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Perfect Domination Polynomial of a Graph

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Abstract — *Graph polynomial is one of the algebraic representation for a graph which relates various graph parameters through algebraic operations. In this paper, we initiate the study of one more algebraic representation of a graph called the perfect domination polynomial. The perfect domination polynomial of a graph G of order n is the polynomial $D_p(G,x)$ having the coefficient of x^i to be $d_p(G,i)$ which denotes the number of perfect dominating sets of G of cardinality i and $\gamma_p(G)$ denotes the perfect domination number of G. We obtain some properties of $D_p(G,x)$ and its coefficients, compute the perfect domination polynomial of some families of standard graphs. Further, we obtain some characterization for some specific graphs.*

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I. INTRODUCTION

Let $G=(V,E)$ be a graph of order n. The concept of perfect domination was first studied by Weichel [4] and Yen and Lee, Cockayne, Hedetnemi and Laskar. Also, the concept of domination polynomial of a graph was introduced and studied by Seid Alikhani and Y. H. Peng[1]. Later, many graph polynomials were introduced and studied by many researchers for the domination parameters like total, connected, independent domination and so on. In this paper, we introduce the perfect domination polynomial and determine the same for some standard graphs. For any graph theoretic definitions and notations not defined here refer to [2]. Throughout this article, by a graph, we mean a finite, undirected graph without loops and multiple edges.

For each vertex $v \in V$, the open neighborhood of v is the set $N(v)$ containing all the vertices u adjacent to v and the closed neighborhood of v is the set $N[v]$ containing v and all the vertices u adjacent to v, $i.e.$, $N[v]=N(v)\cup\{v\}$. Let S be any subset of vertes set $V(G)$, then the open neighborhood of S is given by $N(S)=\cup_{v\in S} N(v)$ and the closed neighborhood of S is $N[S]=N(S)\cup S$.

A set $S\subseteq V(G)$ of vertices is said to be a dominating set of G if $N[S]=V$, or equivalently, every vertex in $V-S$ is adjacent to at least one vertex in S. The cardinality of the minimum dominating set is called the domination number, denoted by $\gamma(G)$. A dominating set S of a graph G is said to be perfect dominating set if every vertex not in S is adjacent to exactly one vertex in S. Any perfect dominating set with minimum cardinality is called a minimum perfect dominating set. The cardinality of the minimum perfect dominating is called the perfect domination number of G, denoted by $\gamma_p(G)$. A perfect dominating set with cardinality $\gamma_p(G)$ is referred as γ_p −set.

The corona of two graphs G_1 and G_2, as defined by Frucht and Harary [3] is the graph $G=G_1\circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2. The *join* of two graphs G_1 and G_2, denoted by $G_1\vee G_2$, is a graph with vertex set $V(G_1)\cup V(G_2)$ and the edge set given by $E(G_1)+E(G_2)\cup\{\,uv \mid u\in V(G_1)\ and\ v\in V(G_2)\}$. A graph G is called a bi-star if it can be constructed from K_2 by touching m edges in one vertex and n in the another vertex, denoted by $B(m,n)$.

Suppose $d(G,i)$ denotes the number of dominating sets of G of cardinality i, then the domination polynomial of a graph G is denoted by $D(G,x)$ and is defined by

$$D(G,x)=\sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$$

where $\gamma(G)$ denotes the domination number of G.

II. PERFECT DOMINATION POLYNOMIAL

Suppose $D_p(G,i)$ denotes the family of perfect dominating sets of a graph G with cardinality i and let $d_p(G,i) = |D_p(G,i)|$. Then the perfect domination polynomial $D_p(G,x)$ of G is defined as:

$$D_p(G,x) = \sum_{i=\gamma_p(G)}^{|V(G)|} d_p(G,i)x^i.$$

where $\gamma_p(G)$ denotes the perfect domination number of G. The set of all roots of the perfect domination polynomial of G is denoted by $Z\left(D_p(G,x)\right)$.

