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Planar and Spatial kinematic for vortex fluid flow behaviour by using perturbation parameter

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Abstract:
Our aim in this research paper is to elaborate two new kinematics of deformation of a particle fluid in a planar and spatial vortex for mathematical modelization of these natural phenomenons. Incompressibility condition, Rotational and divergence of velocity tensor are calculated in every case and in also two examples of fluid flow. The pressure is interpret by using a Bernoulli theorem. As results, we have: the same incompressible condition in the case of horizontal accelerated flow than in the case of a shearing flow and we have the pressure which decreases in the two motion. We have the same rotational between the planar vortex flow and the spatial vortex flow, what means that there is no influence of the z component in the rotional of these two vortex flows when $\varepsilon = 1$. *And we also show that for specific values of* ε *and* Θ, *we have the same values in calculated expressions between the planar vortex flow and the spatial vortex flow.*

Keywords:
Kinematic of deformation , isotropic elementary invariants, Incompressible deformation, Vortex flow, Rotational, Divergence, Bernoulli theorem.

1 Introduction

The notion of fluid refers to the absence of an organized structure of matter at the microscopic scale, thus allowing large-amplitude movements of atoms. It therefore that fluids group together the liquid and gaseous states [1].
The study of fluids plays a very important role nowadays because it allows the development of models for improving the performance of machines in the maritime, land or space fields. A fluid can be viscous, compressible or incompressible.
When we focus about viscosity, an exact analysis of radiative effects on the magnetohydrodynamic (MHD) free convection flow of an electrically conducting incompressible viscous fluid over a vertical plate is studied where the non-dimensional continuity, momentum, and energy equations are solved using appropriate transformation [2]. A solution in the explicit form of the equation of the momentum diffusion for a viscous fluid flowing around a plate taking into account deceleration with three characteristic regions of a viscous flow have been

given in [3].
In the case of incompressible flows, a new scalar projection method presented for simulating incompressible flows with variable density is proposed where the first phase of the projection is purely kinematics. The predicted velocity field is subjected to a discrete Hodge-Helmholtz decomposition [4]. A solution of an incompressible fluid flow is also studied in [5].
However the study of kinematics plays a very important role because it constitutes for most of the time, the starting point of a study in mechanics of continuous medium[6,7,8]
In this paper, we will firstly do a mathematical formulation of a mechanical fluid flow study by definitions of some tensors using in this area. We will secondly apply those formulations in twa examples of kinematics on fluid dynamic to determine incompressible condition and calculate rotational and divergence in any case.
As a contribution we will propose two new kinematics of deformation for vortex flow in order to bring a new tool in the modelization of vortex dynamic. The incompressible condition, the rotational and divergence will also be given for our new kinematics.
The theorem of Bernoulli will be used to analyze the behaviour of the pressure in every case.

2 Mathematical formulation

Let's consider a continuous system where a material point which can be a fluid particle occupies the position $\mathbf{X} = (X_1, X_2, X_3)$ before the deformation and the position $\mathbf{x} = (x_1, x_2, x_2)$ after deformation, where x_1,x_2,x_3 are function of $\mathbf{X}$, we mean defined by the following kinematics:

$$x_1 = x_1\left(\mathbf{X}\right); \quad x_2 = x_2\left(\mathbf{X}\right); \quad x_3 = x_3\left(\mathbf{X}\right); \tag{1}$$

By deforming in space and in time, the fluid paticle obtains a speed $\mathbf{v}$ which depends on the deformed configuration and therefore on the initial configuration. This speed is represented by:

$$v_1 = v_1\left(\mathbf{X}\right); \quad v_2 = v_2\left(\mathbf{X}\right); \quad v_3 = v_3\left(\mathbf{X}\right); \tag{2}$$

where $v_i = \partial x_i/\partial t$.
To describe the local deformation, we use the deformation gradient tensor noted $\mathbf{F}$ which is also called the tangent linear application. This tensor allow to have the volume change from his determinant which is always positive, so that:

$$\mathbf{F} = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix}. \tag{3}$$

In continuum mechanics, it is important to specify that from the deformation gradient tensor $\mathbf{F}$, We can measure the strain rate by calculating the familiar tensors that are: right Cauchy-Green tensor in Lagrangian configuration which is noted $\mathbf{C}$ and left Cauchy-Green tensor in Eulerian configuration which is noted $\mathbf{B}$, with:

$$\mathbf{C} = \mathbf{F}^{\mathbf{T}}\mathbf{F}; \qquad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathbf{T}}. \tag{4}$$

