

# Solution of Modified Hirota-Satsuma (MHS) Coupled KdV-Equations by Variational Iteration Method

Franklin Ogunfiditimi<sup>#1</sup>, Nyore Okiotor<sup>#2</sup>

<sup>#1, #2</sup>Department of mathematics, University of Abuja, Abuja, Nigeria

**Abstract -** In this paper the Variational Iteration Method (VIM) is applied to solve the Hirota-Satsuma coupled KdV equations. The results obtained are compared with the results by the analytical solution and the solutions by other methods such as the “Differential Transform Method” (DTM), the “Decomposition Method” (DM) and “Homotopy Analysis Transform Method” (HATM). The results obtained by using VIM show rapid convergence to the exact solution and are in agreement with results from other methods. A new modified Hirota-Satsuma (mHS) coupled KdV equation is also introduced and solved. The numerical solution of the new (mHS) equation is computed using VIM. Computations are carried using Maple Software.

**Keywords —** Hirota-Satsuma Coupled KdV equation, modified Hirota-Satsuma equation (mHS), Variational Iteration method (VIM).

## I. INTRODUCTION

He's Variational Iteration Method has been shown to solve effectively, easily and accurately a large class of linear and nonlinear Differential Equations problems with approximation converging rapidly to accurate solutions (He, Wu, 2007). The method has been used in solving systems of linear and nonlinear PDEs, (Wazwaz,2007),Voltera's integro differential equations (Abbasbandy Shivanian 2009), Nonlinear Burgers Equation (Mamadu, Njoseh 2016), coupled Burgers equations (Mohyud-Din 2010), the cubic Boussinesq equations (Wazwaz,2007), and the Kuramoto-Sivanshinsky equation (Majeed 2014) to mention but a few. In [13] we have also applied VIM in solving differential equations and systems of differential equations using specially computed exact Lagrange multiplier for the Variational Iteration Method. Hirota and Satsuma (1981) first proposed the well-known coupled KdV equations, which describes the interactions of two long waves with different dispersion relations. The equation is usually referred to as the Hirota-Satsuma coupled KdV equation.

In this study we apply VIM to solve the generalized Hirota-Satsuma coupled KdV equation initiated by Wu *et al* (1999). This equation has the form:

$$u_t = \frac{1}{2}u_{xxx} - 3uu_x + 3vw_x + 3v_xw \quad (1)$$

$$v_t = -v_{xxx} + 3uv_x \quad (2)$$

$$w_t = -w_{xxx} + 3uw_x \quad (3)$$

Where the subscripts  $t$  and  $x$  denotes partial differentiation. Various schemes have been applied in investigating (1)-(3). Fan (2001), constructed two kinds of soliton solution for (1)-(3) by using an “Extended tanh-function and symbolic computation”. Raslan (2007) applied the “Decomposition method” to the problem, Kangalgil and Ayaz (2010) investigated the problem using “Differential transform method”, while Gupta *et al* (2017) solved the problem using “Homotopy Analysis Transform Method”. In this paper we applied the Variational Iteration method (VIM) in computing the approximate solution of the general form of the Hirota-Satsuma Coupled KdV equation (1)-(3).

Furthermore we also introduce a new modified Hirota Satsuma equation (mHS) of the form (15)-(18), and computed the approximate results using VIM.

## II. BASIC IDEA OF VIM

To illustrate the basic concept of the technique, we consider the following nonlinear differential equation:

$$Lu + Nu = g(x) \quad (4)$$

Where:  $L$  is a linear operator,  $N$  is a nonlinear operator and  $g(x)$  is a known function. The variational iteration method presents a correction functional for equation (4) in the form:



$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)(Lu_n(s) + N\tilde{u}_n(s) - g(s))ds \quad (5)$$

Where  $\lambda$  is a general Lagrange's multiplier, which may be a constant or a function, and may be identified optimally via variational theory (Abbasbandy 2007). The subscript  $n$  denotes the  $n$ th approximation, and  $\tilde{u}_n$  is considered a restricted variation (Wazwaz 2014), i.e.  $\delta\tilde{u}_n = 0$ . The VIM correction functional takes the form:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)(u_n^{(n)}(s) + f(u_n(s), u'_n(s), u''_n(s), \dots, u_n^{n-1}(s)) - g(s))ds \quad (6)$$

$$\text{With } \lambda(s) = \frac{(-1)^n}{(n-1)!} (s-x)^{n-1} \quad (7)$$

### VIM for Coupled Systems

For the coupled Hirota-Satuma KDV equation (1)-(3), the VIM correction functional takes the form

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda_1(s) \left( \frac{\partial u_n(x, s)}{\partial s} - \frac{1}{2} \frac{\partial^3 u_n(x, s)}{\partial x^3} + 3u_n(x, s) \frac{\partial u_n(x, s)}{\partial x} - 3v_n(x, s) \frac{\partial w_n(x, s)}{\partial x} - 3 \frac{\partial v_n(x, s)}{\partial x} w_n(x, s) \right) ds \quad (8)$$

$$v_{n+1}(x, t) = v_n(x, t) + \int_0^t \lambda_2(s) \left( \frac{\partial v_n(x, s)}{\partial s} + \frac{\partial^3 v_n(x, s)}{\partial x^3} - 3u_n(x, s) \frac{\partial v_n(x, s)}{\partial x} \right) ds \quad (9)$$

$$w_{n+1}(x, t) = w_n(x, t) + \int_0^t \lambda_3(s) \left( \frac{\partial w_n(x, s)}{\partial s} + \frac{\partial^3 w_n(x, s)}{\partial x^3} - 3u_n(x, s) \frac{\partial w_n(x, s)}{\partial x} \right) ds \quad (10)$$

The Lagrange multiplier is easily determined using (7) as

$$\lambda_1(s) = \lambda_2(s) = \lambda_3(s) = -1 \quad (11)$$

So that we obtain the following iteration formula to compute successive approximations:

