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New Boundary Sets In Fuzzy Topological Spaces

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ABTRACT: *In this paper, boundary closed fuzzy sets have been introduced and studied. Among many other results it is observed that every closed fuzzy set is boundary closed and that every fuzzy boundary space is fuzzy-$T_{1/2}$ but not conversely. Further fuzzy b-closure and fuzzy b-interior concepts have been investigated.*

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INTRODUCTION

N.Levine [1] introduced generalized closed (g-closed) sets in general topology as a generalization` of closed sets. This concept was found to be useful and many results in general topology were improved. For example, it was proved that a g-closed subset of a compact space is compact. Many researchers like S.P.Arya and R.Gupta [2], K.Balachandran, P.Sundaram and H.Maki [3], S.G.Crosseley and S.K.Hilderbrand [4], J.Dontchev [5],H.Maki, J.Umehara and T.Noiri [6], S.R.Malghan [7], N.Palaniappan and K.Chandrasekhara Rao [8], T.Noiri [9], W.Dunham [10] and P.Sundaram [11] have worked on this and related problems in general topology.

This idea of N.Levine motivated us to generalize the concept of closed fuzzy sets in fuzzy topological spaces to a concept called b-closed (boundary-closed) fuzzy sets, using the concept of boundary of a fuzzy set defined by R.H.Warren [15]

1. Preliminaries

1.1 Definition: A fuzzy subset A in a set X is a function $A : X \to [0, 1]$. A fuzzy subset in X is empty iff its membership function is identically 0 on X and is denoted by 0 or μ_ϕ.The set X can be considered as a fuzzy subset of X whose membership function is identically 1 on X and is denoted by μ_x or I_x. In fact every subset of X is a fuzzy subset of X but not conversely. Hence the concept of a fuzzy subset is a generalization of the concept of a subset.

1.2 Defnition: A fuzzy set on X is 'Crisp 'if it takes only the values 0 and 1 on X.

1.3 Defnition: Let X be a set and be a family of fuzzy subsets of (X,τ) is called a fuzzy topology on X iff τ satisfies the following conditions.

(i) μ_ϕ ; $\mu_X \in \tau$: That is 0 and $1 \in \tau$

(ii) If $G_i \in \tau$ for $i \in I$ then $\bigvee_{i \in I} G_i \in \tau$

(iii) If $G,H \in \tau$ then $G \wedge H \in \tau$

The pair (X, τ) is called a fuzzy topological space. The members of τ are called fuzzy open sets and a fuzzy set A in X is said to be closed iff 1 − A is an fuzzy open set in X.

1.4 Remark: Every topological space is a fuzzy topological space but not conversely.

1.5 Definition : If A and B are any two fuzzy subsets of a set X , then A is said to be included in B or A is contained in B iff A(x) ≤ B(x) for all x in X. Equivalently, A ≤ B iff A(x) ≤ B(x) for all x in X.

1.6 Definition: Two fuzzy subsets A and B are said to be equal if A(x) = B(x) for every x in X.
Equivalently A = B if A(x) = B(x) for every x in X.

1.7Defnition: The complement of a fuzzy subset A in a set X, denoted by A′ or 1 − A, is the fuzzy subset of X defined by A′(x) = 1 − A(x) for all x in X. Note that(A′)′ = A.

1.8 Definition: The union of two fuzzy subsets A and B in X, denoted by A ∨ B, is a fuzzy subset in X defined by (A ∨ B)(x) = Max{μ_A(x), μ_B(X)} for all x in X.

1.9 Defnition: The intersection of two fuzzy subsets A and B in X, denoted by A ∧ B, is a fuzzy subset in X defined by (A ∧ B)(x) = Min{A(x), B(x)} for all

2.2 BOUNDARY CLOSED FUZZY SETS IN FUZZY TOPOLOGICAL SPACES

2.2.1 Definition: A fuzzy set A of a fuzzy topological space X is called boundary closed (b-closed) fuzzy set if bd(A) < G whenever A < G and G is an open fuzzy set.

