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Fuzzy Rough Approximations: A Novel Approach

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Abstract - *Rough sets and fuzzy sets are two different but complementary concepts that provide effective mathematical tools for handling imperfect information. Their hybrid form, namely, fuzzy rough sets are very useful in dealing with real world data that involve vagueness and indiscernibility. In this paper, fuzzy rough approximations of a fuzzy set in a fuzzy approximation space are defined using normalized fuzzy divergence measures and their properties are investigated. Also, it is proved that the present approach is a generalization of both the Pawlak's rough set approach and the fuzzy rough set approach. Moreover, the proposed definition gives better approximations to a set than the original fuzzy rough approximations.*

Keywords — *Approximations, Rough set, Fuzzy set, Divergence Measure, Fuzzy Rough Set.*

I. INTRODUCTION

Z. Pawlak [10] put forward the notion of rough set approximations in the early 1980's with an objective to provide mathematical foundations to artificial intelligence. He proposed two approximations for a vague concept in terms of precise concepts defined using the euivalence classes on an approximation space. Later, there have been extensive studies on this theory and many generalizations and applications have been proposed [1,9,18,20]. Being a generalization of set theory, rough set theory has often been compared and contrasted with the fuzzy set theory.

Fuzzy sets [19] and rough sets model two different types of uncertainty namely vagueness and indiscernibility respectively. Fuzzy set theory addresses the problem of ambiguity in the belongingness of an object in a set, whereas rough set theory addresses the problem of ambiguity caused by the existence of a boundary region for the set. A fuzzy set is characterized by a membership function giving individual importance to each element in the universal set, whereas a rough set is characterized by an indiscernibility relation giving importance to equivalence classes.

The first attempt to define fuzzy rough set in a fuzzy approximation space was made by A. Nakamura in 1988 [8]. D. Dubois and H. Prade [3] gave another definition by incorporating the membership values of the fuzzy equivalence relation in the definition of fuzzy rough approximations. Subsequently, substantial research has been done in this direction and many extensions and applications have been proposed [4,6,12,15,16,17]. A comprehensive study of the different approaches can be found in L. D'eer et al [2]. The present authors defined fuzzy rough sets on an information system based on the divergence between the fuzzy sets corresponding to the attributes and applied it to feature selection [13,14].

In this paper, the concept of divergence based fuzzy rough sets is extended to fuzzy approximation spaces. New fuzzy rough approximations of a fuzzy set in a fuzzy approximation space are defined using normalized divergence measures of fuzzy sets. The properties of the divergence based fuzzy rough approximations are explored. Moreover, it is proved that the present approach is a generalization of the crisp rough set approximations and the fuzzy rough approximations. Also, the proposed approximations of a fuzzy set are found to be nearer to the set than the existing fuzzy rough approximations. The rest of the paper is organised as follows: Section 2 gives some preliminary definitions and a brief review of the existing fuzzy rough set models. The concept of divergence based fuzzy rough sets in a fuzzy approximation space are introduced in section 3 and their properties are studied. Section 4 gives the conclusion.

II. PRELIMINARIES

This section gives some preliminary concepts. The basic notions of rough set theory and fuzzy set theory as described in [3] and [9] respectively, are followed throughout this paper.

A. *Fuzzy Set Theory*

Definition 1. Let U be a non-empty finite set of objects and R be a fuzzy equivalence relation on U. The *fuzzy equivalence classes* [5] of R are defined $\forall x \in U$ as, $[x]_R(y) = R(x, y),\ \forall y \in U$.
Definition 2. Let $\mathcal{F}(U)$ be the family of all fuzzy sets on U. Then a function $D: \mathcal{F}(U) \times F(U) \to R$ is a *divergence measure* [7] if and only if $\forall A, B \in \mathcal{F}(U)$,

i) $D(A,B) = D(B,A)$
ii) $D(A,A) = 0$
iii) $max\{D(A \cup C, B \cup C), D(A \cap C, B \cap C)\} \leq D(A,B)$.

B. Rough Set Theory

Definition 3. Let (U,R) be an approximation space, where R is an equivalence relation defined on U. The *lower and upper approximations* [11] of $A \subseteq U$ with respect to R are respectively defined as

$$\underline{R}(A) = \{x \in U : [x]_R \subseteq A\} \quad (1)$$

$$\overline{R}(A) = \{x \in U : [x]_R \cap A \neq \emptyset\}. \quad (2)$$

Definition 4. For $A \subseteq U$, the sets $\underline{R}(A)$, $U - \overline{R}(A)$, and $\overline{R}(A) - \underline{R}(A)$ are respectively called the *positive region* (*POS*(*A*)), the *negative region* (*NEG*(*A*)) and the *boundary region* (*BND*(*A*)) of A with respect to R [11].