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left( \frac{\partial u_n(x, s)}{\partial s} - \frac{1}{2} \frac{\partial^3 u_n(x, s)}{\partial x^3} + 3u_n(x, s) \frac{\partial u_n(x, s)}{\partial x} - 3v_n(x, s) \frac{\partial w_n(x, s)}{\partial x} - 3 \frac{\partial v_n(x, s)}{\partial x} w_n(x, s) \right) ds \quad (12)$$

$$v_{n+1}(x, t) = v_n(x, t) - \int_0^t \left( \frac{\partial v_n(x, s)}{\partial s} + \frac{\partial^3 v_n(x, s)}{\partial x^3} - 3u_n(x, s) \frac{\partial v_n(x, s)}{\partial x} \right) ds \quad (13)$$

$$w_{n+1}(x, t) = w_n(x, t) - \int_0^t \left( \frac{\partial w_n(x, s)}{\partial s} + \frac{\partial^3 w_n(x, s)}{\partial x^3} - 3u_n(x, s) \frac{\partial w_n(x, s)}{\partial x} \right) ds \quad (14)$$

For the newly introduced modified Hirota-Satuma KDV equation (mHS) which has the form:

$$u_t = \frac{1}{2} u_{xxx} - 3uu_x + 3vzw_x + 3wzv_x + 3vwz_x \quad (15)$$

$$v_t = -v_{xxx} + 3uv_x \quad (16)$$

$$w_t = -w_{xxx} + 3uw_x \quad (17)$$

$$z_t = -z_{xxx} + 3uz_x \quad (18)$$

We construct four correction functionals of the form:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda_1(s) \left( \frac{\partial u_n(x, s)}{\partial s} - \frac{1}{2} \frac{\partial^3 u_n(x, s)}{\partial x^3} + 3u_n(x, s) \frac{\partial u_n(x, s)}{\partial x} - 3v_n(x, s)z_n(x, s) \frac{\partial w_n(x, s)}{\partial x} - 3w_n(x, s)z_n(x, s) \frac{\partial v_n(x, s)}{\partial x} - 3v_n(x, s)w_n(x, s) \frac{\partial z_n(x, s)}{\partial x} \right) ds \quad (19)$$

$$v_{n+1}(x, t) = v_n(x, t) + \int_0^t \lambda_2(s) \left( \frac{\partial v_n(x, s)}{\partial s} + \frac{\partial^3 v_n(x, s)}{\partial x^3} - 3u_n(x, s) \frac{\partial v_n(x, s)}{\partial x} \right) ds \quad (20)$$

$$w_{n+1}(x, t) = w_n(x, t) + \int_0^t \lambda_3(s) \left( \frac{\partial w_n(x, s)}{\partial s} + \frac{\partial^3 w_n(x, s)}{\partial x^3} - 3u_n(x, s) \frac{\partial w_n(x, s)}{\partial x} \right) ds \quad (21)$$

$$z_{n+1}(x, t) = z_n(x, t) + \int_0^t \lambda_4(s) \left( \frac{\partial z_n(x, s)}{\partial s} + \frac{\partial^3 z_n(x, s)}{\partial x^3} - 3u_n(x, s) \frac{\partial z_n(x, s)}{\partial x} \right) ds \quad (22)$$

The Lagrange multiplier is easily determined using (7) as

$$\lambda_1(s) = \lambda_2(s) = \lambda_3(s) = \lambda_4(s) = -1 \quad (23)$$

The iteration formula to compute successive approximations takes the form:

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left( \frac{\partial u_n(x, s)}{\partial s} - \frac{1}{2} \frac{\partial^3 u_n(x, s)}{\partial x^3} + 3u_n(x, s) \frac{\partial u_n(x, s)}{\partial x} - 3v_n(x, s)z_n(x, s) \frac{\partial w_n(x, s)}{\partial x} - 3w_n(x, s)z_n(x, s) \frac{\partial v_n(x, s)}{\partial x} - 3v_n(x, s)w_n(x, s) \frac{\partial z_n(x, s)}{\partial x} \right) ds \quad (24)$$

$$v_{n+1}(x, t) = v_n(x, t) - \int_0^t \left( \frac{\partial v_n(x, s)}{\partial s} + \frac{\partial^3 v_n(x, s)}{\partial x^3} - 3u_n(x, s) \frac{\partial v_n(x, s)}{\partial x} \right) ds \quad (25)$$

$$w_{n+1}(x, t) = w_n(x, t) - \int_0^t \left( \frac{\partial w_n(x, s)}{\partial s} + \frac{\partial^3 w_n(x, s)}{\partial x^3} - 3u_n(x, s) \frac{\partial w_n(x, s)}{\partial x} \right) ds \quad (26)$$

$$z_{n+1}(x, t) = z_n(x, t) - \int_0^t \left( \frac{\partial z_n(x, s)}{\partial s} + \frac{\partial^3 z_n(x, s)}{\partial x^3} - 3u_n(x, s) \frac{\partial z_n(x, s)}{\partial x} \right) ds \quad (27)$$

### III. NUMERICAL APPLICATIONS

#### **Problem 1**

Consider the Hirota-Satsuma coupled KDV equation (1)-(3) (Wu *et al* 1999)