Observations: Let G be any graph connected graph of order n, then

1. $d_p(G,i) = \begin{cases} 0 & if \;\; i < \gamma_p(G) \text{ or } n < i \\ 1 & if \;\; i = n \end{cases}$
2. $D_p(G,x)$ is monotonically increasing function.
3. Zero is always a root of $D_p(G,x)$ of multiplicity $\gamma_p(G)$. Further $Z\left(D_p(G,x)\right) = \{0\}$ if G is totally disconnected.
4. For any graph G, we have $0 < \left|Z\left(D_p(G,x)\right)\right| < n$. Second equality holds if all the roots $D_p(G,x)$ are distinct.

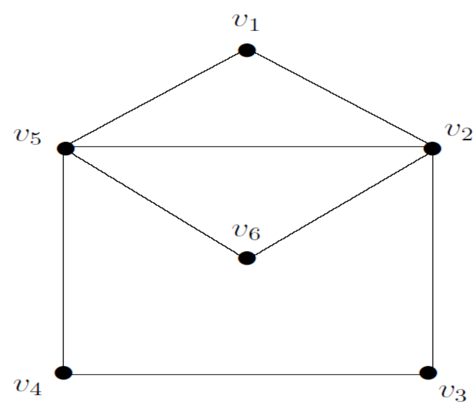

Figure 1: Graph on 6 vertices

Example: Let G be a graph as shown in the Figure 1. Then G has 2 perfect dominating sets of size 2. $i.e.$, $d_p(G,2) = 2$. But there exists no perfect dominating set of size 3. Also there exists 1, 2 perfect dominating sets of size 4, 5 respectively. It Is trivial that $d_p(G,6) = 1$. Therefore, the perfect domination polynomial of G is then $D_p(G,x) = 2x^2 + x^4 + 2x^5 + x^6$.

Proposition: For any path P_n, we have

$$\gamma_p(P_n) = \begin{cases} \frac{n}{2} & if \; n \equiv 1 \; (mod\; 3); \\ \frac{n+2}{2} & if \; n \equiv 2 \; (mod\; 3); \\ \frac{n+1}{2} & if \; n \equiv 3 \; (mod\; 3) \end{cases}$$

Proposition: Let G be any graph of order n. Then $D_p(G,x) = nx + x^n$ if and only if G is a complete graph.

Proof: Let G be any graph of order n. Assume that $D_p(G,x) = nx + x^n$. Then G has n dominating sets of cardinality one implying that each vertex in G constitutes a perfect dominating set and hence G must be connected and each vertex is adjacent to every vertex in G. Therefore, G will be a complete graph of order n. Converse is obvious.

Proposition: Let $G \cong K_{1,n-1}$ be a star graph of order n. Then $D_p(G,x) = x(1+x)^{n-1}$.

Proof: Let $G \cong K_{1,n-1}$ be a star and let the vertex set be $V(K_{1,n-1}) = \{v_1, v_2, \dots v_n\}$, where v_n is the support vertex of G.

Then, $S = \{v_n\}$ is a unique perfect dominating set in G and so $d_p(G,1) = 1$. Next, to obtain the perfect dominating set of cardinality i, $(2 \leq i \leq n)$, we proceed as follows. Fix the vertex v_n and the select $i-1$ vertices from $n-1$ pendant vertices. Then $\{v_n, v_1, v_2, \dots, v_i\}$ will be the perfect dominating set of size i. Clearly, number of perfect dominating sets of size i will be $\binom{n-1}{i-1}$. Therefore, $D_p(G,x) = \sum_{i=1}^{n-1}\binom{n-1}{i-1}x^i = x(1+x)^{n-1}$.

Proposition: Let $G \cong K_m(a_1, a_2, \dots, a_m)$ be a multi-star graph of order N. Then $D_p(G, x) = x^m(1+x)^{N-m}$.

Proof: Let $G \cong K_m(a_1, a_2, \dots, a_m)$ be a multi-star graph of order N. Clearly $N = a_1 + a_2 + \cdots + a_n + m$. The set of all support vertices of the stars forms a perfect dominating set of minimum cardinality and so $\gamma_p(G) = m$. Since, there is only one such set, it follows that $d_p(G, m) = 1$. The γ_p −set in G can be extended to perfect dominating set of cardinality $m+k$, where $1 \leq k \leq N-m$ by choosing k vertices among the $N-m$ pendant vertices. Hence, the co-efficient of x^k will be $\binom{N-m}{k}$ and so $D_p(G,x) = x^m(1+x)^{N-m}$.