C. Fuzzy Rough Sets

Fuzzy rough sets incorporate the two distinct but related concepts of vagueness and indiscernibility. A fuzzy approximation space is a pair (U,R), where U is a non-empty set of objects and R is a fuzzy equivalence relation. The first attempt to define fuzzy rough sets in a fuzzy approximation space was done by A. Nakamura [8]. He defined the lower and upper approximations of a fuzzy set A on U as the fuzzy sets on U respectively given by $\mu_{\underline{R}(A)}(x) = inf_{R(x,y)\geq\alpha}\{\mu_A y\}$ and $\mu_{\overline{R}(A)}(x) = sup_{R(x,y)\geq\alpha}\{\mu_A(y)\}$.

Later, Dubois and Prade [2] defined fuzzy rough approximations as

$$\mu_{\underline{R}(A)}(x) = inf_{y\in U}\{\max[1 - R(x,y), \mu_A(y)]\} \quad (3)$$

$$\mu_{\overline{R}(A)}(x) = sup_{y\in U}\{\min[R(x,y), \mu_A(y)]\} \quad (4)$$

Throughout this paper, fuzzy rough approximations refer to the approximations given by equations (3) and (4).

III. DIVERGENCE BASED FUZZY ROUGH APPROXIMATIONS

Consider a fuzzy approximation space (U,R), U being a non-empty finite set of objects and R being a fuzzy equivalence relation on U. Let $D(A,B)$ be a normalized measure of divergence between fuzzy sets. Then, $D([x]_R,[y]_R)$ measure the extent of dissimilarity between the objects x and y, with respect to the corresponding fuzzy equivalence classes.

We define a function $D_x : U \to [0,1]$ for each $x \in U$, by

$$D_x(y) = D([x]_R,[y]_R), \forall y \in U. \quad (5)$$

Definition 5. The DFR-lower and upper approximations $\underline{R_D}(A)$ and $\overline{R_D}(A)$ of $A \in \mathcal{F}(U)$, corresponding to the divergence measure D are respectively defined $\forall x \in U$ as

$$\mu_{\underline{R_D}(A)}(x) = inf_{y\in U}\{\max[D_x(y), \mu_A(y)]\} \quad (6)$$

$$\mu_{\overline{R_D}(A)}(x) = sup_{y\in U}\{\min[1 - D_x(y), \mu_A(y)]\}. \quad (7)$$

The following proposition shows that the above approximations are fuzzy subsets of U.

Proposition 1. *In a fuzzy approximation space, the DFR-lower and upper approximations of a fuzzy set are fuzzy subsets of* U.

Proof: Since $\mu_A(y)$, $D_x(y) \in [0,1]$, $\forall x,y \in U$, we get, $max[D_x(y), \mu_A(y)] \in [0,1]$.
Using equation (6), $\mu_{\underline{R_D}(A)}(x) \in [0,1], \forall x \in U$. Similarly, $\mu_{\overline{R_D}(A)}(x) \in [0,1], \forall x \in U$.

The DFR-approximations are different from the original fuzzy rough approximations. This fact is illustrated in the following example.

Example 1. Consider $U = \{a_1, a_2, a_3, a_4\}$ and define a fuzzy equivalence relation R as

R	a_1	a_2	a_3	a_4
a_1	1	.8	0	.4

a_2	.8	1	0	.4
a_3	0	0	1	0
a_4	.4	.4	0	1

Consider the t-conorm given by $\mathfrak{T}(m,n) = \min(1, m+n)$ for $m,n \in [0,1]$ and let $D(A,B) = \mathfrak{T}_{x\in U}|A(x) - B(x)|$. Consider the fuzzy set $A = \{(a_1,.2),(a_2,.1),(a_3,.6),(a_4,.9)\}$. The DFR-approximations of A with respect to D are given by $\underline{R_D}(A) = \{(a_1,.2),(a_2,.1),(a_3,.6),(a_4,.9)\}$ and $\overline{R_D}(A) = \{(a_1,.2),(a_2,.2),(a_3,.6),(a_4,.9)\}$ respectively. The fuzzy rough approximations of A computed using equations (3) and (4) are given by $\underline{R}(A) = \{(a_1,.2),(a_2,.1),(a_3,.6),(a_4,.6)\}$ and $\overline{R}(A) = \{(a_1,.4),(a_2,.4),(a_3,.6),(a_4,.9)\}$ respectively. It is obvious that, $\underline{R_D}(A) \neq \underline{R}(A)$ and $\overline{R_D}(A) \neq \overline{R}(A)$. Also, $\underline{R_D}(A) \supset \underline{R}(A)$ and $\overline{R_D}(A) \subset \overline{R}(A)$.