With initial conditions (Fan 2001)

$$u(x, 0) = \frac{1}{3}(\beta - 8k^2) + 4k^2 \tanh^2(kx) \quad (28)$$

$$v(x, 0) = \frac{-4(3k^4c_0 - 2\beta k^2 c_2 + 4k^2 c_2)}{3c_2^2} + \frac{4k^2}{c_2} \tanh^2(kx) \quad (29)$$

$$w(x, 0) = c_0 + c_2 \tanh^2(kx) \quad (30)$$

#### **Solution**

To solve the coupled system (1)-(3) w.r.t the initial conditions (28)-(30), we proceed by constructing three (3) correction functional (8)-(10). The iteration formula is determined as (12)-(14). Using (12) – (14) and the initial conditions (28)-(30) as our initial approximation, we obtain the following results:

$$u_0(x, t) = u(x, 0) = \frac{1}{3}(\beta - 8k^2) + 4k^2 \tanh^2(kx)$$

$$v_0(x, t) = v(x, 0) = \frac{-4(3k^4c_0 - 2\beta k^2 c_2 + 4k^2 c_2)}{3c_2^2} + \frac{4k^2}{c_2} \tanh^2(kx)$$

$$w_0(x, t) = w(x, 0) = c_0 + c_2 \tanh^2(kx)$$

$$\begin{aligned} u_1(x, t) = & \frac{1}{3} \frac{1}{c_2} \left( 144 \tanh(kx)^5 k^5 t c_2 - 144 \tanh(kx)^5 k^3 t c_2 + 72 \tanh(kx)^3 k^5 t c_0 \right. \\ & - 144 k^5 \tanh(kx)^3 t c_2 - 24 \tanh(kx)^3 \beta k^3 t c_2 - 72 \tanh(kx)^3 k^3 t c_0 \\ & + 144 \tanh(kx)^3 k^3 t c_2 - 72 \tanh(kx) k^5 t c_0 + 24 \tanh(kx) \beta k^3 t c_2 + 72 \tanh(kx) k^3 t c_0 \\ & \left. + 12 k^2 \tanh(kx)^2 c_2 - 8 k^2 c_2 + \beta c_2 \right) \end{aligned}$$

$$\begin{aligned} v_1(x, t) = & \\ & - \frac{4}{3} \frac{1}{c_2^2} \left( k^2 \left( 6 \tanh(kx)^3 \beta k t c_2 - 6 \tanh(kx) \beta k t c_2 - 3 \tanh(kx)^2 c_2 + 3 k^2 c_0 \right. \right. \\ & \left. \left. + 4 k^2 c_2 - 2 \beta c_2 \right) \right) \end{aligned}$$

$$w_1(x, t) = -2 c_2 \tanh(kx)^3 k t \beta + 2 c_2 \tanh(kx) k t \beta + c_2 \tanh(kx)^2 + c_0$$

...and so on

We compare the result of second approximate from VIM i.e.  $u_2(x, t)$ ,  $v_2(x, t)$ , &  $w_2(x, t)$  with that of the analytical solution (Fan 2001), the DM solution, Raslan (2007), the DTM solution, Kangalgil and Ayaz (2010), and the HATM solution Gupta *et al* (2017) in the following Tables for  $c_0 = 1$ ,  $c_2 = 1$ ,  $k = 0.1$ ,  $\beta = 1$ ,  $t = 1$

**Table 1:**

$x$	$u_{exact}$	$u_{VIM.}$	$u_{DM.}$	$u_{DTM}$	$u_{HATM.}$	Absolute Error $u_{exact} - u_{VIM.}$
-50.00	0.3466577953	0.3466448909	0.346645	0.3466607106	0.3466527	1.2904e-05
-40.00	0.3466011627	0.3465059930	0.346505	0.3466012375	0.3465251	9.5170e-05
-30.00	0.3461851777	0.3454918231	0.345487	0.3461857110	0.3453124	6.9335e-04
-20.00	0.3432422696	0.3386237847	0.338603	0.3432452281	0.3430654	4.6185e-03
-10.00	0.3271899722	0.3107504212	0.310816	0.3271845676	0.3204578	1.6440e-02
0.00	0.3070640150	0.3071854666	0.307183	0.3070666666	0.3071245	1.2145e-04
10.00	0.3322986140	0.3486686782	0.348596	0.3323021676	0.3405664	1.6370e-02
20.00	0.3443377450	0.3489698283	0.348992	0.3443349776	0.3457081	4.6321e-03
30.00	0.3463432716	0.3470393543	0.347044	0.3463427870	0.3465342	6.9608e-04
40.00	0.3466227462	0.3467183028	0.346719	0.3466226784	0.3466493	9.5557e-05
50.00	0.3466607198	0.3466736769	0.346674	0.3466607106	0.3466642	1.2957e-05