It should be noted that these two tensors although they differ by their formula describe the study in an equivalent way.
These Cauchy-Green tensors have adjoints tensors which are noted $\mathbf{C}^*$ and $\mathbf{B}^*$ defined by:

$$\begin{aligned} \mathbf{C}^* &= det\,(\mathbf{C})\,\mathbf{C}^{-1}; \\ \mathbf{B}^* &= det\,(\mathbf{B})\,\mathbf{B}^{-1}. \end{aligned} \tag{5}$$

In isotropic deformation which concern fluid flow, we can calculate the three first elementary invariants which are:

$$\begin{aligned} I_1 &= tr(\mathbf{C}) = tr(\mathbf{B}); \\ I_2 &= tr(\mathbf{C}^*) = tr(\mathbf{B}^*); \\ I_3 &= det\,(\mathbf{C}) = det\,(\mathbf{B})\,. \end{aligned} \tag{6}$$

In mechanics, a deformation is said to be incompressible if the third isotropic invariant is equal to 1.

$$I_3 = 1. \tag{7}$$

These previous mathematical tensors play a very important role in mechanics and engineering.
As you can see, we have defined only the isotropic elementary invariants, this is due to the fact that fluids are considered to be isotropic media due to the ability to move in all directions (there is no preferred direction for movement).

For what's coming, we give two examples of kinematics of deformation on fluid flow where we will study the speed, the condition of incompressibility, the rotational and the divergence. The same will be done for two new kinematics which represents our contribution in the modeling of planar and spatial vortex phenomenons. For the interpretation of the pressure from the speed we will use the theorem of Bernoulli which says that during the flow of a fluid the pressure decreases with the increase of the speed.

3 Some fluid flow kinematic

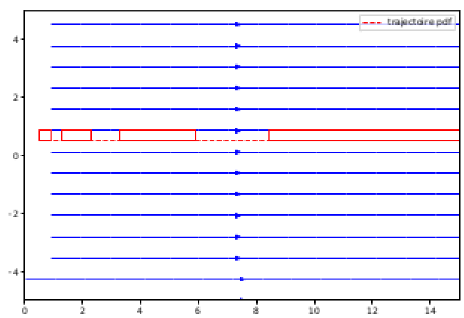
In this section, we give some kinematics of fluid flow find in the literature. We focus on the study of incopressibility behaviour condition and calculation of some expressions as rotational and divergence. the trajectory and vilocity are also similated.

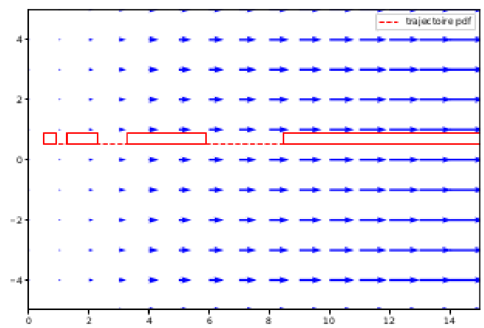
3.1 Accelerated horizontal flow

In fluid dynamic, a fluid particle can follow a rectilinear trajectory in a constant, accelerated or decelereted motion.
Let's study a fuid particle flow accelerated horizontaly as a water droplet on water surface discribing a trajectory given by the following kinematic:

$$x = \alpha X t;\quad y = Y;\quad z = Z; \tag{8}$$

where α is the initial vilocity and t the time variable. The behaviour is given by the following graphics where we can see the rectilinear motion.





As usually, to calculate the vilocity components, we derivate by the time every component of the kinematic of defomation above. From that we obtain:

$$v_x = \alpha X;\quad v_y = 0;\quad v_z = 0; \tag{9}$$

Here we can see that the speed depends only on the one component which is that of the variable X but also we have the others components which are null. The deformation gradient of this transformation becomes:

$$\mathbf{F} = \begin{pmatrix} \alpha t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

According to our kinematic, the gradient tensor has only the main components which are not null; mean its gives a digonal matrix.
The Langrangian tensor of Cauchy-Green given from the gradient tensor by

$$\mathbf{C} = \begin{pmatrix} (\alpha t)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

In order to find the condition of incompressibility which mean that there is no volume change of the fluid paticle, we calculate the isotropic invariants:

$$\begin{aligned} I_1 &= (\alpha t)^2 + 2; \\ I_2 &= 2(\alpha t)^2 + 1; \\ I_3 &= (\alpha t)^2. \end{aligned} \tag{12}$$