2.2.2Theorem: Every closed fuzzy set is a b-closed fuzzy set in any Fuzzy topological spaces X. **Proof :** Let A be a closed fuzzy set in a Fuzzy topological spaces X. Let A < G, where G is an open fuzzy set in X. Since A is closed, cl(A) = A < G which implies cl(A) < G. Also since
bd (A) < cl (A) < G, it follows that bd(A) < G. Hence A is a b-closed fuzzy set.

The converse of the above theorem need not be true as seen from the following example.

2.2.3 Example: Let X = [0, 1] and A be a fuzzy subset of X defined by

$$A(x) = \begin{cases} 0.5 & \text{if } x = 2/3 \\ 0 & \text{otherwise} \end{cases}$$

Consider T = {0,1, A}. Then (X, T) is a Fuzzy topological space. Let B be a fuzzy subset of X defined by

$$B(x) = \begin{cases} 0.6 & \text{if } x = 2/3 \\ 1 & \text{otherwise} \end{cases}$$

Then B is a b-closed fuzzy set.
For, B < 1 where 1 is an open fuzzy set, we have bd (B) < 1. Further B is not a closed fuzzy set. Hence B is a b-closed fuzzy set which is not closed fuzzy set.

2.2.4 Definition: Let X be a Fuzzy topological spaces. A fuzzy set A in X is said to be generalized closed (g-closed) fuzzy set in X if cl (A) < U whenever A < U and U is an open fuzzy set in X.

2.2.5 Theorem: Every g-closed fuzzy set is b-closed fuzzy set.

Proof: Let A be g-closed fuzzy set in a Fuzzy topological spaces X. To prove that A is b-closed fuzzy set in X. Let A < G, where G is an open fuzzy set in X. Since A is g-closed fuzzy set, by definition it follows, cl (A) < G. But we have bd (A) < cl (A), Therefore bd (A) < cl(A) < G which implies bd(A) < G. Hence A is b-closed fuzzy set in X. The converse of the above theorem is true if cl(A) ∧ cl(l - A) > 0 for any fuzzy set A.

2.2.6 Theorem: If A is a b-closed fuzzy set in a fuzzy topological spaces X and
cl(A) ∧ cl(l - A) > 0, then A is a g-closed fuzzy set.

Proof: Suppose A is a b-closed fuzzy set and cl(A) ∧ cl(l –A) > 0. To prove that A is g-closed fuzzy set. Let A < G, where G is an open fuzzy set in X. Since A is b-closed fuzzy set, by definition it follows that bd (A) < G. Also since cl(A)∧ cl(l - A) > 0,bd(A) = cl(A),Therefore cl(A) = bd(A) < G which implies cl(A) < G.Hence A is a g-closed fuzzy set.

2.2.7 Definition: A fuzzy subset A of a Fuzzy topological space X is said to be

(i) A regular open fuzzy set in X if int(cl(A)) = A and

(ii) A regular closed fuzzy set in X if cl (int(A)) = A.

2.2.8 Theorem: If a fuzzy set A of a fuzzy topological space X is both open fuzzy set and
b-closed fuzzy set, then it is a closed fuzzy set.

Proof: Let A be a fuzzy subset of a Fuzzy topological spaces X which is both open fuzzy set and b-closed fuzzy set. Now, we have A < A, where A is an open fuzzy set. Since A is b-closed fuzzy set, we have bd(A) < A. Therefore from, A is a closed fuzzy set in X.

2.2.9 Corollary: If a fuzzy set A in a Fuzzy topological spaces X is both open fuzzy set and
b- Closed fuzzy set then it is g-closed fuzzy set.

2.2.10 Corollary: If A is both open fuzzy set and b-closed fuzzy set in a Fuzzy topological spaces X then it is regular open fuzzy set and regular closed fuzzy set in X.

Proof: Since A is an open fuzzy set A = int(A). But from the main theorem 2.2.8, it follows that A is closed and hence A = cl(A).Therefore A = int(A) = int(cl(A)). Hence A is regular open fuzzy set. Similarly, cl(int(A)) = cl(A) = A as A is open and closed fuzzy set. Hence A is also regular closed fuzzy set.