The following two theorems present the properties of the DFR-approximations.

Theorem 1. *Let φ and U denote the fuzzy empty set and the fuzzy universal set respectively and let $A,B \in \mathcal{F}(U)$. Then,*

i) $\underline{R_D}(\varphi) = \varphi = \overline{R_D}(\varphi)$, $\underline{R_D}(U) = U = \overline{R_D}(U)$

ii) $\underline{R_D}(A) \subseteq A \subseteq \overline{R_D}(A)$

iii) $A \subseteq B \Rightarrow \underline{R_D}(A) \subseteq \underline{R_D}(B)$ *and* $\overline{R_D}(A) \subseteq \overline{R_D}(B)$

iv) $\underline{R_D}(\hat{\alpha}) = \hat{\alpha} = \overline{R_D}(\hat{\alpha})$, $\forall \alpha \in [0,1]$

v) $\left(\underline{R_D}(A^C)\right)^C = \overline{R_D}(A)$ *and* $\left(\overline{R_D}(A^C)\right)^C = \underline{R_D}(A)$

Proof.

i) We have, $\mu_\varphi(x) = 0$ and $D_x(x) = 0, \forall x \in U$. Hence, $max[D_x(x), \mu_\varphi(x)] = 0$.
Therefore, $\mu_{\underline{R_D}(\varphi)}(x) = inf_{y\in U}\{max[D_x(y), \mu_\varphi(y)]\} = 0$.
Also, $min[1 - D_x(x), \mu_\varphi(x)] = 0,\ \forall x \in U$. So, $\mu_{\overline{R_D}(\varphi)}(x) = sup_{y\in U}\{min[1 - D_x(y), \mu_\varphi(y)]\} = 0$.
Thus, $\underline{R_D}(\varphi) = \varphi = \overline{R_D}(\varphi)$.
Also, $\mu_U(y) = 1, \forall y \in U \Rightarrow max[D_x(y), \mu_U(y)] = 1, \forall y \in U$.
Therefore, $\mu_{\underline{R_D}(U)}(x) = inf_{y\in U}\ max[D_x(y), \mu_U(y)] = 1$.
Since $min[1 - D_x(y),\ \mu_U(y)] = 1 - D_x(y), \forall y \in U$, $\mu_{\overline{R_D}(U)}(x) = sup_{y\in U}\{1 - D_x(y)\}$.
Also, $D_x(x) = 0$. Therefore, $\mu_{\overline{R_D}(U)}(x) = 1, \forall x \in U$. Thus, $\underline{R_D}(U) = U = \overline{R_D}(U)$.

ii) $D_x(x) = 0 \Rightarrow max[D_x(x), \mu_A(x)] = \mu_A(x), min[1 - D_x(x), \mu_A(x)] = \mu_A(x), \forall x \in U$.
Hence, $\mu_{\underline{R_D}(A)}(x) \leq \mu_A(x), \mu_{\overline{R_D}(A)}(x) \geq \mu_A(x), \forall x \in U$.
Thus, $\underline{R_D}(A) \subseteq A \subseteq \overline{R_D}(A), \forall A \in \mathcal{F}(U)$.

iii) If $A \subseteq B$, then $\mu_A(y) \leq \mu_B(y), \forall y \in U$.
So, $\forall x \in U$, $max[D_x(y), \mu_A(y)] \leq max[D_x(y), \mu_B(y)]$ and $min[D_x(y), \mu_A(y)] \leq min[D_x(y), \mu_B(y)]$.
It follows that $\mu_{\underline{R_D}(A)}(x)) \leq \mu_{\underline{R_D}(B)}(x)$ and $\mu_{\overline{R_D}(A)}(x)) \leq \mu_{\overline{R_D}(B)}(x)$.
Hence, $\underline{R_D}(A) \subseteq \underline{R_D}(B)$ *and* $\overline{R_D}(A) \subseteq \overline{R_D}(B)$