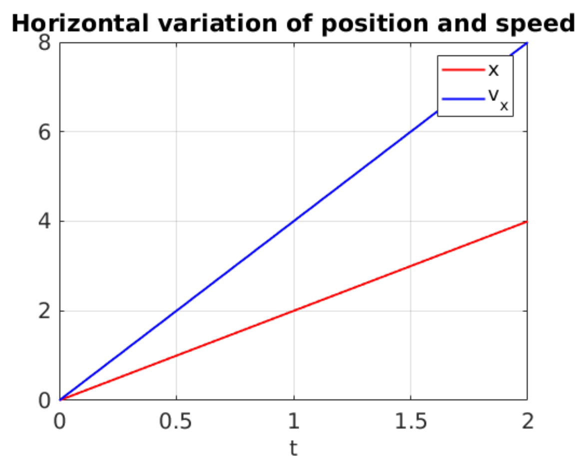
The fact that α and t are positive gives us the following incompressibility condition of a horizontal accelerated fluid flow :

$$t = \frac{1}{\alpha} \tag{13}$$

Even if we give the condition of incompressibility, we can see if that is easy or complicate or even impossible to have it by calculating the rotational and the divergence of the speed, that give:

$$\overrightarrow{rot}\,(\overrightarrow{v}) = \overrightarrow{0}; \qquad div\overrightarrow{v} = \alpha. \tag{14}$$

The rotational is null which goes in the same way than the kinematic which represents a rectilinear motion. The divergence is equal to $\alpha > 0$ and mathematically, that means that the volume change during the motion with a increasing volume.



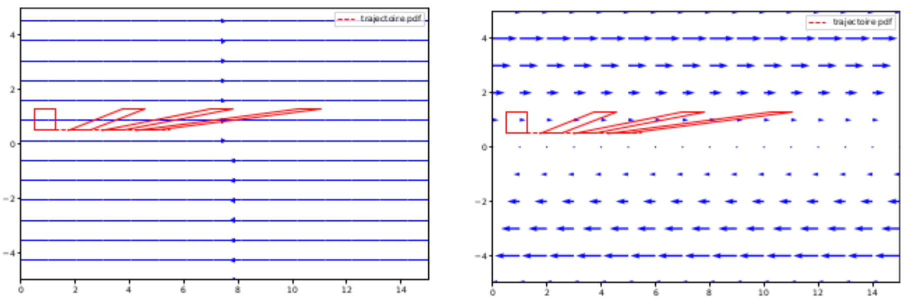
The graghic shows how the particle is accelerated during the motion with the path of the velocity, a behaviour which increases the volume of the fluid particle. As a consequence, the pressure of a horizontally accelerated fluid flow decreases horizontally during motion according to the theorem of Bernoulli.

3.2 Shearing flow

Now we consider a fluid particle which follows a shearing motion. As an example many cases can be given in aerodynamic.
Let's study a fuid particle flow with a shear represented by the kinematic below:

$$x = \alpha Y t; \quad y = X; \quad z = Z; \tag{15}$$

where also α is the initial vilocity and t the time variable. The kinematic is illustrated by the graphics below where we can see the red rectangles representing the trajectory.



In our shear motion we have the first cinematique component in deformed configuration which depends only to the second variable in initial configuration. what gives the following velocity components:

$$v_x = \alpha Y; \quad v_y = 0; \quad v_z = 0; \tag{16}$$

It is this previous dependence which allows us to have a shearing motion.
It follows the calculation of the deformation gradient and we obtain:

$$\mathbf{F} = \begin{pmatrix} 0 & \alpha t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{17}$$

Differently to what happen in the accelerated horizontal motion the deformation gradient tensor is not now a digonal matrix.
The Langrangian tensor of Cauchy-Green is given from the gradient tensor by

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\alpha t)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{18}$$

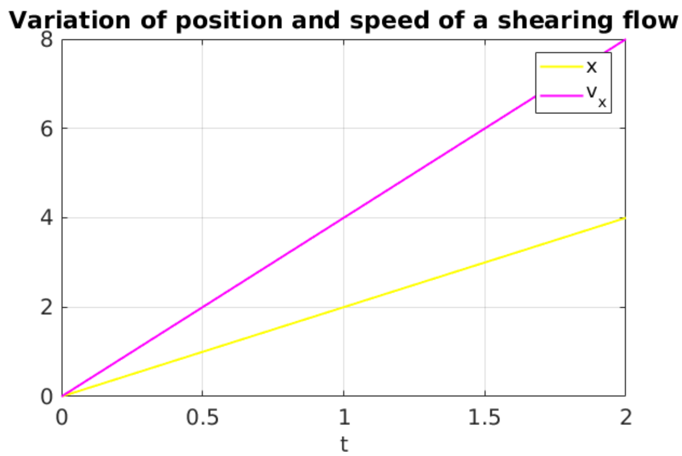
We can note that even if the deformation gradient tensor is not a digonal matrix, the Langragian Cauchy-Green tensor gives a digonal matrix.
The successif determination of the isotropic invariants and the incompressible condition gives:

$$\begin{aligned} I_1 &= (\alpha t)^2 + 2; \\ I_2 &= 2\,(\alpha t)^2 + 1; \\ I_3 &= (\alpha t)^2\,; \\ t &= \frac{1}{\alpha}. \end{aligned} \tag{19}$$

The difference of kinematic than in the case of the horizontal accelerated fluid flow have no influence because we obtain the same third isotropic invariant and then the same imcompressibilty condition.
The determination of the rotational and the divergence of the speed gives:

$$\overrightarrow{rot}\,(\overrightarrow{v}) = -\alpha \overrightarrow{e_z}; \qquad\qquad div\,\overrightarrow{v} = 0. \tag{20}$$

The result shows that there exist a rotation of our fluid particle but no divergence.
The lack of the divergence means that it more easy for the fluid particle to conserve is volume during all the motion.



The graphic shows that the velocity increases during the motion which mathematically means that we are in the case of an shearing motion with an acceleration.
The analysis of the speed shows that in the kinematics of shearing, we have the pressure of the fluid which decreases during the motion according to the theorem of Bernoulli.

Mathematically, these results mean that if α has the same value than t in all the period of the motion, the fluid particle will stay with the same volume in any time. As an interesting result, this part shows that when we can control an initial velocity of a fluid particle by increasing it in same way that the time, the particle will stay with the same volume in the two previous kinematics.

4 New kinematics for planar and spatial vortex

A vortex is a phenomenon which can be natural or generate by a technological engine and has many. A rotative flow is in fluid dynamics, a region of a fluid in which the flow is primarily a rotational movement around an axis, rectilinear or curved. This type of movement is called vortex flow.
In this section we propose two new kinematics of modelization of a particle inside a vortex depeding to the way that the radius decreases, increase or being constant and in the way we are in the planar or spatial vortex. Our aim is to give a kinematic which generalize a bihaviour of a fluid particle by using a pertubation parameter to controle the varition of the radius we need. We will

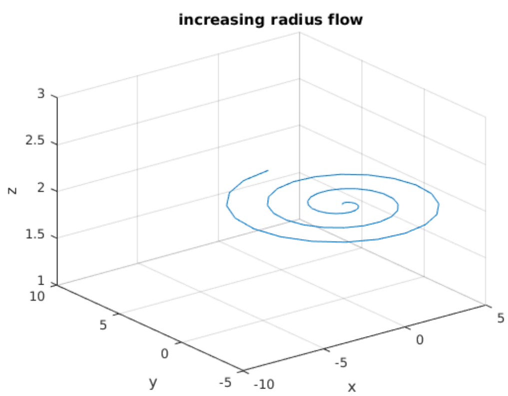
establish the condition on the paramater that gives an incompressible transformation on the particle. some similations will also be done.

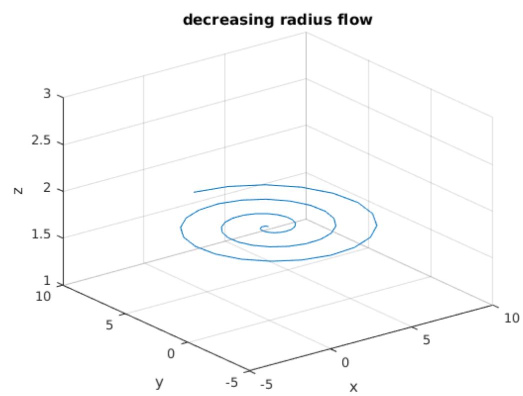
4.1 planar vortex flow

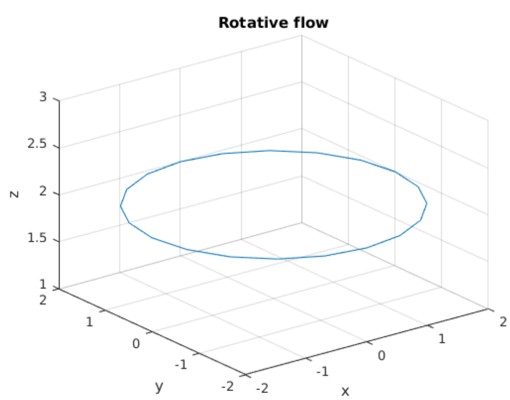
In this subsection, we consider a fuid particle with a rotative planar flow given by the kinematic below:

$$x = \varepsilon R cos\left(\Theta t\right); \quad y = \varepsilon R sin\left(\Theta t\right); \quad z = Z; \tag{21}$$

where t the time variable, R represents the radius supposed to be constant and ε the perturbation parameter.
The trajectory of the motion is given by the following graphics where we can see how the fluid particle rotate around an axis.