2.2.11 Theorem: Let X be a Fuzzy topological spaces and A be a fuzzy subset of X such that bd(A) ∧ (1 – bd(A)) = 0. Then A is b-closed fuzzy set iff bd(A) ∧ (1 - A) contains no non -zero closed fuzzy set.

Proof: Let F be any closed fuzzy set such that F < bd(A) ∧ (1-A). Now F < 1 - A implies

A < 1 - F, 1 - F is open fuzzy set. Since A is b-closed, we have by definition bd(A) < 1 - F, which implies that F < 1 - bd(A). Thus F < bd(A) and F < 1 - bd(A). Therefore

F < bd(A) ∧ (1 - bd(A)) = 0 by hypothesis. Therefore F = 0.Conversely, suppose the condition holds. Let A < U, where U is an open fuzzy set. If bd(A) > U, then bd(A) ∧ (1 - U) is a closed fuzzy set and bd(A) ∧ (1 - U) < bd(A) ∧ (1 - A), which contradicts the hypothesis. Therefore bd(A) < U. Hence A is b-closed fuzzy set.

2.2.12 Theorem: The union of any two b-closed fuzzy sets of Fuzzy topological spaces X is b-closed fuzzy set.

Proof: Let A and B be two b-closed fuzzy sets of a Fuzzy topological spaces X. To prove A v B is a b-closed fuzzy set. Let Av B < G, where G is an open fuzzy set. Then A < G and B < G. Then by hypothesis, bd(A) < G and bd(B) < G , as A and B are b-closed fuzzy sets. Therefore bd(A) v bd(B) < G. Now from, we have bd(A v B) < bd(A) v bd(B) < G.

Therefore bd(Av B) < G. Hence A v B is b-closed fuzzy set.

2.2.13 Remarks:

1. Finite union of b-closed fuzzy sets is a b-closed fuzzy set.

2. Intersection of b-closed fuzzy sets need not be b-closed

2.2.14 Example: Let X = {a, b, c} Fuzzy sets A,B and C be defined as follows: A = {(a, .3), (b, .5),(c, .6)}, B = {(a, .5),(b, .4),(c, .6)} and C = {(a, .2),(b, .5),(c, .7)}. Consider T = {0, 1, A}. Then {X, T) is Fuzzy topological spaces. Fuzzy sets B and C are b-closed. Now B < 1 implies bd(B) = 1 < 1 and C < 1 implies bd(C) = 1 < 1.Now D = B ∧ C = {(a, .2),(b, .4),(c, .6)} is not a b-closed fuzzy set. Hence the intersection of any two b-closed fuzzy sets need not be a b-closed fuzzy set.

2.2.15 Definition: A Fuzzy topological spaces (X,T) is compact iff each open cover of X has a finite sub cover. S.S.Benchalli and Jenifer Rodrigues proved that a closed crisp subspace of compact Fuzzy topological spaces is compact. Therefore it follows from 2.2.8 that an open b-closed crisp subspace of a compact Fuzzy topological spaces is also compact.

2.2.16 Definition : A fuzzy set A of a Fuzzy topological spaces X is called b-open fuzzy set if its complement (1 - A) is b-closed fuzzy set.

2.2.17 Theorem : Every open fuzzy set is b - open fuzzy set.

Proof: Let A be an open fuzzy set. Then 1 - A is closed fuzzy set in X.

And so 1 - A is b-closed fuzzy set. Hence A is b-open fuzzy set in Fuzzy topological spaces X.

The converse of the above theorem need not be true as seen from the following example.