Corollary: Let $G \cong B(m,n)$ be a bi-star graph with $m+n+2$ vertices. Then $D_p(G,x) = x^2(1+x)^{m+n}$.

Proposition: Let $G \cong K_{m,n}$ be a complete bipartite graph with $m \leq n$. Then $D_p(G,x) = \begin{cases} 2x + x^2 & if \;\; m = n = 1 \\ mx^2 + x^{m+n} & if \;\; m,n \geq 2 \end{cases}$.

Proof: Let $G = (V_1, V_2, E)$ be a complete bipartite graph with $|V_1| = m$ and $|V_2| = n$ and assume that $2 \leq m \leq n$. Any pair (u,v) of vertices taken from each of V_1 and V_2 respectively, constitutes a perfect dominating set of size 2. As the pair of vertices taken from the same set will not be a dominating set, there are exactly m dominating sets of cardinality two. $i.e.$, $d_p(G,2) = 2$ and it is trivial that $d_p(G, m+n) = 1$. Further, G contains no perfect dominating set of cardinality i, $3 \leq i \leq m+n-1$, since any dominating set having at least two vertices from V_1 or V_2 fails to be a perfect dominating set. Therefore, $D_p(G,x) = mx^2 + x^{m+n}$.

Proposition: Let $G \cong K_{m_1, m_2, \dots, m_r}$ be a complete multipartite graph with $m_1 \leq m_2 \leq \cdots \leq m_r$ and $r \geq 3$. Then $D_p(G,x) = kx + x^{m_1+m_2+\cdots+m_r}$, where k is the smallest integer such that $m_{k+1} \geq 2$.

Proof: Let $G \cong K_{m_1, m_2, \dots, m_r}$ be a complete multipartite graph with $m_1 \leq m_2 \leq \cdots \leq m_r$ and let $\{V_{m_1}, V_{m_2}, \dots, V_{m_r}\}$ be the partition of the vertex set of G. Suppose k is the smallest integer such that $m_{k+1} \geq 2$. Then, for $1 \leq i \leq k$, the partite set $V_{m_i} = \{v_{m_i}\}$ itself a perfect dominating set of G. Therefore G contains k number of perfect dominating sets each of cardinality one and so $d_p(G,1) = k$. Further, let $\{u,v\}$ be any arbitrary pair of vertices in G. If u, v are taken from same set V_i, then for any vertex $w \in V_j$ with $i \neq j$, w will be adjacent to both u and v. Thus $u, v \notin V_i$ for any i. Next, suppose $i \neq j$ and assume $u \in V_i$ and $v \in V_j$ respectively. Since $r \geq 3$, as in the previous case, we may find a vertex w adjacent to both u, v. This argument shows that G cannot have any perfect dominating set of size i where $2 \leq i \leq m_1 + m_2 + \cdots + m_r - 1$. Therefore, $D_p(G,x) = kx + x^{m_1+m_2+\cdots+m_r}$, where k is the smallest integer such that $m_{k+1} \geq 2$.

Corollary: Let $G \cong K_{m_1, m_2, \dots, m_r}$ be a complete multipartite graph with $2 \leq m_1 \leq m_2 \leq \cdots \leq m_r$. Then $D_p(G,x) = x^{m_1+m_2+\cdots+m_r}$.

Corollary: Let G be any graph of order n. Then $D_p(G,x) = x^n$ if and only if G is either totally disconnected or complete r −partite graph with $r \geq 3$.

Theorem: Let G be a disconnected graph with components $G_1, G_2, \dots, G_m$. Then $D_p(G,x) = \prod_{i=1}^{m} D_p(G_i, x)$.

Proof: We prove this by using mathematical induction on m. The result is trivial if $m = 1$. Suppose $m = 2$ and $G = G_1 \cup G_2$. For any integer $r > \gamma_p(G)$, a perfect dominating set of size r arises by choosing a perfect dominating set of size s in G_1 and a perfect dominating set of size $r - s$ in G_2. Then, the co-efficient of x^r in $D_p(G_1,x)D_p(G_2,x)$ will be precisely the number of ways of choosing perfect dominating of size s which varies from $\gamma_p(G_1)$ to $|V(G_1)|$. Thus, both side of the equation have same co-efficient, which proves that the polynomials are identical.