iv) $\mu_{\hat{\alpha}}(y) = \alpha,\ \forall x \in U \Rightarrow max[D_x(x), \mu_{\hat{\alpha}}(y)] = max[D_x(x), \alpha] \geq \alpha = \mu_{\hat{\alpha}}(x)\ ,\ \forall y \in U$.
So, $\mu_{\underline{R_D}(\hat{\alpha})}(x) = inf_{y\in U}\{max[D_x(y), \mu_{\hat{\alpha}}(y)]\} \geq \mu_{\hat{\alpha}}(x)$. Hence, $\underline{R_D}(\hat{\alpha}) \supseteq \hat{\alpha}$.
Using property *(ii)*, $\underline{R_D}(\hat{\alpha}) \subseteq \hat{\alpha}$. Thus, $\underline{R_D}(\hat{\alpha}) = \hat{\alpha}$.
Similarly, $\overline{R_D}(\hat{\alpha}) = \hat{\alpha}$.

v) $\forall x \in U,\ \mu_{\left(\underline{R_D}(A^C)\right)^C}(x) = 1 - \mu_{\underline{R_D}(A^C)}(x) = 1 - inf_{y\in U}\ max[D_x(y), \mu_{A^C}(y)]$

$$= sup_{y\in U}\{1 - max[D_x(y), \mu_{A^C}(y)]\} = sup_{y\in U}\{min[1 - D_x(y), 1 - \mu_{A^C}(y)]\}$$

$= sup_{y\in U}\{min[1 - D_x(y), \mu_A(y)]\} = \mu_{\overline{R_D}(A)}(x))$.

Therefore, $\left(\underline{R_D}(A^C)\right)^C = \overline{R_D}(A), \ \forall A \in \mathcal{F}(U)$.

Similarly, $\left(\overline{R_D}(A^C)\right)^C = \underline{R_D}(A), \forall A \in \mathcal{F}(U)$.

Theorem 2. For all $A, B \in \mathcal{F}(U)$,

i) $\underline{R_D}(A \cap B) = \underline{R_D}(A) \cap \underline{R_D}(B)$

ii) $\overline{R_D}(A \cap B) \subseteq \overline{R_D}(A) \cap \overline{R_D}(B)$

iii) $\underline{R_D}(A \cup B) \supseteq \underline{R_D}(A) \cup \underline{R_D}(B)$

iv) $\overline{R_D}(A \cup B) = \overline{R_D}(A) \cup \overline{R_D}(B)$

v) $\overline{R_D}(A \cap \hat{\alpha}) = \overline{R_D}(A) \cap \hat{\alpha}$

vi) $\underline{R_D}(A \cup \hat{\alpha}) = \underline{R_D}(A) \cup \hat{\alpha}$

Proof.

i) $\mu_{\underline{R_D}(A\cap B)}(x) = inf_{y\in U}\{max[D_x(y), \mu_{A\cap B}(y)]\} = inf_{y\in U}\left\{max\left[D_x(y), \min\left(\mu_A(y), \mu_B(y)\right)\right]\right\}$

$= inf_{y\in U}\left\{min\left[max\left(D_x(y), \mu_A(y)\right), max\left(D_x(y), \mu_B(y)\right)\right]\right\}$

$= min\left\{inf_{y\in U}\left[max\left(D_x(y), \mu_A(y)\right)\right], inf_{y\in U}\left[max\left(D_x(y), \mu_B(y)\right)\right]\right\}$

$= min\{\mu_{\overline{R_D}(A)}(x), \mu_{\overline{R_D}(B)}(x)\} = \mu_{\left(\underline{R_D}(A)\cap\underline{R_D}(B)\right)}(x)$.

Thus, $\underline{R_D}(A \cap B) = \underline{R_D}(A) \cap \underline{R_D}(B)$

ii) $\mu_{\overline{R_D}(A\cap B)}(x)) = sup_{y\in U}\{min[1 - D_x(y), \mu_{A\cap B}(y)]\} = sup_{y\in U}\left\{min\left[1 - D_x(y), \min\left(\mu_A(y), \mu_B(y)\right)\right]\right\}$

$= sup_{y\in U}\left\{min\left[min\left(1 - D_x(y), \mu_A(y)\right), min\left(1 - D_x(y), \mu_B(y)\right)\right]\right\}$

$\leq min\left\{sup_{y\in U}\left[min\left(1 - D_x(y), \mu_A(y)\right)\right], sup_{y\in U}\left[min\left(1 - D_x(y), \mu_B(y)\right)\right]\right\}$