2.2.18 Example: Let X = [0, 1] and A be a fuzzy subset of X defined by

$$A(x) = \begin{cases} 0.5 & \text{if } x = 2/3 \\ 0 & \text{otherwise} \end{cases}$$

Consider T = (0, 1, A}. Then (X, T) is a fuzzy topological space. Let B be a fuzzy subset of X defined by

$$B(x) = \begin{cases} 0.4 & \text{if } x = 2/3 \\ 0 & \text{otherwise} \end{cases}$$

Then B is a b-open fuzzy set. We show that 1 - B is a b-closed fuzzy set. Now

$$1 - B(x) = \begin{cases} 0.6 & \text{if } x = 2/3 \\ 0 & \text{otherwise} \end{cases}$$

and 1 - B < 1 where 1 is an open fuzzy set. Then we have bd (1 - B) < 1. And so 1 - B is b-closed fuzzy set. Therefore B is b-open fuzzy set. Further B is not an open fuzzy set. Hence B is a b-open fuzzy set which is not an open fuzzy set.

2.2.19 Remark: Every g-open fuzzy set is b-open fuzzy set, which follows from 2.2.5.

2.2.20 Theorem: The intersection of any two b-open fuzzy sets is a b-open fuzzy set.

Proof: Let A, B be two b-open fuzzy sets in a Fuzzy topological spaces X. Then 1 - A, 1 - B are two b-closed fuzzy sets in X. From theorem 2.2.11, it follows that (1 - A) v (1 - B) is a b - closed fuzzy set which implies 1 - (A ∧ B) is a b-closed fuzzy set. Therefore A ∧ B is a b-open fuzzy set.

2.2.21 Remark: It can be verified from example 2.2.14, that union of two b-open fuzzy sets need not be a b-open fuzzy set. Boundary- closure and boundary interior of a fuzzy set are defined as follows.

2.2.22 Definition: Let A be any fuzzy set in a Fuzzy topological spaces X. We define boundary closure (B cl) and boundary interior (B int) of A as follows:

B cl (A) = ∧ {U : U is b-closed fuzzy set and A < U}

B int (A) = V {V : V is b-open fuzzy set and A > V)

2.2.23 Theorem: Let A be any fuzzy set in a Fuzzy topological spaces (X, T).

Then B cl (1 - A) = 1 - B int (A) and B int (1 - A) = 1 - B cl (A).

Proof: We see that a b-open fuzzy set U < A is precisely the complement of a b-closed fuzzy set V < 1 - A. Thus

B int (A) = V { l-V : V is b-closed and V > 1 - A}

= 1 - ∧ {V: V is b-closed and V > 1 - A}

= 1 - B cl (1 - A)

So, B cl (1 - A) = 1 - B int (A).

Let g be any b-open fuzzy set. Then for any b-closed fuzzy set f > A, g = 1 - f < 1 - A .

Now B cl (A) = ∧ {1 - g: g is b-open and g < 1 - A)

= 1 - v (g: g is b-open and g < 1 - A}

= 1 – B int (1 - A)

Thus B int (l - A) = 1 - B cl (A).

2.2.24 Theorem: In a Fuzzy topological spaces X, a fuzzy set A is b-closed iff A = B cl (A).

Proof : Let A be a b-closed fuzzy set in Fuzzy topological spaces X. Since A < A and A is b-closed fuzzy set, A ϵ {f: f is a b-closed fuzzy set and A < f} and A < f implies that A = ∧ {f : f is a b-closed fuzzy set and A < f}. That is A = B cl (A).

Conversely, suppose that A = B cl (A). Then A = ∧ {f : f is b-closed fuzzy set and A < f}.

This implies that, A ϵ {f: f is a b-closed fuzzy set and A < f}. Hence A is b-closed fuzzy set

2.2.25 Theorem: In a Fuzzy topological spaces X, the following hold for b-closure.

1. B cl (0) = 0
2. B cl (A) is a b-closed fuzzy set in X.
3. B cl (B cl (A)) < B cl (A).

Proof: The straight forward proof is omitted.

2.2.26 Theorem: in a Fuzzy topological space X, a fuzzy set A is b-open fuzzy set Iff A = B int (A).

Proof: Let A be b-open fuzzy set in X. Since A < A and A is b-open fuzzy set,

A ϵ {f : f is b-open fuzzy set and A > f } and **A** > f implies that A = v {f : f is a b-open fuzzy set and A > f } = B int (A). That is A = B int(A). Conversely, suppose that A = B int(A).

That is A = v {f: f is b-open fuzzy set and A > f}.

