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Certain Subclasses of Bi-Univalent and Meromorphic Functions Associated With Al-Oboudi Differential Operator

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Abstract - *In this paper, we introduce two new subclasses of meromorphic and bi-univalent functions defined by Al-Oboudi differential operator on $\Delta = \{z \in \mathbb{C}: 1 < |z| < \infty\}$. Also we obtain bounds of coefficients $|b_0|$ and $|b_1|$ for functions in this subclasses.*

Keywords -- *Al-Oboudi differential operator, Bi-univalent functions, Coefficient bounds, Meromorphic Bi-univalent functions.*

I. INTRODUCTION

Let Σ be the class of functions f of the form

$$f(z) = z + \sum_{l=0}^{\infty} \frac{b_l}{z^l}\ , \tag{1}$$

which are meromorphic univalent in the domain $\Delta = \{z \in \mathbb{C}: 1 < |z| < \infty\}$. Since every function f belong to Σ has an inverse function f^{-1} exist. Inverse function satisfies conditions:

$$f^{-1}(f(z)) = z, (z \in \Delta),$$

and

$$f(f^{-1}(w)) = w, w \in \Delta \quad (M < |w| < \infty, M > 0),$$

where

$$f^{-1}(w) = q(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} - \frac{b_3 + 2b_0 b_2 + b_0^2 b_1 + b_1^2}{w^3} + \cdots. \tag{2}$$

If f and $f^{-1}$are meromorphic univalent in Δ then $f \in \Sigma$ is said to be meromorphic bi-univalent in Δ. The class of meromorphic bi-univalent functions of the form (1) in Δ is denoted by Σ_M.

Recently, several researcher has been studied various subclasses of meromorphic univalent functions and estimates the bounds of coefficients of meromorphic univalent functions and inverse of meromorphic univalent functions in Δ ([2], [3], [5], [9], [10], [11]).

Let $\mathcal{A}$ be the class of functions h of the form

$$h(z) = z + \sum_{l=2}^{\infty} a_l z^l \tag{3}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$.

Now, Al-Oboudi [1] introduced the Al-Oboudi operator $D_\delta^k: \mathcal{A} \to \mathcal{A}$ and defined as

$$D^k h(z) = D_\delta^k h(z) = z + \sum_{l=2}^{\infty} [1 + (l-1)\delta]^k a_l z^l, \quad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \delta \geq 0, \text{ where } h \in \mathcal{A} \text{ of the form (3).}$$

Amol Patil et.al. [7] extend the Al-Oboudi operator $D_\delta^k: \Sigma \to \Sigma$ and defined as

$$D^k f(z) = D_\delta^k f(z) = z + (1-\delta)^k b_0 + \sum_{l=1}^{\infty} [1 - (l+1)\delta]^k b_l z^{-l}, \quad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \delta > 1,$$

where $f \in \Sigma$ of the form (1).

In 2014, H. Orhan et.al. [6] define the subclass $\Sigma_M^*(\mu, \lambda, \beta)$ consisting of meromorphic functions $f(z)$ of the form (1) satisfies the following conditions:

$$f \in \Sigma_M, \Re\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right] > \beta \quad (z \in \Delta),$$

and

$$\Re\left[(1-\lambda)\left(\frac{q(w)}{w}\right)^{\mu} + \lambda q'(w)\left(\frac{q(w)}{w}\right)^{\mu-1}\right] > \beta \quad (w \in \Delta),$$

where $\lambda \geq 1, \lambda > \mu, \mu \geq 0$, $0 \leq \beta < 1$ and q is function given by (2).

