(2n-1)j_{-n+5} - 2j_{-n+4} + (2n-3)j_{-n+3} + (4n-2)j_{-n+2} + (6n+1)j_{-n+1} + (8n+6)j_{-n} + 27)$.
- **(c):** $\sum_{k=1}^{n} kK_{-k} = \frac{1}{12}(-(2n-1)K_{-n+5} - 2K_{-n+4} + (2n-3)K_{-n+3} + (4n-2)K_{-n+2} + (6n+1) K_{-n+1} + (8n+6)K_{-n} + 17)$.
- **(d):** $\sum_{k=1}^{n} kQ_{-k} = \frac{1}{12}(-(2n-1)Q_{-n+5} - 2Q_{-n+4} + (2n-3)Q_{-n+3} + (4n-2)Q_{-n+2} + (6n+1) Q_{-n+1} + (8n+6)Q_{-n} + 10)$.
- **(e):** $\sum_{k=1}^{n} kS_{-k} = \frac{1}{12}(-(2n-1)S_{-n+5} - 2S_{-n+4} + (2n-3)S_{-n+3} + (4n-2)S_{-n+2} + (6n+1)S_{-n+1} + (8n+6)S_{-n} + 7)$.

(f): $\sum_{k=1}^{n} kR_{-k} = \frac{1}{12}(-(2n-1)R_{-n+5} - 2R_{-n+4} + (2n-3)R_{-n+3} + (4n-2)R_{-n+2} + (6n+1)R_{-n+1} + (8n+6)R_{-n} - 11)$.

Taking $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$ in Theorem 3.1, we obtain the following proposition.

PROPOSITION 3.8. *If $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$ then for $n \geq 1$ we have the following formula:*

$\sum_{k=1}^{n} kW_{-k} = \frac{1}{200}(-(5n-2)W_{-n+5} + (5n-7)W_{-n+4} + (20n-8)W_{-n+3} + (45n+2)W_{-n+2} + (80n+33)W_{-n+1} + (135n+91)W_{-n} - 2W_5 + 7W_4 + 8W_3 - 2W_2 - 33W_1 - 91W_0)$.

From the last proposition, we have the following corollary.

For 6-primes numbers, take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 0, G_4 = 1, G_5 = 2$ in the last proposition. For Lucas 6-primes numbers, take $W_n = H_n$ with $H_0 = 6, H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150, H_5 = 542$. For modified 6-primes numbers, take $W_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 0, E_4 = 1, E_5 = 1$.

COROLLARY 3.9. *For $n \geq 1$, we have the following properties:*

(a): $\sum_{k=1}^{n} kG_{-k} = \frac{1}{200}(-(5n-2)G_{-n+5} + (5n-7)G_{-n+4} + (20n-8)G_{-n+3} + (45n+2)G_{-n+2} + (80n+33)G_{-n+1} + (135n+91)G_{-n} + 3)$.

(b): $\sum_{k=1}^{n} kH_{-k} = \frac{1}{200}(-(5n-2)H_{-n+5} + (5n-7)H_{-n+4} + (20n-8)H_{-n+3} + (45n+2)H_{-n+2} + (80n+33)H_{-n+1} + (135n+91)H_{-n} - 338)$.

(c): $\sum_{k=1}^{n} kE_{-k} = \frac{1}{200}(-(5n-2)E_{-n+5} + (5n-7)E_{-n+4} + (20n-8)E_{-n+3} + (45n+2)E_{-n+2} + (80n+33)E_{-n+1} + (135n+91)E_{-n} + 5)$.

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