$= min\{\mu_{\overline{R_D}(A)}(x)), \mu_{\overline{R_D}(B)}(x))\} = \mu_{\left(\overline{R_D}(A)\cap\overline{R_D}(B)\right)}(x)$

Thus, $\overline{R_D}(A \cap B) \subseteq \overline{R_D}(A) \cap \overline{R_D}(B)$.

iii) $\mu_{\underline{R_D}(A\cup B)}(x) = inf_{y\in U}\{max[D_x(y), \mu_{A\cup B}(y)]\} = inf_{y\in U}\left\{max\left[D_x(y), max\left(\mu_A(y), \mu_B(y)\right)\right]\right\}$

$= inf_{y\in U}\left\{max\left[max\left(D_x(y), \mu_A(y)\right), max\left(D_x(y), \mu_B(y)\right)\right]\right\}$

$\geq max\left\{inf_{y\in U}\left[max\left(D_x(y), \mu_A(y)\right)\right], inf_{y\in U}\left[max\left(D_x(y), \mu_B(y)\right)\right]\right\}$

$= max\left\{\mu_{\underline{R_D}(A)}(x), \mu_{\underline{R_D}(B)}(x)\right\} = \mu_{\left(\underline{R_D}(A)\cup\underline{R_D}(B)\right)}(x)$.

Thus, $\underline{R_D}(A \cup B) \supseteq \underline{R_D}(A) \cup \underline{R_D}(B)$.

iv) $\mu_{\overline{R_D}(A\cup B)}(x) = sup_{y\in U}\{min[1 - D_x(y), \mu_{A\cup B}(y)]\} = sup_{y\in U}\left\{min\left[1 - D_x(y), max\left(\mu_A(y), \mu_B(y)\right)\right]\right\}$

$= sup_{y\in U}\left\{max\left[min\left(1 - D_x(y), \mu_A(y)\right), min\left(1 - D_x(y), \mu_B(y)\right)\right]\right\}$

$= max\left\{sup_{y\in U}\left[min\left(1 - D_x(y), \mu_A(y)\right)\right], sup_{y\in U}\left[min\left(1 - D_x(y), \mu_B(y)\right)\right]\right\}$

$= max\{\mu_{\overline{R_D}(A)}(x), \mu_{\overline{R_D}(B)}(x)\} = \mu_{\left(\overline{R_D}(A)\cup\overline{R_D}(B)\right)}(x)$.

Thus, $\overline{R_D}(A \cup B) = \overline{R_D}(A) \cup \overline{R_D}(B)$.

v) $\mu_{\overline{R_D}(A\cap\hat{\alpha})}(x)) = sup_{y\in U}\{min[1-D_x(y),\mu_{A\cap\hat{\alpha}}(y)]\} = sup_{y\in U}\left\{min\left[1-D_x(y),\min(\mu_A(y),\alpha)\right]\right\}$

$= sup_{y\in U}\left\{min\left[min\left(1-D_x(y),\mu_A(y)\right),\alpha\right]\right\}$

$= min\left\{sup_{y\in U}\left[min\left(1-D_x(y),\mu_A(y)\right)\right],\alpha\right\}$

$= min\{\mu_{\overline{R_D}(A)}(x)),\mu_{\hat{\alpha}}(x))\} = \mu_{\overline{R_D}(A)\cap\hat{\alpha}}(x)$

Thus, $\overline{R_D}(A\cap\hat{\alpha}) = \overline{R_D}(A)\cap\hat{\alpha}$

vi) The proof is similar to that of *(v)*

Theorem 3. *If D and D' are two measures of divergence of fuzzy sets and if $D(A,B) \le D'(A,B), \forall A,B \in \mathcal{F}(U)$, then, $\underline{R_D}(A) \le \underline{R_{D'}}(A)$ and $\overline{R_D}((A) \ge \overline{R_{D'}}((A)$.*

Proof.

Given, $D(A,B) \le D'(A,B), \forall A,B \in \mathcal{F}(U)$. So, $D_x(y) \le D'_x(y)$, and $1-D_x(y) \ge 1-D'_x(y), \forall\, x,y \in U$.
It follows that, $max\left(D_x(y),\mu_A(y)\right) \le max\left(D'_x(y),\mu_A(y)\right)$ and $min\left(1-D_x(y),\mu_A(y)\right) \ge min\left(1-D'_x(y),\mu_A(y)\right)$.