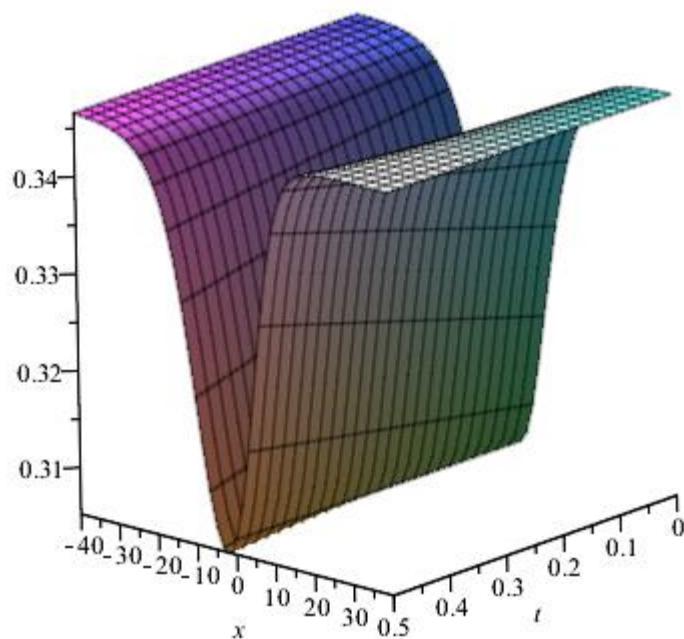
**Table 2:**

$x$	$v_{exact}$	$v_{VIM.}$	$v_{DM.}$	$v_{DTM}$	$v_{HATM.}$	Absolute Error $v_{exact} - v_{VIM.}$
-50.00	0.0657244621	0.0657244723	0.0657245	0.06572738652	0.06572451	1.0210e-08
-40.00	0.0656678294	0.0656679060	0.0656678	0.06566782943	0.06566784	7.6620e-08
-30.00	0.0652518444	0.0652524718	0.0652520	0.06525184445	0.06525212	6.2732e-07
-20.00	0.0623089364	0.0623161984	0.0632137	0.06230893638	0.06231247	7.2620e-06
-10.00	0.0462566389	0.0463189587	0.0463266	0.04625663889	0.04630012	6.2320e-05
0.00	0.0261306817	0.0261333333	0.0261317	0.02613068170	0.02613213	2.6516e-06
10.00	0.0513652807	0.0514271227	0.0514202	0.05136528069	0.00224021	6.1842e-05
20.00	0.0634044118	0.0634049027	0.0634074	0.06340441178	0.06340742	4.9093e-07
30.00	0.0654099384	0.0654095237	0.0654100	0.06540993837	0.06540994	4.1470e-07
40.00	0.0656894129	0.0656893464	0.0656894	0.06568941289	0.06568942	6.6480e-08
50.00	0.0657273865	0.0657273773	0.0657274	0.06572738652	0.06572746	9.1900e-09

**Table 3:**

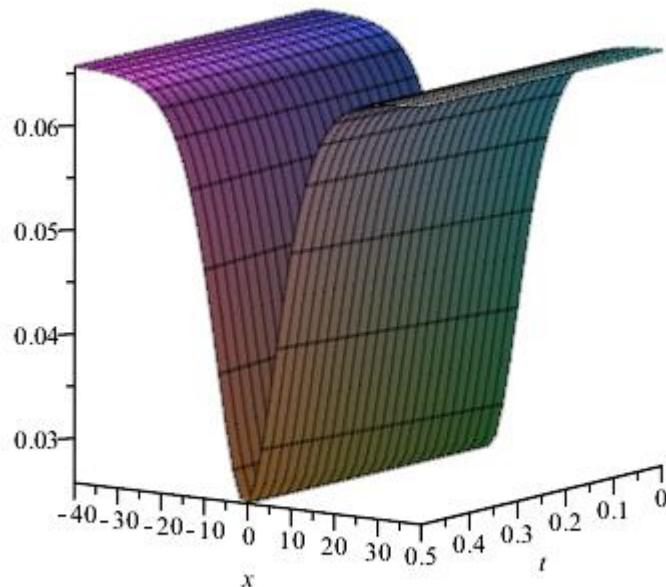
$x$	$w_{exact}$	$w_{VIM.}$	$w_{DM.}$	$w_{DTM}$	$w_{HATM.}$	Absolute Error $w_{exact} - w_{VIM.}$
-50.00	1.9997782180	1.9997784730	1.99978	1.999851099	1.99978	2.5500e-07
-40.00	1.9983624020	1.9983643170	1.99836	1.998364273	1.99836	1.9150e-06
-30.00	1.9879627780	1.9879784610	1.98797	1.987976113	1.98797	1.5683e-05
-20.00	1.9879627780	1.9145716260	1.91451	1.914464035	1.91451	1.8155e-04
-10.00	1.5130826390	1.5146406330	1.51483	1.512947524	1.51483	1.5580e-03
0.00	1.0099337090	1.0100000000	1.00996	1.101000000	1.00996	6.6291e-05
10.00	1.6407986840	1.6423447350	1.64217	1.640887525	1.64217	1.5461e-03
20.00	1.9417769610	1.9417892340	1.94185	1.941707773	1.94185	1.2273e-05
30.00	1.9919151260	1.9919047590	1.99192	1.991903011	1.99192	1.0367e-05
40.00	1.9989019890	1.9989003270	1.99890	1.998900294	1.99890	1.6620e-06
50.00	1.9998513300	1.9998511010	1.99985	1.999851099	1.99985	2.2900e-07

**Fig. 1**



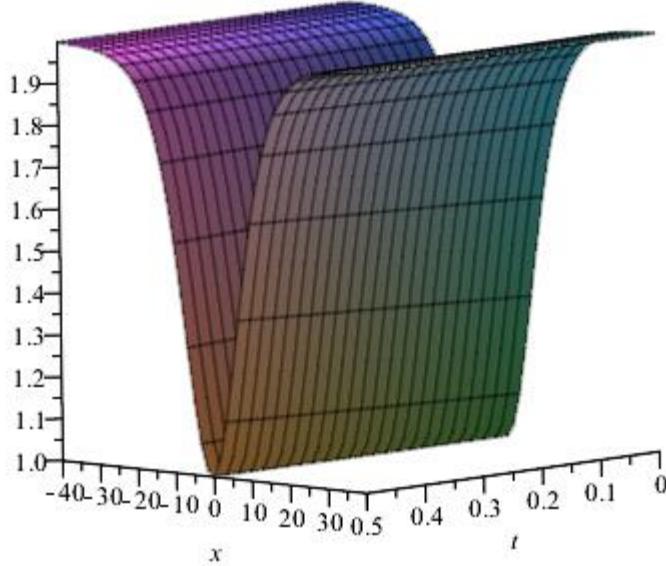
$$u(x,t): c_0 = 1, c_2 = 1, k = 0.1, \beta = 1$$

Fig. 2



$$v(x,t): c_0 = 1, c_2 = 1, k = 0.1, \beta = 1$$

Fig. 3



$$w(x,t): c_0 = 1, c_2 = 1, k = 0.1, \beta = 1$$

Table 1, 2 & 3 compares the solitary wave solution of (1)-(3) using the initial conditions (28)-(30) for  $c_0 = 1, c_2 = 1, k = 0.1, \beta = 1$  and  $t = 1$ .

It is obvious from the results on the table that the computed results from VIM, from the analytical solution and from the other methods are in agreement. We observe that VIM has a very small error margin for the problem with the given initial conditions (28) - (30). Figures 1, 2 & 3 show that the soliton solution of the system (1) - (3) are all bell shaped as depicted. When the exact solutions are plotted on the same figure with the result from VIM, the curves of the exact and the numerical result are indistinguishable.

