International Journal of Mathematics Trends and Technology
ISSN: 2231 – 5373 /doi:10.14445/22315373/IJMTT-V67I5P516

Volume 67 Issue 5, 150-156, May, 2021


# Some Results on Fixed Point Theory in G-metric Space

Samjhana Koirala [1] & Narayan Prasad Pahari[2]

[1]*Tribhuvan University, Department of Mathematics, Butwal Multiple Campus, Butwal, Nepal*
[2]*Tribhuvan University, Central Department of Mathematics, Kritipur Kathmandu, Nepal*

***Abstract:***
*This paper deals with the brief introduction and some existing results in G-metric space as the usual notion of a metric space . In fact, this study generalizes some recent results in the literature about metric space. We also study some fixed point theorems for contraction mapping in the context of G-metric space as the further extension and generalization of metric space.*

**Keywords:** *Metric space, G-metric space, Fixed point and Contraction mapping*

## I. Introduction

So far, there have been several endeavors to generalize the definition of a metric space to obtain various fixed point results. Also, the development of the fixed point theory in these generalized metric spaces may be a consequence of the fixed point theory in previously generalized metric spaces. Moreover, the generalized metric spaces are topologically equivalent to previously generalized metric spaces. In different times different generalizations of the usual notion of a metric space were proposed by several mathematicians such as Dutta and Chaudhary **[2],** Gahler **[5]** proposed 2-metric spaces as the generalization of classical metric spaces. Ha, Cho and White **[2]** have pointed out that the results cited by Gahler are independent, rather than generalizations of the corresponding results in metric space. Also Mustafa and Sims **[10]** and Naidu., Rao and Raw **[7]** demonstrated that most of the claims concerning the fundamental topological structures of D-metric spaces are incorrect. Alternatively Mustafa and Sims introduced more appropriate notion of generalized metric space which is called G-metric space and obtained some topological properties.

The concept of G-metric space was introduced by Mustafa and Sims in order to extend and generalize the notion of metric spaces. There is closed relation between a usual metric space and G-metric space. In fact the nature of a G-metric is to corresponding the geometry of three points instead of two points. Since then many fixed point results have been developed by different authors in G-metric spaces.

Later Mustafa, Obiedat and Awawdeh **[12],** Mustafa and Sims **[11],** Shatanwi **[8]** and Mustafa , Shanta and Bataineh **[13]** obtained some fixed point theorems for a single map in G-metric space. Many results are obtained by contractive condition. Further Gupta **[1]** in 2014 introduced the notion of *F*-contraction and Chauh ,Kadian, Rani and Rhoods **[4]** have studied Property *P* in G-metric space.

The propose of this paper is to study the development of fixed point theory in *G*-metric space.

## II. Preliminaries

First we begin by briefly some basic definitions, properties and fixed point theorem in *G*-metric space.

**Definition 2.1 : G-metric space: See [10]:**

Let $X$ be a nonempty set and $G: X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties.

$G_1$: $G(x, y, z) = 0$ if $x = y = z$

$G_2$ : $G(x, y, z)$ $\forall\, x, y, z \in X$ with $x \neq y$

$G_3$: $G(x, x, y) \leq G(x, y, z)$ $\forall\, x, y, z \in X$ with $y \neq z$

$G_4$ : $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots = \ldots$ (Symmetry in all three variables.

$G_5$ : $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ , $\forall$ $x, y, z \in X$ (Rectangle inequality)

Then the function $G$ is called a generalized metric or more specially a $G$-metric on $X$ and the pair $(X, G)$ is a $G$-metric space.

A $G$-metric space $(X, G)$ is called symmetric if

$$G(x, y, y) = G(y, x, x) \ \forall \ x, y \in X.$$

**Example 2.1 : (See [10])**

Let $\mathbb{R}$ be the set of real numbers. Define $G$: $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ by

$$G(x, y, z) = |x - y| + |y - z| + |z - x| \ \ \forall \ x, y, z \in X.$$

Then it is clear that $(\mathbb{R}, G)$ is a $G$-metric space.