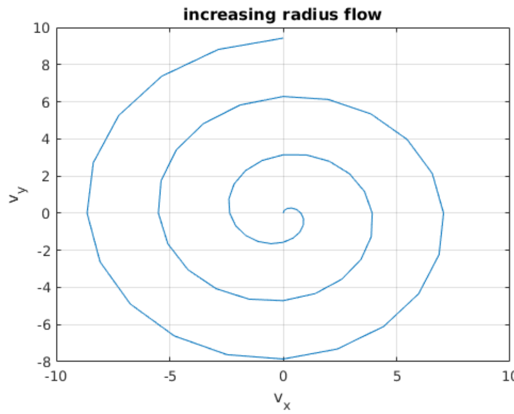


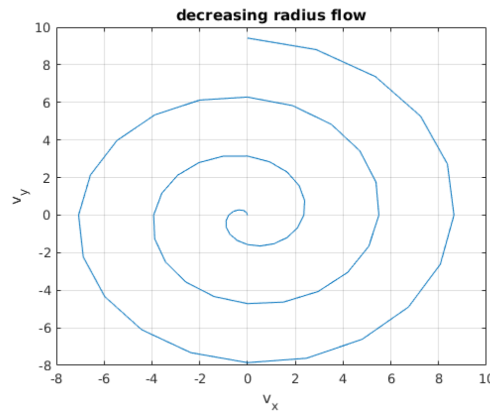


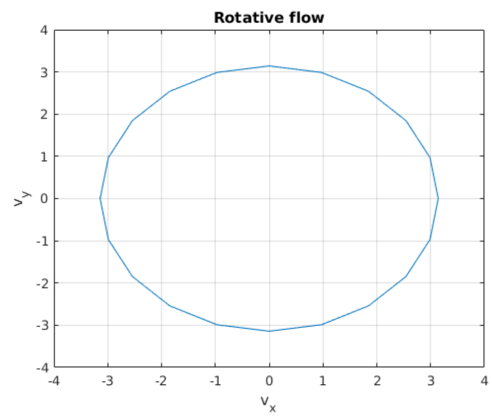


To see what kind of motion we have, we calculate the velocity from the kinematic above, so it yields:

$$v_x = -\varepsilon R\Theta sin\left(\Theta t\right); \quad v_y = \varepsilon R\Theta cos\left(\Theta t\right); \quad v_z = 0; \tag{22}$$







It is very important to specify that $\mathbf{V} = \varepsilon R\Theta$ that is why the velocity will follows the same path of ε. So according to the Bernoulli theorem, the pressure has a contrary variation with ε and it is constant when parameter is constant.

With our transformation kinematic, the deformation gradient tensor becomes a symetric matrix given by:

$$\mathbf{F} = \begin{pmatrix} \varepsilon cos\left(\Theta t\right) & -\varepsilon R\Theta sin\left(\Theta t\right) & 0 \\ \varepsilon sin\left(\Theta t\right) & \varepsilon R\Theta cos\left(\Theta t\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{23}$$

And as in previous cases, we calculate the Langrangian tensor of Cauchy-Green given from the gradient tensor by:

$$\mathbf{C} = \begin{pmatrix} \varepsilon^2 & 0 & 0 \\ 0 & \left(\varepsilon R\Theta\right)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{24}$$

We can note also that the deformation gradient tensor is not a digonal ma- trix, but the Lagrangian Cauchy-Green tensor gives a digonal matrix.