### **Problem 2**

We repeat computation of the solution of above problem (1) - (3) using another initial conditions as follow:

$$u(x, 0) = \frac{1}{3}(\beta - 2k^2) + 2k^2 \tanh^2(kx) \quad (31)$$

$$v(x, 0) = \frac{-4k^2 c_0 (\beta + k^2)}{3c_1^2} + \frac{4k^2 (\beta + k^2)}{3c_1} \tanh(kx) \quad (32)$$

$$w(x, 0) = c_0 + c_1 \tanh(kx) \quad (33)$$

### **Solution**

To solve the coupled system (1)-(3) w.r.t the initial conditions (31)-(33), we proceed again by constructing three (3) correction functionals (8)-(10). The iteration formula is determined as (12)-(14). Using (12) – (14) and the initial conditions (31)-(33) as our initial approximation, we obtain the following results:

$$u_0(x, t) = u(x, 0) = \frac{1}{3}(\beta - 2k^2) + 2k^2 \tanh^2(kx)$$

$$v_0(x, t) = v(x, 0) = \frac{-4k^2 c_0 (\beta + k^2)}{3c_1^2} + \frac{4k^2 (\beta + k^2)}{3c_1} \tanh^2(kx)$$

$$w_0(x, t) = w(x, 0) = c_0 + c_1 \tanh(kx)$$

$$u_1(x, t) = 2k^2 \tanh(kx)^2 - \frac{2}{3}k^2 + \frac{1}{3}\beta - 4 \tanh(kx)^3 \beta k^3 t + 4 \tanh(kx) \beta k^3 t$$

$$v_1(x, t) = -\frac{4}{3} \frac{k^2 (k^2 + \beta) (\tanh(kx)^2 \beta k t c_1 - \beta k t c_1 - c_1 \tanh(kx) + c_0)}{c_1^2}$$

$$w_1(x, t) = -c_1 k t \tanh(kx)^2 \beta + c_1 k t \beta + c_1 \tanh(kx) + c_0$$

$$\begin{aligned}
 u_2(x, t) = & 2k^4\beta^2t^2 - \frac{2}{3}k^2 - 4\tanh(kx)^3\beta k^3t + 4\tanh(kx)\beta k^3t + 48\tanh(kx)^7\beta^2k^7t^3 \\
 & - \frac{352}{3}\tanh(kx)^5\beta^2k^7t^3 - \frac{16}{3}\tanh(kx)^5\beta^3k^5t^3 + \frac{272}{3}\tanh(kx)^3\beta^2k^7t^3 \\
 & + \frac{32}{3}\tanh(kx)^3\beta^3k^5t^3 - \frac{64}{3}\tanh(kx)\beta^2k^7t^3 - \frac{16}{3}\tanh(kx)\beta^3k^5t^3 \\
 & + 6\tanh(kx)^4\beta^2k^4t^2 - 8\tanh(kx)^2\beta^2k^4t^2 + 2k^2\tanh(kx)^2 + \frac{1}{3}\beta
 \end{aligned}$$

$$\begin{aligned}
 v_2(x, t) = & -\frac{4}{3}\frac{1}{c_1^2}\left(k^2(k^2 + \beta)\left(8\tanh(kx)^6\beta^2k^5t^3c_1 - 16\tanh(kx)^4\beta^2k^5t^3c_1\right.\right. \\
 & \left.+ 8k^5c_1\tanh(kx)^2\beta^2t^3 - \tanh(kx)^3\beta^2k^2t^2c_1 + k^2\tanh(kx)c_1\beta^2t^2 + \tanh(kx)^2\beta ktc_1\right. \\
 & \left.- \beta ktc_1 - c_1\tanh(kx) + c_0\right)
 \end{aligned}$$

$$\begin{aligned}
 w_2(x, t) = & -8\tanh(kx)^6\beta^2k^5t^3c_1 + 16\tanh(kx)^4\beta^2k^5t^3c_1 - 8k^5c_1\tanh(kx)^2\beta^2t^3 \\
 & + \tanh(kx)^3\beta^2k^2t^2c_1 - c_1k^2\tanh(kx)\beta^2t^2 - \tanh(kx)^2\beta ktc_1 + \beta ktc_1 + c_1\tanh(kx) \\
 & + c_0
 \end{aligned}$$

...and so on.

We compare the result of second approximate from VIM i.e.  $u_2(x, t)$ ,  $v_2(x, t)$ , &  $w_2(x, t)$  with that of the analytical solution (Fan 2001), the DM solution, Raslan (2007), the DTM solution, Kangalgil and Ayaz (2010), and the HATM solution Gupta et al (2017) in the following Tables for  $c_0 = 1$ ,  $c_2 = 1$ ,  $k = 0.1$ ,  $\beta = 1$ ,  $t = 2$

**Table 4:**

$x$	$u_{exact}$	$u_{VIM.}$	$u_{DM.}$	$u_{DTM}$	$u_{HATM.}$	Absolute Error $u_{exact} - u_{VIM.}$
-50.00	0.3466612490	0.3466612918	0.346661	0.3466612919	0.3466617	4.2800e-08
-40.00	0.3466266705	0.3466266705	0.346627	0.3466269860	0.3466269	3.1630e-07
-30.00	0.3463730133	0.3463752952	0.346373	0.3463752559	0.3463734	2.2819e-06
-20.00	0.3445954991	0.3446096772	0.344595	0.3446077137	0.3445957	1.4178e-05
-10.00	0.3354855632	0.3355163433	0.335485	0.3354597292	0.3354856	3.0780e-05
0.00	0.3274458069	0.3274666666	0.327445	0.3274666666	0.3274462	2.0860e-05
10.00	0.3405662667	0.3405207149	0.340567	0.3405773292	0.3405664	4.5552e-05
20.00	0.3457081597	0.3456954998	0.345708	0.3456974632	0.3457081	1.2660e-05
30.00	0.3465341826	0.3465322926	0.346534	0.3465323318	0.3465342	1.8900e-06
40.00	0.3466486853	0.3466484260	0.346649	0.3466484268	0.3466493	2.5930e-07
50.00	0.3466642322	0.3466641970	0.346664	0.3466641969	0.3466642	3.5200e-08