Finally, Suppose the result is true for $m-1$. Then by induction hypothesis and the case $m = 2$, the result follows for m.

Corollary: Let G_1 and G_2 be any two graphs. Then $D_p(G_1 \cup G_2, x) = D_p(G_1,x)D_p(G_2,x)$.

Lemma: Let G_1 and G_2 be any two graphs. Then $\gamma_p(G_1 \vee G_2) < n$ if and only if G_1 and G_2 are either totally disconnected or $\gamma(G_i) = 1$ for some i.

Proposition: Let G_1 and G_2 be any two connected graphs of order n_1, n_2 respectively. Then

$$\gamma_p(G_1 \vee G_2) = \begin{cases} x + x^{n_1+n_2} & if \;\; \gamma_p(G_1) = 1 \; or \; \gamma_p(G_2) = 1 \\ 2x + x^{n_1+n_2} & if \;\; \gamma_p(G_1) = \gamma_p(G_2) = 1 \\ x^{n_1+n_2} & if \;\; \gamma_p(G_1), \quad \gamma_p(G_2) \geq 2 \end{cases}$$

Proof: Let G_1 and G_2 be any two connected graphs of order n_1, n_2 respectively. First, suppose $\gamma_p(G_1) = 1$ or $\gamma_p(G_2) = 1$. For definiteness, assume $\gamma_p(G_1) = 1$. Then γ_p −set of G_1 is also γ_p −set of $G_1 \vee G_2$. Hence, $d_p(G_1 \vee G_2, 1) = 1$. Let (u, v) be any pair of vertices in $G_1 \vee G_2$. As the graphs G_1 and G_2 are connected, there is at least one vertex w adjacent to both u and v. Thus, $G_1 \vee G_2$ contains no perfect dominating set of size i, $2 \leq i \leq n-1$. Hence $d_p(G_1 \vee G_2, i) = 0$ for $i = 2, 3, \dots, n-1$. Therefore $D_p(G_1 \vee G_2, x) = x + x^{n_1+n_2}$. Similarly, if $\gamma_p(G_1) = \gamma_p(G_2) = 1$, then $G_1 \vee G_2$ contains two perfect dominating sets of size one. Therefore, $D_p(G_1 \vee G_2, x) = 2x + x^{n_1+n_2}$. Finally, suppose $\gamma_p(G_1)$, $\gamma_p(G_2) \geq 2$. Then $G_1 \vee G_2$ contains no perfect dominating sets of size less than n. $i.e$, $d_p(G, i) = 0$ for $1 \leq i \leq n-1$. Hence, $D_p(G_1 \vee G_2) = x^{n_1+n_2}$.

Corollary: Let $G \cong W_{1,n}$ be a wheel graph with $n \geq 4$ vertices. Then $D_p(G, x) = x + x^n$.

Proof: Let $G \cong W_{1,n}$ be a wheel graph. Then $G \cong K_1 + C_{n-1}$. Since $\gamma_p(K_1) = 1$, from the above theorem, it follows that $D_p(G, x) = x + x^n$.

Graphs with $Z\left(D_p(G, x)\right) = \{0\}$.

In this section, we the consider the class of graphs having unique root. Since, zero is always a root of $D_p(G, x)$ of multiplicity γ_p, we study the graphs without non-zero roots. Clearly, if G is totally disconnected, then $Z\left(D_p(G, x)\right) = \{0\}$. It is obvious that $Z\left(D_p(G, x)\right) = \{0\}$ if only if G contains no dominating sets of different cardinalities.

Proposoition: Suppose $G \cong P_n \circ K_1$, then $Z\left(D_p(G, x)\right) = \{0\}$.

Proof: Suppose $G \cong P_n \circ K_1$. Then $V(P_n)$ and its complement are the only perfect dominating sets of G, each of cardinality n. Thus, $D_p(G, x) = 2x^n$.

Proposition:

1. Let G be a complete graph of order n. Then $D_p(G, -1) = \begin{cases} 1 - n, & if \;\; n \;\; even; \\ -1 - n, & if \;\; n \;\; odd \end{cases}$
2. If G is either a star or a bi-star, then $D_p(G, -1) = 0$. In fact $D_p(G, -1) = 0$ for any multi-star graph.

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