We have the isotropic invariants of the Lagrangian Cauchy-Green tensor which are defined as by:

$$\begin{aligned} I_1 &= \varepsilon^2 + \left(\varepsilon^2 R\Theta\right)^2 + 1; \\ I_2 &= \left(\varepsilon R\Theta\right)^2 + \varepsilon^2 + \left(\varepsilon^2 R\Theta\right)^2; \\ I_3 &= \left(\varepsilon^2 R\Theta\right)^2. \end{aligned} \tag{25}$$

And then the incompressibility condition from the third isotropic elementary invariant is:

$$\Theta = \frac{1}{\varepsilon^2 R}. \tag{26}$$

A condition which not now depends on the time variable compared with the two previous kinematics.
The rotational and the divergence of the speed are

$$\begin{aligned} \overrightarrow{rot}\left(\overrightarrow{v}\right) &= \varepsilon\Theta cos\left(\Theta t\right)\left(1 + R\Theta\right)\overrightarrow{e_z}; \\ div\overrightarrow{v} &= -\varepsilon\Theta sin\left(\Theta t\right)\left(1 + R\Theta\right). \end{aligned} \tag{27}$$

And with the condition $\Theta = -1/R$, the previous equations become:

$$\overrightarrow{rot}\left(\overrightarrow{v}\right) = \overrightarrow{0}; \qquad div\overrightarrow{v} = 0. \tag{28}$$

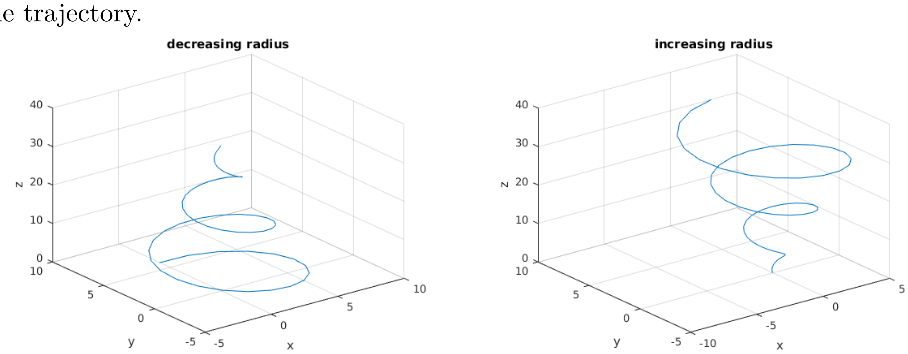
The null value of the rotational not means that there is no rotation component but there is the oposition of the existing components which makes the sum null. The divergence is equal to zero means that there is no divergence.

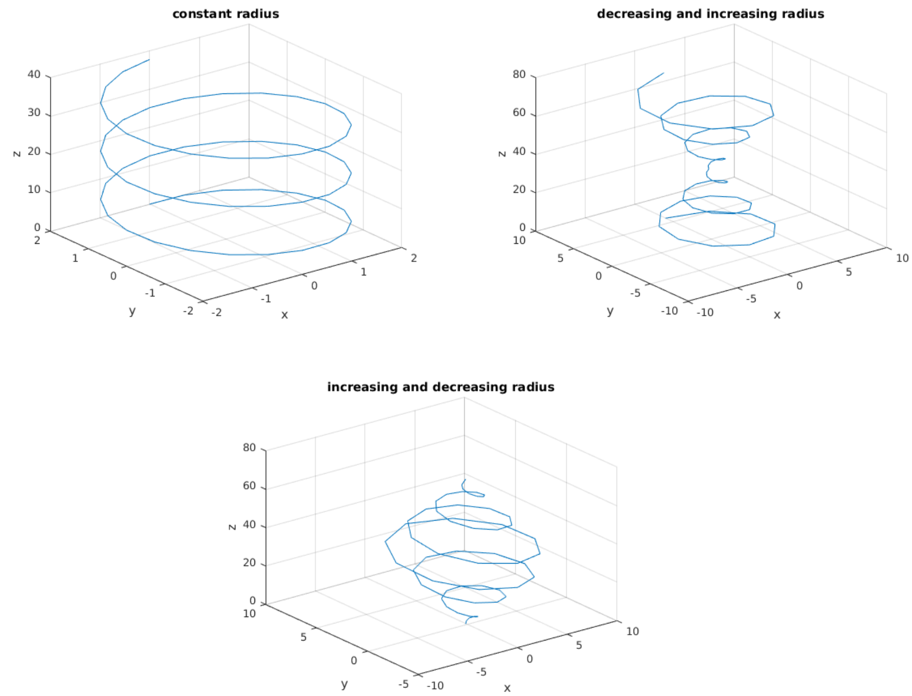
4.2 Spatial vortex flow

Let us now consider a fluid particle inside a vortex with a helical trajectory discribed by the following kinematic:

$$x = Rcos\left(\Theta t\right)\varepsilon\left(Z\right); \quad y = Rsin\left(\Theta t\right)\varepsilon\left(Z\right); \quad z = Zt; \tag{29}$$