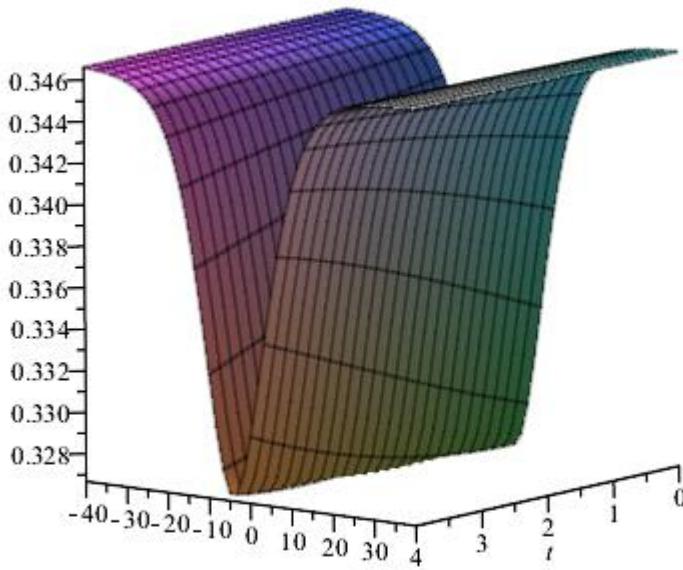
**Table 5:**

$x$	$v_{exact}$	$v_{VIM.}$	$v_{DM.}$	$v_{DTM}$	$v_{HATM.}$	Absolute Error $v_{exact} - v_{VIM.}$
-50.00	-0.0269315093	-0.0269315238	-0.0269315	-0.02693150931	-0.02693157	1.4460e-08
-40.00	-0.0269198613	-0.0269199678	-0.0269199	-0.02691986125	-0.02692010	1.0653e-07
-30.00	-0.0268341045	-0.0268348774	-0.0268341	-0.02668341045	-0.02683408	7.7295e-07
-20.00	-0.0262169876	-0.0262219703	-0.0262170	-0.02621698753	-0.02621705	4.9826e-06

-10.00	-0.0224090285	-0.0224202596	-0.0224088	-0.02240902919	-0.02240884	1.1231e-05
0.00	-0.0108086790	-0.0107733333	-0.0108092	-0.01080866987	-0.01080864	3.5346e-05
10.00	-0.0022401180	-0.0022525754	-0.0022399	-0.00224011816	-0.00224021	1.2457e-05
20.00	-0.0003266592	-0.0003308706	-0.0003266	-0.00032665908	-0.00032653	4.2114e-06
30.00	-0.0000446770	-0.0000453125	-0.0000446	-0.00004467705	-0.000044600	6.3548e-07
40.00	-0.0000060551	-0.0000061423	-0.000000605	-0.0000006055	-0.00000605	8.7269e-08
50.00	-0.0000008196	-0.0000008315	-0.00000081	-0.0000008196	-0.00000081	1.1840e-08

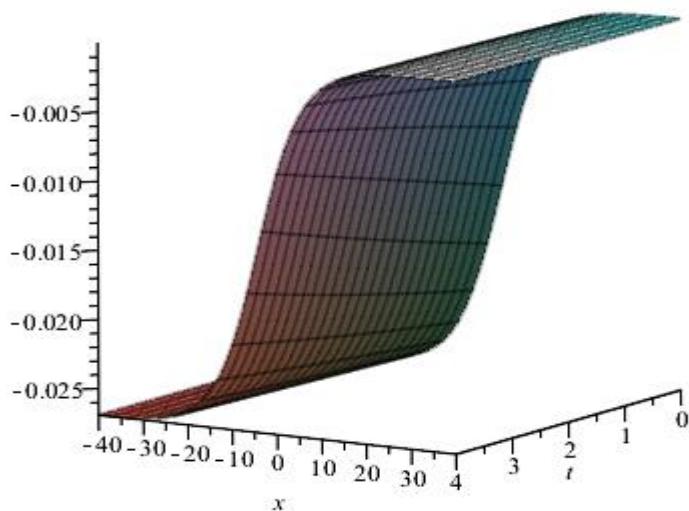
**Table 6:**

$x$	$w_{exact}$	$w_{VIM.}$	$w_{DM.}$	$w_{DTM}$	$w_{HATM.}$	<i>Absolute Error</i> $w_{exact} - w_{VIM.}$
-50.00	0.0001354483	0.0001343750	0.00013544	0.000134375116	0.00013544	1.0733e-06
-40.00	0.0010004022	0.0009924914	0.00100034	0.000992492788	0.00100382	7.9108e-06
-30.00	0.0073684798	0.0073110819	0.00736806	0.007311643521	0.00736824	5.7398e-05
-20.00	0.0531939872	0.0528239897	0.05319250	0.05282695872	0.05319317	3.7000e-04
-10.00	0.3359632297	0.3351292379	0.33597900	0.03351947124	0.33596004	8.3399e-04
0.00	0.3359632297	1.2000000000	1.19733000	1.2000000	1.19736401	2.6247e-03
10.00	1.8336546070	1.8327295500	1.83367000	1.832795024	1.83364823	9.2506e-04
20.00	1.9757431300	1.9754304020	1.97574000	1.975433371	1.97574000	3.1273e-04
30.00	1.9966823980	1.9966352090	1.99668000	1.996635271	1.99668000	4.7189e-05
40.00	1.9995503660	1.9995438870	1.99955000	1.999543888	1.99955000	6.4790e-06
50.00	1.9999391370	1.9999382580	1.99994000	1.999938258	1.99994000	8.7900e-07