By the property of infimum and supremum, $\underline{R_D}(A) \le \underline{R_{D'}}(A)$ and $\overline{R_D}(A) \ge \overline{R_{D'}}(A)$.

The DFR-approximations are generalizations of the crisp rough approximations. This is proved in the next theorem.

Lemma 1. *Consider a crisp equivalence relation R on U. Let $\mathfrak{T}$ be a fuzzy t-conorm and $D(A,B) = \mathfrak{T}_{x\in U}|\mu_A(x)-\mu_B(x)|$. Then, $D_x(y) = 0, \forall y \in [x]_R$ and $D_x(y) = 1, \forall y \notin [x]_R$.*

Proof.

We have, $D_x(y) = D([x]_R,[y]_R) = \mathfrak{T}_{z\in U}|\mu_{[x]_R}(z)-\mu_{[y]_R}(z)| = \mathfrak{T}_{z\in U}|R(x,z)-R(y,z)|$.

The characteristic function of R acts as the corresponding fuzzy equivalence relation. That is;

$$R(x,y) = \chi_R(x,y) = \begin{cases} 1, if\ (x,y)\in R \\ 0, otherwise \end{cases}$$

Also, $y \in [x]_R \Rightarrow [x]_R = [y]_R$. Hence, $D_x(y) = D([x]_R,[y]_R) = D([x]_R,[x]_R) = 0$.

If $y \notin [x]_R$, then $[x]_R \cap [y]_R = \varphi$.

So, for $z \in U$, there are three cases namely, $z \in [x]_R$, $z \in [y]_R$ and $z \in ([x]_R \cup [y]_R)^C$.

If $z \in [x]_R$, then $z \notin [y]_R$. So, $R(x,z) = 1$ and $R(y,z) = 0$. Therefore, $|R(x,z)-R(y,z)| = 1$.

Similarly, if $z \in [y]_R$, $|R(x,z)-R(y,z)| = 1$. If $z \in ([x]_R \cup [y]_R)^C$, then $R(x,z) = R(y,z) = 0$.

Hence, $|R(x,z) - R(y,z)| = 0$.

Therefore $D_x(y) = \mathfrak{T}_{z\in([x]_R\cup[y]_R)}|R(x,z)- R(y,z)| = 1$.

Theorem 4. *In the crisp case, the divergence based fuzzy rough approximations with respect to the divergence measure $D(A,B) = \mathfrak{T}_{x\in U}|\mu_A(x)-\mu_B(x)|$ coincides with Pawlak's rough set approximations.*

Proof.

Let U be a non-empty finite set of objects and R be a crisp equivalence relation defined on U. The fuzzy equivalence relation corresponding to R is given by equation (6). Let $A \subseteq U$ be a crisp set. The fuzzy set corresponding to A is given by the characteristic function,

$$\mu_A(x) = \chi_A(x) = \begin{cases} 1, if\ x \in A \\ 0, otherwise \end{cases} \tag{7}$$

The fuzzy set corresponding to the crisp rough lower approximation of A is given by

$$\mu_{\underline{R}(A)}(x) = \chi_{\underline{R}(A)}(x) = \begin{cases} 1, if\ [x]_R \subseteq A \\ 0, otherwise \end{cases} \tag{8}$$

The DFR-lower approximation of A is obtained using equation (4) as $\mu_{\underline{R_D}(A)}(x) = inf_{y\in U}\{max[D_x(y),\mu_A(y)]\}$.

Let $[x]_R \subseteq A$. If $y \in [x]_R$, then $y \in A$ and $\mu_A(y) = 1$. So, $max[D_x(y),\mu_A(y)] = 1$.

If $y \notin [x]_R$, then $D_x(y) = 1$, using lemma 1. Hence, $max[D_x(y),\mu_A(y)] = 1$.

Thus, $max[D_x(y), \mu_A(y)] = 1, \forall y \in U$.
Therefore, $\mu_{\underline{R_D}(A)}(x) = 1$.

Now let $[x]_R \nsubseteq A$. Then there exists $z \in [x]_R$ such that $z \notin A$. Clearly, $\mu_A(z) = 0$. Also, $D_x(z) = 0$ by lemma 1.
Thus $max[D_x(z), \mu_A(z)] = 0$. By the property of infimum, $\mu_{\underline{R_D}(A)}(x) = 0$.