**Proposition 2.1:( See[10]) :-**

Let $(X, G)$ be a $G$-metric space. Then for any $x, y, z$ and $a \in X$, it follows that

(i) If $G(x, y, z) = 0$ then $x = y = z$;

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$;

(iii) $F(x, y, y) \leq 2 G(y, x, x)$;

(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$;

(v) $G(x, y, z) \leq \frac{2}{3} G(x, y, a) + G(x, a, z) + G(a, y, z)$; and

(vi) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

**Definition 2.2 *G*-convergent: See[10]:-**

Let $(X, G)$ be a $G$-metric space and let $\{x_n\}$ be a sequence of points of $X$. We say that $\{x_n\}$ is $G$-convergent to $x \in X$ if

$$\lim_{n, m \to \infty} G(x, x_n, x_m) = 0.$$

**Proposition 2.2 : see [10]:**

Let $(X, G)$ be a $G$-metric space. Then the following are equivalent:

(i) $\{x_n\}$ is $G$-convergent to $x$

(ii) $\lim_{n \to \infty} G(x_n, x, x) = 0$

(iii) $\lim_{n \to \infty} G(x_n, x_n, x) = 0$

(iv) $\lim_{n, m \to \infty} G(x_n, x_m, x) = 0$

**Definition 2.3 *G*-Cauchy: (see ([10]):-**

Let $(X, G)$ be a $G$-metric space. A sequence $\{x_n\}$ is called a $G$- Cauchy sequence if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m \geq N$, that is

$$G(x_n, x_m, x_l) \to 0 \text{ as } n, m, l \to \infty.$$

**Proposition 2.3: (See [10]):-**

Let $(X, G)$ be a $G$-metric space. Then the following statements are equivalent:

(i) the sequence $\{x_n\}$ is Cauchy

(ii) $\forall \ \varepsilon > 0, \exists \ N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ , $\forall m, n \geq$ N.

**Lemma 2.1: (See [10]):-**

Every $G$-metric space $(X, G)$ will define a metric space $(X, d_G)$ by

$$d_G(x, y) = G(x, y, y) + G(y, x, x)\ , \ \forall\ x, y \in X.$$

**Definition 2.4 G-complete: See[11]:**

A $G$-metric space $(X, G)$ is called $G$-complete if every $G$-Cauchy sequence is $G$-convergent in $(X, G)$

**Example 2.2:** Let $X = [0, \infty)$, $G: X \times X \times X \to \mathbb{R}^+$ be defined by

$G(x, y, z) = 0$ if $x = y = z$

and $G(x, y, z) = \max\{x,y,z\}$, otherwise.

Then $(X, G)$ is a complete $G$-metric space.

**Proposition 2.4 : (See [11]):-**

A $G$-metric space $(X, G)$ is complete $\Leftrightarrow$ $(X, d_G)$ is a complete metric space.

**Definition 2.5 G-continuous: (see [10]):-**

Let $(X, G)$ and $(X', G')$ be $G$-metric spaces and let $f : (X, G) \to (X', G')$ be a function.

Then $f$ is said to be $G$-continuous function at a point $a \in X$ if

$$\forall\ \varepsilon > 0, \exists \text{ a } \delta > 0 \text{ such that } x, y \in X,\ G(a, x, y) < \delta \Rightarrow G'(f(a), f(x), f(y)) < \varepsilon$$

A function $f$ is $G$-continuous on $X$ if and only if it is $G$-continuous $\forall\ a \in X$.

**Proposition 2.5: (See [10]) :-**

Let $(X, G)$ be a $G$-metric space. Then the function $G(x, y, z)$ is continuous in all variables.

**Definition 2.6. Contraction mapping: (See[9]) :-**

Let $(X, G)$ be a $G$-metric space and let $S: X \to X$ be a mapping. Then it is called contraction mapping in $X$ if

$$G(Sx, Sy, Sz) \le \lambda\ G(x, y, z)\ \forall x, y, z \in X \text{ and whenever } 0 \le \lambda < 1.$$

**Definition 2.7. Fixed point:-**

Let $(X, d)$ be a metric space and $S: X \to X$ be a mapping. Then any $x \in X$ is called fixed point of $S$ if $Sx = x$.

In particular, the function $S$ such that $S(x) = 4x(1 - x)$ has $x = 0$ and $x = \frac{3}{4}$ are the fixed points.

## III. Some Generalized Fixed Point Theorems in *G* - Metric Space

In this section we shall present some fixed point theorems for self - mapping satisfying various contractive conditions in complete $G$ - metric space introduced by Mustafa, Obiedat and Awawdeh (2008). This idea has been fascinated by many workers and various types of fixed point theorems for mappings on $G$-metric spaces have been studied and generalized. The proofs of these results seem more difficult than the proof of fixed point theorems for maps on ordinary metric spaces.

**Theorem 3.1:** Let $(X, G)$ be a complete G-metric space, and let $S : X \to X$ be a mapping. Then $S$ has a unique fixed point in $X$ if $S$ satisfies the condition :

$$G(Sx, Sy, Sz) \le \lambda \max\{G(x, Sx, Sx), G(y, Sy, Sy), G(z, Sz, Sz)\}\ \forall\ x,y,z \in X \quad \ldots(1)$$

whenever $0 \le \lambda < 1$.