Also, H. Orhan et.al. [6] define the subclass $\Sigma_M^*(\mu, \lambda, \alpha)$ consisting of meromorphic functions $f(z)$ of the form (1) satisfies the following conditions:

$$f \in \Sigma_M, \left| arg\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right]\right| < \frac{\alpha\pi}{2} \quad (z \in \Delta),$$

and

$$\left| arg\left[(1-\lambda)\left(\frac{q(w)}{w}\right)^{\mu} + \lambda q'(w)\left(\frac{q(w)}{w}\right)^{\mu-1}\right]\right| < \frac{\alpha\pi}{2} \quad (w \in \Delta),$$

where $\lambda \geq 1, \lambda > \mu, \mu \geq 0$, $0 < \alpha \leq 1$ and q is function given by (2).

Motivated from above work, we introduced new subclasses of bi-univalent and meromorphic functions by using Al-Oboudi Differential operator. Also obtain the coefficient bounds $|b_0|$ and $|b_1|$ for functions in this new subclasses.

Lemma 1.1. [8] If $p \in \mathcal{P}$ then $|p_n| \leq 2 \ (n \in \mathbb{N})$, where $\mathcal{P}$ is the family of all analytic functions $p(z)$ in $\mathbb{U}$, for which

$$\Re(p(z)) > 0 \quad (z \in \mathbb{U}) \quad \text{and} \quad p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \quad (z \in \mathbb{U}).$$

II. COEFFICIENT ESTIMATES

Definition 2.1. A function $f(z)$ of the form (1) is said to be in the class $\Sigma_M^*(k, \delta, \mu, \lambda, \beta)$, if

$$f \in \Sigma_M, \Re\left[(1-\lambda)\left(\frac{D_\delta^k f(z)}{z}\right)^{\mu} + \lambda\left(D_\delta^k f(z)\right)'\left(\frac{D_\delta^k f(z)}{z}\right)^{\mu-1}\right] > \beta \quad (z \in \Delta), \tag{4}$$

and

$$\Re\left[(1-\lambda)\left(\frac{D_\delta^k q(w)}{w}\right)^{\mu} + \lambda\left(D_\delta^k q(w)\right)'\left(\frac{D_\delta^k q(w)}{w}\right)^{\mu-1}\right] > \beta \quad (w \in \Delta), \tag{5}$$

where $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda \geq 1, \lambda > \mu, \mu \geq 0, \delta > 1$and $0 \leq \beta < 1$.

If we put $k = 0$ in the class $\Sigma_M^*(k, \delta, \mu, \lambda, \beta)$, then we get class $\Sigma_M^*(\mu, \lambda, \beta)$, studied by H. Orhan et.al.[6] .

If we put $k = 0, \mu = 0, \lambda = 1$ in the class $\Sigma_M^*(k, \delta, \mu, \lambda, \beta)$, then we get class $\Sigma_M^*(\beta)$, studied byHalim et. al. [4].

Definition 2.2. A function $f(z)$ of the form (1) is said to be in the class $\Sigma_M^*(k, \delta, \mu, \lambda, \alpha)$, if

$$f \in \Sigma_M, \left| arg\left[(1-\lambda)\left(\frac{D_\delta^k f(z)}{z}\right)^{\mu} + \lambda\left(D_\delta^k f(z)\right)'\left(\frac{D_\delta^k f(z)}{z}\right)^{\mu-1}\right]\right| < \frac{\alpha\pi}{2} \quad (z \in \Delta), \tag{6}$$

and

$$\left| arg\left[(1-\lambda)\left(\frac{D_\delta^k q(w)}{w}\right)^{\mu} + \lambda\left(D_\delta^k q(w)\right)'\left(\frac{D_\delta^k q(w)}{w}\right)^{\mu-1}\right]\right| < \frac{\alpha\pi}{2} \quad (w \in \Delta), \tag{7}$$

where $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda \geq 1, \lambda > \mu, \mu \geq 0, \delta > 1$and $0 < \alpha \leq 1$.

If we put $k = 0$ in the class $\Sigma_M^*(k, \delta, \mu, \lambda, \alpha)$, then we get class $\Sigma_M^*(\mu, \lambda, \alpha)$, studied by H. Orhan et.al [6] .