Thus, we get, $\mu_{\underline{R_D}(A)}(x) = \begin{cases} 1, if\ [x]_R \subseteq A \\ 0, otherwise \end{cases} = \mu_{\underline{R}(A)}(x)$.

Similarly, we can prove that $\mu_{\overline{R_D}(A)}(x) = \mu_{\overline{R}(A)}(x)$.

Theorem 5. *Let $\mathfrak{T}$ be a fuzzy t-conorm and consider the class of divergence measures $D(A,B) = \mathfrak{T}_{x\in U}|\mu_A(x) - \mu_B(x)|$. Then,* $\underline{R}(A) \subseteq \underline{R_D}(A) \subseteq A \subseteq \overline{R_D}(A) \subseteq \overline{R}(A)$.

Proof.
We have, $D_x(y) = D([x]_R, [y]_R) = \mathfrak{T}_{z\in U}|R(x,z) - R(y,z)|$.
Since R is reflexive and symmetric, for $z = x$, $|R(x,z) - R(y,z)| = |R(x,x) - R(y,x)| = 1 - R(x,y)$.
So, $D_x(y) \geq 1 - R(x,y)$, as $\mathfrak{T}$ is a t-conorm.
Hence, $max[D_x(y), \mu_A(y)] \geq max[1 - R(x,y), \mu_A(y)], \forall y \in U$.
It follows that, $inf_{y\in U}\left[max\big(D_x(y), \mu_A(y)\big)\right] \geq inf_{y\in U}\left[max\big(1 - R(x,y), \mu_A(y)\big)\right], \forall y \in U$.

Using equations (1) and (4) we get, $\underline{R_D}(A) \supseteq \underline{R}(A)$. Similarly, $\overline{R_D}(A) \subseteq \overline{R}(A)$.

Therefore, from property *(ii)* of theorem 3.4, we get $\underline{R}(A) \subseteq \underline{R_D}(A) \subseteq A \subseteq \overline{R_D}(A) \subseteq \overline{R}(A)$.

Next, we prove that the fuzzy rough approximations are the particular case of the proposed DFR-approximations corresponding to the divergence measure using the max operator as t-conorm. Thus, the DFR-approximations are the generalizations of the fuzzy rough approximations as well.

Theorem 6. *For the divergence measure of fuzzy sets defined by $D(A,B) = max_{x\in U}|\mu_A(x) - \mu_B(x)|$, $\underline{R}(A) = \underline{R_D}(A)$ and $\overline{R_D}(A) = \overline{R}(A)$.*

Proof.
Let $A \in \mathcal{F}(U)$. It is enough to prove that $D_x(y) = 1 - R(x,y),\ \forall x,y \in U$.
We have, $D_x(y) = D([x]_R, [y]_R) = max_{z\in U}|R(x,z) - R(y,z)|$.
As U is finite, this maximum corresponds to some $z' \in U$.
We may also assume that $R(x,z') \geq R(y,z')$. So, $D_x(y) = R(x,z') - R(y,z')$.
Then, R is transitive $\Rightarrow R(x,y) \geq max_{z\in U} min(R(x,z), R(z,y))$
$\Rightarrow R(x,y) \geq min(R(x,z), R(z,y)), \forall z \in U$
$\Rightarrow R(x,y) \geq min(R(x,z'), R(z',y)) \Rightarrow R(x,y) \geq R(z',y)$.
At this point, there arise two cases:
Case I- $R(x,y) \geq R(x,z') \geq R(y,z')$. R being transitive, $R(y,z') \geq min(R(y,x), R(x,z'))$.
This can happen only if $R(x,z') = R(y,z')$. Then, $x \in [z']_R$ and $y \in [z']_R$.
So, $D_x(y) = 0$ and as $D_x(y) \geq 1 - R(x,y)$, $1 - R(x,y) = 0$.
Therefore, $D_x(y) = 1 - R(x,y)$.
Case II- $R(x,z') \geq R(x,y) \geq R(y,z')$. R being transitive, $R(y,z') \geq min(R(y,x), R(x,z'))$.
This can happen only if $R(x,y) = R(y,z')$.
So, $D_x(y) = R(x,z') - R(y,z') = R(x,z') - R(x,y) \leq 1 - R(x,y)$.
Since $D_x(y) \geq 1 - R(x,y)$, $D_x(y) = 1 - R(x,y)$.
It follows that $D_x(y) = 1 - R(x,y), \forall x,y \in U$ and hence, $\underline{R}(A) = \underline{R_D}(A)$ *and* $\overline{R_D}(A) = \overline{R}(A)$.