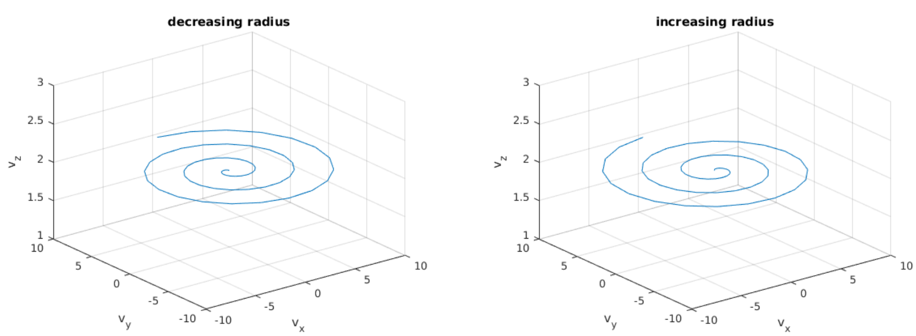
where R is the initial radius, Θ the angle before deformation and $\varepsilon = \varepsilon(Z)$ a function of Z representing here the perturbation parameter. And according to the values of the parameter, we can have the following gaphics which illustrate the trajectory.

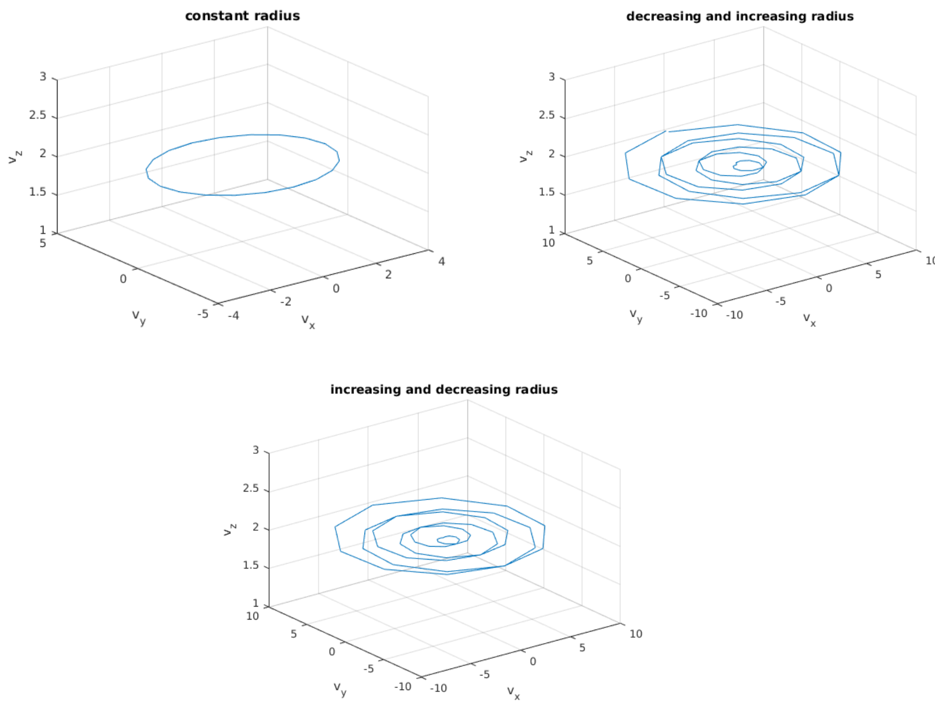




From this kinematics, we can find the particule speed components by derivation of the deformation kinematic components according to the time. what gives:

$$V_x = -\Theta R sin\left(\Theta t\right) \varepsilon\left(Z\right); \;\; V_y = \Theta R cos\left(\Theta t\right) \varepsilon\left(Z\right); \;\; V_z = Z; \qquad (30)$$





As it were predict by the velocity components, variation of speed is done only in the plane (x, y). And as previously the velocity follows the same path of ε. So according to the Bernoulli theorem, the pressure follows a contrary variation with ε and is constant when parameter is constant.
The deformation gradient tensor of the this tranformation becomes:

$$\mathbf{F} = \begin{pmatrix} cos\left(\Theta t\right)\varepsilon & -\Theta Rsin\left(\Theta t\right)\varepsilon & \varepsilon^{'}Rcos\left(\Theta t\right) \\ sin\left(\Theta t\right)\varepsilon & \Theta Rcos\left(\Theta t\right)\varepsilon & \varepsilon^{'}Rsin\left(\Theta t\right) \\ 0 & 0 & 1 \end{pmatrix}. \tag{31}$$