If we put $k = 0, \mu = 0, \lambda = 1$ in the class $\Sigma_M^*(k, \delta, \mu, \lambda, \alpha)$, then we get class $\Sigma_M^*(\alpha)$, studied by Halim et. al. [4]

Theorem 2.3. Let function $f(z)$ of the form (1) be in the class $\Sigma_M^*(k, \delta, \mu, \lambda, \beta)$.Then

$$|b_0| \leq \frac{2(1-\beta)}{(\lambda-\mu)(\delta-1)^k} \tag{8}$$

and

$$|b_1| \leq \frac{2(1-\beta)}{(2\delta-1)^k}\sqrt{\frac{(1-\mu)^2(1-\beta)^2}{(\lambda-\mu)^4} + \frac{1}{(2\lambda-\mu)^2}}. \tag{9}$$

Proof. From conditions (4) and (5), we have

$$(1-\lambda)\left(\frac{D_\delta^k f(z)}{z}\right)^{\mu} + \lambda\left(D_\delta^k f(z)\right)'\left(\frac{D_\delta^k f(z)}{z}\right)^{\mu-1} = \beta + (1-\beta)h(z), (z \in \Delta) \tag{10}$$

and

$$(1-\lambda)\left(\frac{D_\delta^k q(w)}{w}\right)^{\mu} + \lambda\left(D_\delta^k q(w)\right)'\left(\frac{D_\delta^k q(w)}{w}\right)^{\mu-1} = \beta + (1-\beta)p(w), (w \in \Delta), \tag{11}$$

where $h(z)$ and $p(w)$ are functions such that it's real part positive in Δ and have forms

$$h(z) = 1 + \frac{h_1}{z} + \frac{h_2}{z^2} + \frac{h_3}{z^3} + \cdots \tag{12}$$

and

$$p(w) = 1 + \frac{p_1}{w} + \frac{p_2}{w^2} + \frac{p_3}{w^3} + \cdots. \tag{13}$$

Implies

$$(1-\lambda)\left(\frac{D_\delta^k f(z)}{z}\right)^{\mu} + \lambda\left(D_\delta^k f(z)\right)'\left(\frac{D_\delta^k f(z)}{z}\right)^{\mu-1} = 1 + \frac{(1-\beta)h_1}{z} + \frac{(1-\beta)h_2}{z^2} + \frac{(1-\beta)h_3}{z^3} + \cdots, (z \in \Delta) \tag{14}$$

and

$$(1-\lambda)\left(\frac{D_\delta^k q(w)}{w}\right)^\mu + \quad \lambda\left(D_\delta^k q(w)\right)'\left(\frac{D_\delta^k q(w)}{w}\right)^{\mu-1} = 1 + \frac{(1-\beta)p_1}{w} + \frac{(1-\beta)p_2}{w^2} + \frac{(1-\beta)p_3}{w^3} + \cdots (w \in \Delta). \tag{15}$$

Now, equating the coefficients in equation (14) and (15), we obtain

$$(\mu-\lambda)(1-\delta)^k b_0 = (1-\beta)h_1, \tag{16}$$

$$(\mu-2\lambda)\left[(1-2\delta)^k b_1 + \left(\frac{\mu-1}{2}\right)(1-\delta)^{2k} b_0^2\right] = (1-\beta)h_2, \tag{17}$$

$$-(\mu-\lambda)(1-\delta)^k b_0 = (1-\beta)p_1 \tag{18}$$

and

$$(\mu-2\lambda)\left[-(1-2\delta)^k b_1 + \left(\frac{\mu-1}{2}\right)(1-\delta)^{2k} b_0^2\right] = (1-\beta)p_2. \tag{19}$$

From equation (16) and equation (18), we get

$$h_1 = -p_1 \tag{20}$$

and

$$2(\mu-\lambda)^2(1-\delta)^{2k} b_0^2 = (1-\beta)^2[h_1^2 + p_1^2]. \tag{21}$$

Therefore, From equation (21), we get

$$b_0^2 = \frac{(1-\beta)^2[h_1^2+p_1^2]}{2(\mu-\lambda)^2(1-\delta)^{2k}} \tag{22}$$