VI. CONCLUSIONS

The fuzzy rough set theory has been found to be a very effective tool for dealing with imperfect knowledge. In this paper, the DFR-approximations of a fuzzy set in a fuzzy approximation space have been defined using measures of divergence between fuzzy sets. Further, it has been verified that the proposed approach is a generalization of the fuzzy rough approximations proposed by Dubois and Prade and they coincide with Pawlak's rough set approximations in the crisp case. Moreover, the divergence based fuzzy rough approximations of a fuzzy set were found to be nearer to the set than the existing fuzzy rough approximations. The future work includes further studies on divergence based fuzzy rough sets and their applications to decision making.

REFERENCES

[1] Rough Sets on Topological Spaces, K. Anitha , P. Venkatesan, International Journal of Mathematics Trends and Technology (IJMTT) 10 (1) (2014) 47-50.
[2] L. D'eer, N. Verbiest, C. Cornelis and L. Godo, A Comprehensive Study of Implicator Conjunctor-based and Noise-Tolerant Fuzzy Rough Sets: Definitions, Properties and Robustness Analysis, Fuzzy Sets and Systems 275 (2015) 1-38.
[3] D. Dubois and H. Prade, Rough Fuzzy Sets and Fuzzy Rough Sets, International Journal of General Systems 17 (1990) 191-209.
[4] R. Jensen and Q. Shen, New Approaches to Fuzzy-Rough Feature Selection, IEEE Transactions on Fuzzy Systems, 17(4) (2009) 824-838.
[5] J. Klir and B. Yuan, Fuzzy Sets and Fuzzy Logic, Prentice Hall, New Jersey, 1995.
[6] J.S. Mi, Y. Leung, H.Y. Zhao and T. Feng, Generalized Fuzzy Rough Sets Determined by a TriangularNorm, Information Sciences 178 (2008) 3203-3213.
[7] S. Montes, I. Couso, P. Gil and C. Bertoluzza, Divergence Measure Between Fuzzy Sets, International Journal of Approximate Reasoning 30 (2002) 91-105.
[8] A. Nakamura, "Fuzzy Rough Sets", Note on Multiple-Valued Logic in Japan 9(8) (1988) 1-8.
[9] N. R. Paul, Rough Topology Based Decision Making in Medical Diagnosis, International Journal of Mathematics Trends and Technology (IJMTT). 18(2015) 40-43.
[10] Z. Pawlak, Rough Sets, International Journal of Computer and Information Sciences 11(5), (1982) 341 - 356.
[11] Z. Pawlak, Rough Sets - Theoretical Aspect of Reasoning About Data, Kluwer Academic Publishers The Netherlands, 1991.
[12] A. M. Radzikowska and E. E. Kerre, A Comparative Study of Fuzzy Rough Sets, Fuzzy Sets and Systems 126(2) (2002) 137-155.
[13] Sheeja. T. K, Sunny Kuriakose A., Divergence Based Generalized Fuzzy Rough Sets, International Journal of Computer Sciences and Engineering 6(6) (2018) 809-815.
[14] Sheeja. T. K, Sunny Kuriakose A., A Novel Feature Selection Method Using Fuzzy Rough Sets, Computers in Industry 97 (2018) 111-116.
[15] Sheeja. T. K, Sunny Kuriakose A., Image Segmentation Using a Fuzzy Roughness Measure, International Journal of Engineering and Advanced Technology 8(6) (2019) 307-311.
[16] S. Vluymans, L. D'eer, Y. Saeys and C. Cornelis, Applications of Fuzzy Rough Set Theory in Machine Learning: a Survey, Fundamenta Informaticae, 142 (2015) 53-86.
[17] W. Z. Wu, J. S. Mi and W. X. Zhang, Generalized Fuzzy Rough Sets, Information Sciences, 151 (2003) 263-282.
[18] Y. Y. Yao, S. K. M. Wong and T. Y. Lin, A Review of Rough Set Models, in Rough Sets and Data mining analysis for Imprecise Data, eds T. Y. Lin, and N. Cerone, Kluwer Academic Publishers, Boston (1997) 47-75.
[19] L. A. Zadeh, Fuzzy Sets, Information and Control 8(3) (1965) 338-353.
[20] Q. Zhang, Q. Xie and G. Y. Wang, "A Survey on Rough Set Theory and its Applications, CAAI Transactions on Intelligence Technology 1(4) (2016) 323 - 333.