This implies that A ϵ {f : f is b-open fuzzy set and A > f }. Hence A is b-open fuzzy set in X.

2.2.27 Theorem: In a Fuzzy topological spaces X, the following hold for b-interior

1. B int (0) = 0.

2. B int (A) is a b-open fuzzy set in X.

3.B int (B int (A)) < B int (A).

Proof: The easy verification is omitted

2.2.28 Definition : A Fuzzy topological spaces X is said to be fuzzy $-T_{1/2}$ space if every g-closed fuzzy set is a closed fuzzy set in X.

We introduce the following.

2.2.29 Definition: A Fuzzy topological spaces X is called a fuzzy b-space (f_b-space) if every b-closed fuzzy set is dosed fuzzy set.

2.2.30Theorem: A Fuzzy topological spaces X is f_b-space iff every b-open fuzzy set is open fuzzy set in X.

Proof: Suppose the space X is f_b-space. Let V be b-open fuzzy set in X.Then 1 - V is b-closed.

Since X is f_b-space, 1 - V is closed in X. Therefore V is open in X.

Conversely, assume that every b-open fuzzy set in X is open in X. Let F be b-closed fuzzy set in X, then 1 - F is b-open in X. By hypothesis, 1 - F is open in X. Therefore F is closed in X. Hence X is f_b-space.

2.2.31 Theorem: Every f_b-space is fuzzy $-T_{1/2}$ space.

Proof: Let X be a f_b-space. Let A be g-closed fuzzy set in X. Then A is b-closed fuzzy set in X. Since X is f_b-space, A is closed fuzzy set in X. Hence X is fuzzy $-T_{1/2.}$

2.2.32 Definition: A Fuzzy topological spaces X is called a fuzzy boundary T-space (f_bT-space) if every b-closed fuzzy set is g-closed fuzzy set in X.

2.2.33 Theorem: A Fuzzy topological spaces X is f_bT-space iff every b-open fuzzy set is g-open fuzzy set in X.

Proof: Suppose Fuzzy topological spaces X is f_b T-space. Let V be b-open fuzzy set in X. Then 1 - V is b-closed fuzzy set in X. Since X is f_b T-space, 1 - V is a g-closed fuzzy set in X. Therefore V is g-open fuzzy set in X. Conversely, assume that every b-open fuzzy set in X is
g-open in X. Let F be b-closed fuzzy set in X, then 1 - F is b-open fuzzy set in X. By hypothesis,
1 - F is g-open in X. Therefore F is g-closed fuzzy set in X. Hence X is f_b T-space

2.2.34 Theorem: Every f_b-space is f_b T-space.

Proof: Let Fuzzy topological spaces X be a f_b-space. Let A be a b-closed fuzzy set in X. Then A is closed fuzzy set in X since X is f_b-space. Therefore A is g-closed fuzzy set in X. Hence X is

fb T-space.

The converse of the above theorem need not be true as shown from the following example.

2.2.35 Example: Let X = (a, b, c}. Fuzzy sets A, B and C be defined as follows: A = {(a, 1),

(b, 0), (c, 0)}, B = {(a, 0), (b, 1), (c, 0)} and C = {(a, 0), (b, 1), (c, 1)}. Let T = {0, 1, A, C}.

Then (X, T) is Fuzzy topological spaces. (X, T) is f_b T- space but not f_b-space as the fuzzy set B

is b-closed and not a closed fuzzy set in X.

2.2.36 Theorem: A Fuzzy topological spaces X is f_b-space iff it is fuzzy-T1/2 and f_b T-space.

Proof: Suppose Fuzzy topological spaces X is f_b - space. Then by the theorems 2.2.33 and 2.2.36, the space X is fuzzy-$T_{1/2}$ and f_b T-space

Conversely, X is fuzzy-$T_{1/2}$ and f_b T-space. Let A be b-closed fuzzy set in X, then A is g-closed fuzzy set in X since X is f_b T-space. Again since X is fuzzy-$T_{1/2}$, A is closed fuzzy set in X. Therefore X is f_b-space.

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