Since $\Re(h(z)) > 0$ in Δ, the function $h(1/z) \in \mathcal{P}$ and hence the coefficients h_l and similarly the coefficients p_l of the function h and p satisfy the inequality in Lemma 1.1. Hence $|h_l| \le 2$, $|p_l| \le 2$ and apply on (22), we get

$$|b_0| \le \frac{2(1-\beta)}{(\lambda-\mu)(\delta-1)^k}.$$

Taking product of equations (17) and (19),we get

$$b_1^2 = \frac{(\mu-1)^2(1-\delta)^{4k} b_0^4}{4(1-2\delta)^{2k}} - \frac{(1-\beta)^2 h_2 p_2}{(1-2\delta)^{2k}(u-2\lambda)^2}. \tag{23}$$

By using equation (22) in equation (23) and by Lemma 1.1, $|h_l| \le 2$, $|p_l| \le 2$, we get

$$|b_1| \le \frac{2(1-\beta)}{(2\delta-1)^k}\sqrt{\frac{(1-\mu)^2(1-\beta)^2}{(\lambda-\mu)^4} + \frac{1}{(2\lambda-\mu)^2}}.$$

This complete the proof.

If we take $k = 0$ in Theorem (2.3), then we get following corollary:

Corollary 2.4. [6] Let function $f(z)$ of the form (1) be in the class $\Sigma_M^*(\mu, \lambda, \beta)$.Then

$$|b_0| \le \frac{2(1-\beta)}{(\lambda-\mu)}$$

and

$$|b_1| \le 2(1-\beta)\sqrt{\frac{(1-\mu)^2(1-\beta)^2}{(\lambda-\mu)^4} + \frac{1}{(2\lambda-\mu)^2}}.$$

If we take $k = 0, \mu = 0, \lambda = 1$ in Theorem (2), then we get following corollary:

Corollary 2.5. [4] Let function $f(z)$ of the form (1) be in the class $\Sigma_M^*(\beta)$. Then

$$|b_0| \le 2(1-\beta) \quad \text{and} \quad |b_1| \le (1-\beta)\sqrt{4\beta^2 - 8\beta + 5}.$$

Theorem 2.6. Let function $f(z)$ of the form (1) be in the class $\Sigma_M^*(k, \delta, \mu, \lambda, \alpha)$.Then

$$|b_0| \le \frac{2\alpha}{(\lambda-\mu)(\delta-1)^k} \tag{24}$$

and

$$|b_1| \le \frac{2\alpha^2}{(2\delta-1)^k}\sqrt{\frac{(1-\mu)^2}{(\lambda-\mu)^4} + \frac{1}{(2\lambda-\mu)^2}}. \tag{25}$$

Proof. From conditions (6) and (7), we have

$$(1-\lambda)\left(\frac{D_\delta^k f(z)}{z}\right)^\mu + \lambda\left(D_\delta^k f(z)\right)'\left(\frac{D_\delta^k f(z)}{z}\right)^{\mu-1} = [h(z)]^\alpha, (z \in \Delta) \tag{26}$$

and

$$(1-\lambda)\left(\frac{D_\delta^k q(w)}{w}\right)^\mu + \lambda\left(D_\delta^k q(w)\right)'\left(\frac{D_\delta^k q(w)}{w}\right)^{\mu-1} = [p(w)]^\alpha, (w \in \Delta), \tag{27}$$

where $h(z)$ and $p(w)$ are functions such that it's real part positive in Δ and have forms

$$h(z) = 1 + \frac{h_1}{z} + \frac{h_2}{z^2} + \frac{h_3}{z^3} + \cdots \tag{28}$$

and

$$p(w) = 1 + \frac{p_1}{w} + \frac{p_2}{w^2} + \frac{p_3}{w^3} + \cdots. \tag{29}$$