It should be remembered that the two tensors of Cauchy-Green are equivalent in terms of result, the only difference is at the level of the configuration which is Eulerian in one tensor and Lagrangian in the other. It is for this reason that we will continue our work using the Langrangian tensor of Cauchy-Green given from the gradient tensor by:

$$\mathbf{C} = \begin{pmatrix} \varepsilon^2 & 0 & R\varepsilon\varepsilon^{'} \\ 0 & R^2\Theta^2\varepsilon^2 & 0 \\ R\varepsilon\varepsilon^{'} & 0 & R^2\left(\varepsilon^{'}\right)^2+1 \end{pmatrix}. \tag{32}$$

By calculating the elementary isotropic invariants, we obtain:

$$\begin{aligned} I_1 = \varepsilon^2 + (R\Theta\varepsilon)^2 + \left(R\varepsilon'\right)^2 + 1; \\ I_2 = \varepsilon^2 + R^2\Theta^2\varepsilon^4 + (R\Theta\varepsilon)^2\left(\left(R\varepsilon'\right)^2 + 1\right); \\ I_3 = \varepsilon^2 \end{aligned} \tag{33}$$

The condition of incompressibility on fluid gives:

$$\varepsilon = 1 \tag{34}$$

To know if we are in the presence of a rotative flow or a variable density (volume) or not, we calculate the divergence. This gives us:

$$\begin{aligned} \overrightarrow{rot}\left(\overrightarrow{v}\right) = -\Theta R\varepsilon' cos\left(\Theta t\right)\left(\overrightarrow{e_r} + \overrightarrow{e_\theta}\right) + \varepsilon\Theta cos\left(\Theta t\right)\left(1 + R\Theta\right)\overrightarrow{e_z}; \\ div\overrightarrow{v} = -\varepsilon\Theta sin\left(\Theta t\right)\left(1 + R\Theta\right) + 1. \end{aligned} \tag{35}$$

In the particular case of a rotative flow with a constant value of the perturbation parameter ($\varepsilon = 1$), we have:

$$\begin{aligned} \overrightarrow{rot}\left(\overrightarrow{v}\right) = \Theta cos\left(\Theta t\right)\left(1 + R\Theta\right)\overrightarrow{e_z}; \\ div\overrightarrow{v} = -\Theta sin\left(\Theta t\right)\left(1 + R\Theta\right) + 1. \end{aligned} \tag{36}$$

From the previous result we can see that we have the same rotational than in the case of a planar rotative flow. Then if we consider that our initial component z is independent of time ($z = Z$) and with the condition $\Theta = -1/R$, we end up with:

$$\begin{aligned} \overrightarrow{rot}\left(\overrightarrow{v}\right) = \overrightarrow{0}; \\ div\overrightarrow{v} = 0. \end{aligned} \tag{37}$$

If $\varepsilon = 1$ is constant and for $\Theta = -1/R$, then we get the model of the planar rotative flow studied previously.

Remark:
We have the same incompressible condition in the case of horizontal accelerated flow than in the case of a shearing flow. We also have the pressure which decreases in these two motions.
We have the same rotational between the planar rotative flow and the spatial rotative flow when $\varepsilon = 1$. That's mean there is no influence of the z component in the rotional of these two motions.
Another important result of our contribution is that if ε is constant ($\varepsilon = 1$) and for $\Theta = -1/R$, then we get the same expressions values in the two vortex kinematics.

5 Conclusion

In this mathematical research, we have proposed as a work to elaborate two new kinematics for vortex modelization. Conditions of incompressibilty of two fluid flows, rotational and divergence of velocity tensor have been given. Pressure is interpret by using a Bernoulli theorem.
We start by a mathematical formulation with the calculation of some mechanical tensors and isotropic invariants. In application, we give two examples of fluid flow where the mathematical formulation was applied. We also prppose two new kinematics and applyed the mathematical formulation.
As a first result, we have the same incompressible condition in the case of horizontal accelerated flow than in the case of a shearing flow and we have the pressure which decreases in the two motions.
As a second result, we have the same rotational between the planar rotative flow and the spatial rotative flow when $\varepsilon = 1$, so there is no influence of the z component in the rotional of these two motion.
And the third result of our contribution is that if ε is constant ($\varepsilon = 1$) and for $\Theta = -1/R$, then we get results between the rotative planar flow and the spatial rotative flow.

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