**Fig. 4**


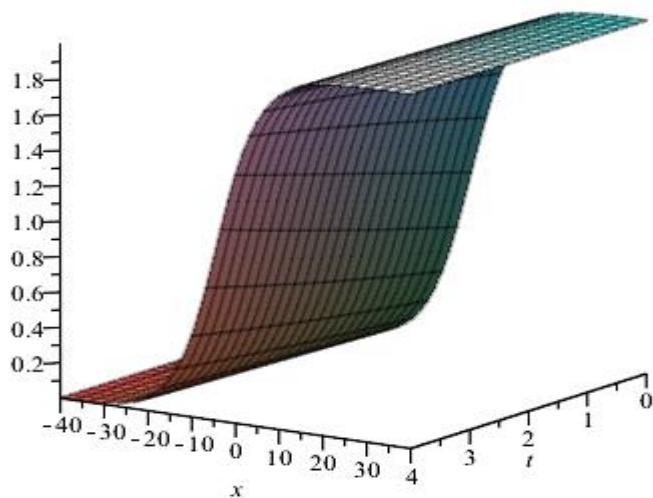
$$u(x,t) : c_0 = 1, c_2 = 1, k = 0.1, \beta = 1$$

**Fig. 5**



$$v(x,t): c_0 = 1, c_2 = 1, k = 0.1, \beta = 1$$

**Fig.6**



$$w(x,t): c_0 = 1, c_2 = 1, k = 0.1, \beta = 1$$

Table 4, 5 & 6 compares the solitary wave solution of (1)-(3) using the initial conditions (31)-(33) for  $c_0 = 1, c_2 = 1, k = 0.1, \beta = 1$  and  $t = 2$ . Again It is obvious from the results on the table that the results from using VIM, and the result from the analytical solution and from other methods are in agreement. We observe that VIM has a very small error margin for the problem with the initial conditions (31)-(33). Figures 4, 5 & 6 show that the soliton solution of the system (1)-(3) is bell shaped for  $u(x, t)$ , and kink-shaped for  $v(x, t)$  and  $w(x, t)$ . Likewise, when the exact solutions are plotted on the same figure with the results from VIM, the curves of the exact and the numerical results are indistinguishable.

### **Problem 3**

We introduce a new modified Hirota-Satsuma (mHS) equation of the form (15)-(18):

$$\begin{aligned} u_t &= \frac{1}{2}u_{xxx} - 3uu_x + 3vzw_x + 3wzv_x + 3vwz_x \\ v_t &= -v_{xxx} + 3uv_x \\ w_t &= -w_{xxx} + 3uw_x \\ z_t &= -z_{xxx} + 3uz_x \end{aligned}$$

We solve the above using the following initial conditions

$$\begin{aligned} u(x, 0) &= 2k^2 \operatorname{sech}^2(kx), \quad v(x, 0) = 4k^2 \tanh^2(kx), \quad w(x, 0) = 4k^2 \tanh^2(kx), \\ z(x, 0) &= \tanh(kx) \end{aligned} \tag{34}$$

### **Solution**

To solve the coupled system (15) - (18) w.r.t the initial conditions (34), we construct four (4) correction functional (19)-(22). The iteration formula is determined as (24)-(22). Using the following initial conditions as our initial approximation:

$$\begin{aligned} u_0(x, 0) &= 2k^2 \operatorname{sech}^2(kx) \\ v_0(x, 0) &= 4k^2 \tanh^2(kx) \\ w_0(x, 0) &= 4k^2 \tanh^2(kx) \\ z_0(x, 0) &= \tanh(kx) \end{aligned}$$

We obtain the following results after the first iteration:

$$\begin{aligned} u_1(x, t) &= 2k^2 \operatorname{sech}(kx)^2 - 8k^5 \operatorname{sech}(kx)^2 \tanh(kx)^3 t + 16k^5 \operatorname{sech}(kx)^2 \tanh(kx) \\ &\quad - \tanh(kx)^2 t + 24k^5 \operatorname{sech}(kx)^4 \tanh(kx) t + 240k^5 \tanh(kx)^4 (1 - \tanh(kx)^2) t \end{aligned}$$

$$\begin{aligned} v_1(x, t) &= 4k^2 \tanh(kx)^2 + 64k^5 (1 - \tanh(kx)^2)^2 \tanh(kx) t - 32k^5 \tanh(kx)^3 (1 \\ &\quad - \tanh(kx)^2) t + 48k^5 \operatorname{sech}(kx)^2 \tanh(kx) (1 - \tanh(kx)^2) t \end{aligned}$$

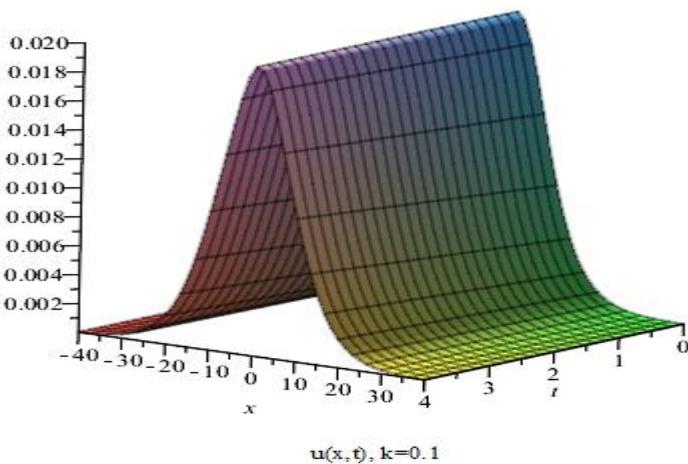
$$\begin{aligned} w_1(x, t) &= 4k^2 \tanh(kx)^2 + 64k^5 (1 - \tanh(kx)^2)^2 \tanh(kx) t - 32k^5 \tanh(kx)^3 (1 \\ &\quad - \tanh(kx)^2) t + 48k^5 \operatorname{sech}(kx)^2 \tanh(kx) (1 - \tanh(kx)^2) t \end{aligned}$$