Now, equating the coefficients in equations (26) and (27), we obtain

$$(\mu - \lambda)(1-\delta)^k b_0 = \alpha h_1, \tag{30}$$

$$(\mu - 2\lambda)\left[(1-2\delta)^k b_1 + \left(\frac{\mu-1}{2}\right)(1-\delta)^{2k} b_0^2\right] = \frac{1}{2}[\alpha(\alpha-1)h_1^2 + 2\alpha h_2], \tag{31}$$

$$-(\mu - \lambda)(1-\delta)^k b_0 = \alpha p_1 \tag{32}$$

and

$$(\mu - 2\lambda)\left[-(1-2\delta)^k b_1 + \left(\frac{\mu-1}{2}\right)(1-\delta)^{2k} b_0^2\right] = \frac{1}{2}[\alpha(\alpha-1)p_1^2 + 2\alpha p_2]. \tag{33}$$

From equation (30) and equation (32), we get

$$h_1 = -p_1 \tag{34}$$

and

$$2(\mu-\lambda)^2(1-\delta)^{2k} b_0^2 = \alpha^2[h_1^2 + p_1^2]. \tag{35}$$

Therefore, From equation (35), we get

$$b_0^2 = \frac{\alpha^2[h_1^2 + p_1^2]}{2(\mu-\lambda)^2(1-\delta)^{2k}} \tag{36}$$

As discussed in the proof of Theorem 2.3, by Lemma 1.1, $|h_l| \le 2$, $|p_l| \le 2$ and apply on (36), we get

$$|b_0| \le \frac{2\alpha}{(\lambda-\mu)(\delta-1)^k}.$$

Now, squaring and adding equations (31) and (33),we get

$$\begin{aligned} 2(2\lambda-\mu)^2 \quad & [4(1-2\delta)^{2k} b_1^2 + (\mu-1)^2(1-\delta)^{4k} b_0^4] = \alpha^2(\alpha-1)^2[p_1^4 + q_1^4] + \\ & 4\alpha^2[p_1^2 + q_1^2] + \alpha^2(\alpha-1)[p_1^2 p_2 + q_1^2 q_2] \end{aligned}. \tag{37}$$

From equation (36) and equation (37), we get

$$b_1^2 = \frac{\alpha^2(\alpha-1)^2[p_1^4+q_1^4] + 4\alpha^2[p_1^2+q_1^2] + \alpha^2(\alpha-1)[p_1^2 p_2 + q_1^2 q_2]}{8(\mu-2\lambda)^2(1-2\delta)^{2k}} - \frac{(\mu-1)^2\alpha^4[p_1^2+q_1^2]^2}{16(1-2\delta)^{2k}(u-\lambda)^4}. \tag{38}$$

By Lemma 1.1, $|h_l| \le 2$ and $|p_l| \le 2$. Hence

$$|b_1| \le \frac{2\alpha^2}{(2\delta-1)^k}\sqrt{\frac{(1-\mu)^2}{(\lambda-\mu)^4} + \frac{1}{(2\lambda-\mu)^2}} \; .$$

This complete the proof.

If we take $k = 0$ in Theorem (2.6), then we get following corollary:

Corollary 2.7. [6] Let function $f(z)$ of the form (1) be in the class $\Sigma_M^*(\mu, \lambda, \alpha)$.Then

$$|b_0| \le \frac{2\alpha}{(\lambda-\mu)} \quad \text{and} \quad |b_1| \le 2\alpha^2\sqrt{\frac{(1-\mu)^2}{(\lambda-\mu)^4} + \frac{1}{(2\lambda-\mu)^2}}.$$

If we take $k = 0, \mu = 0, \lambda = 1$ in Theorem (2.6), then we get following corollary:

Corollary 2.8. [4] Let function $f(z)$ of the form (1) be in the class $\Sigma_M^*(\alpha)$.Then

$$|b_0| \le 2\alpha \quad \text{and} \quad |b_1| \le \sqrt{5}\alpha^2.$$

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