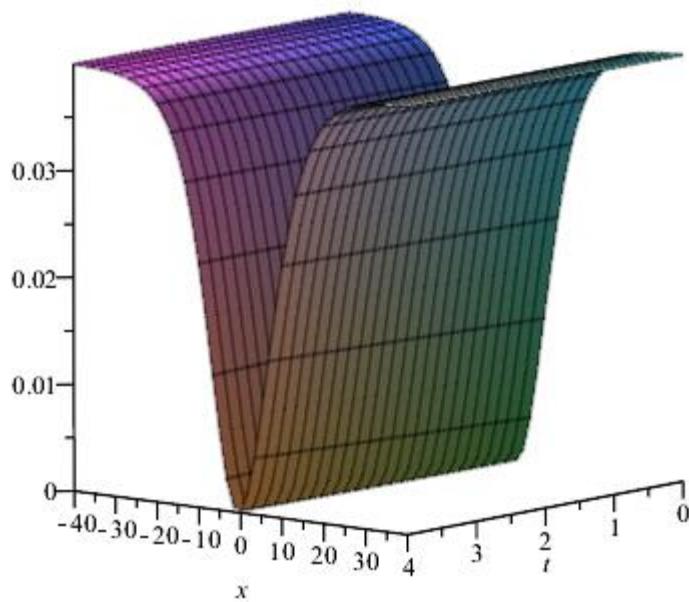
$$\begin{aligned} z_1(x, t) &= \tanh(kx) + 2(1 - \tanh(kx)^2)^2 k^3 t - 4 \tanh(kx)^2 (1 - \tanh(kx)^2) k^3 t \\ &\quad + 6k^3 \operatorname{sech}(kx)^2 (1 - \tanh(kx)^2) t \end{aligned}$$

...and so on.

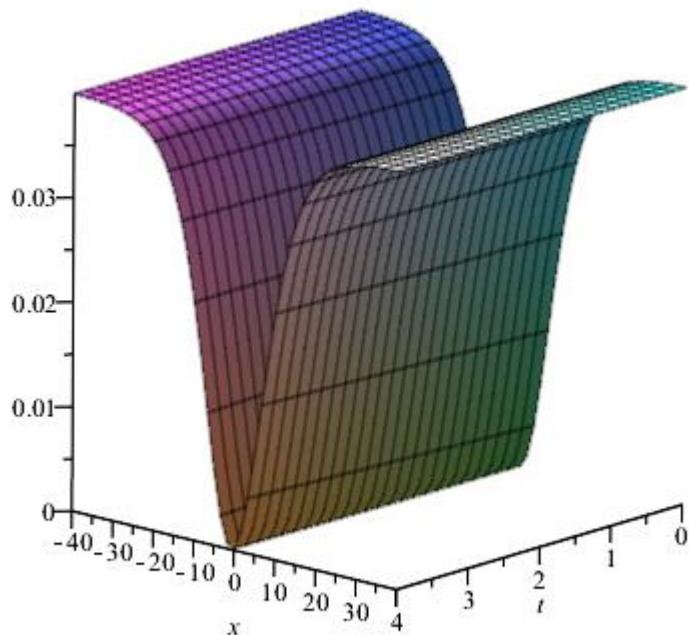
We plot below the result of the second iteration i.e.  $u_2(x, t)$ ,  $v_2(x, t)$ ,  $w_2(x, t)$  and  $z_2(x, t)$

**Fig. 7**

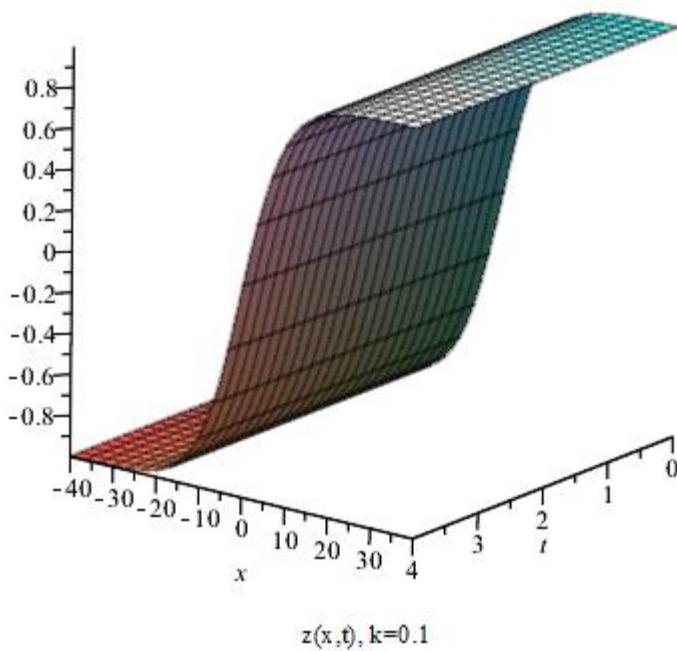




$v(x,t)$ ,  $k=0.1$



$w(x,t)$ ,  $k=0.1$



## VI. CONCLUSIONS

In this study, the Variational Iteration Method has been applied in finding the soliton solution of a generalized Hirota-Satsuma coupled KDV equation with two different types of initial conditions. The results of the study presented in tables and figures demonstrate that the result of the study is in agreement with the result of the analytical soliton solution derived by using an extended tanh-function and symbolic computation. The tables and figures show that there is very small or no difference between the results obtained here using VIM and those obtained using the “Differential Transform Method”, the “Decomposition Method”, and the “Homotopy Analysis Transform Method” (HATM). The error margin using VIM is also very small. We note that the problem is easily solved using VIM, with numerical results converging rapidly to the exact result after few iterations. Furthermore we have introduced a new modified Hirota- Satsuma (mHS) coupled KDV equation and computed its approximate solutions using VIM. The results as plotted were also very good.

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