**Proof :**

Suppose that $S$ satisfies condition (1), then $\forall\, x,y \in X$, we have

$$G(Sx,Sy,Sy) \leq \lambda \max \{G(x,Sx,Sx), G(y,Sy,Sy)\}$$

$$\text{and } G(Sy,Sx,Sx) \leq \lambda \max \{G(y,Sy,Sy), G(x,Sx,Sx)\}$$

Suppose that $(X,G)$ is symmetric, then by the definition of the metric $(X,d_G)$ and given condition, we get

$$d_G(Sx,Sy) \leq \lambda \max \{d_G(x,Sx), d_G(y,Sy)\} \ \forall\, x,y \in X.$$

Since $0 \leq \lambda < 1$, then the existence and uniqueness of the fixed point in metric space $(X,d_G)$. However, if $(X, G)$ is not symmetric, then by the definition of the metric $(X,d_G)$, we get

$$d_G(Sx,Sy) \leq \frac{4\lambda}{3} \max \{d_G(x, Sx), d_G(y,Sy)\}, \forall x, y \in X.$$

This metric condition gives no information about this map since $\frac{4\lambda}{3}$ need not be less than 1, but we will prove it by $G$-meric.

Let $x_0 \in X$ be arbitrary point and define the sequence $\{x_n\}$ by $x_n = S^n(x_n)$. By given condition we can verify that

$$G(x_n, x_{n+1}, x_{n+1}) \leq \lambda \max \{G(x_{n-1},x_n,x_n), G(x_n, x_{n+1}, x_{n+1})\}$$

$$= \lambda\, G(x_{n-1}, x_n, x_n) \text{ since } 0 \leq \lambda < 1.$$

Continuing in the same argument, we will find

$$G(x_n, x_{n+1}, x_{n+1}) \leq \lambda^n G(x_0, x_1, x_1)$$

Now $\forall\, n, m \in \rceil$, $n < m$, we have by rectangle inequality that

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) + \ldots + G(x_{m-1}, x_m, x_m)$$

$$\leq (\lambda^n + \lambda^{n+1} + \ldots + \lambda^{m-1})\, G(x_0, x_1, x_1)$$

$$\leq \frac{\lambda^n}{1-\lambda}\, G(x_0, x_1, x_1).$$

Then, $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0$, and thus $\{x_n\}$ is $G$-Cauchy sequence.

Since $(X,G)$ is complete, □ $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Suppose that $S(u) \neq u$, then

$G(x_{n+1}, Su, Su) \leq k \max \{ G(x_{n+1}, x_{n+2}, x_{n+2}), G(u, Su, Su)\}$

Taking the limit as $\boldsymbol{n} \to \infty$, and using the continuity of $G$ ,we get

$$G(u, Su, Su) \leq \lambda\, G(u, Su, Su).$$

This contradiction implies that $u = Su$.

Finally to prove uniqueness, suppose that $u \neq v$ such that $S(v) = v$, then we have

$$G(u, v, v) \leq \lambda \max \{G(v, v, v), G(u, u, u)\} = 0$$

Which implies that $u = v$. Hence the proof is complete. □

There are bulk numbers of fixed point theorems for mapping on $G$-metric spaces which have been proved complicatedly by using various contraction conditions. Now we show that the proofs of these results may be simplified by using the metric $d_G$, as follows analogously to introduced by Mustafa, Obiedat and Awawdeh [8].

**Theorem 3.2:** Let $(X, G)$ be a complete $G$-metric space and let $S: X \rightarrow X$ be a mapping. Then $S$ has a unique fixed point in $X$ if $S$ satisfies the following condition:

$$G(Sx, Sy, Sz) \leq \lambda \max\{G(x, Sy, Sy), G(y, Sx, Sx), G(y, Sy, Sy)\} \ \ldots (1)$$

$\forall\, x, y, z \in X$ where $\lambda \in [0,1)$.

**Proof :**

Suppose that $S$ satisfies the given condition (1), then all $x, y \in X$

$$G(Sx, Sy, Sy) \leq \lambda \max\{G(x, Sy, Sy), G(y, Sx, Sx), G(y, Sy, Sy)\}$$

$$G(Sy, Sx, Sx) \leq \lambda \max\{G(x, Sy, Sy), G(y, Sx, Sx), G(x, Sx, Sx)\}$$

Suppose that $(X, G)$ is symmetric, then by the definition of the matric $(X, d_G)$ we have

$$d_G(Sx, Sy) \leq \frac{\lambda}{2} \max\{d_G(x, Sy), d_G(y, Sx), d_G(y, Sy)\} + \frac{\lambda}{2} \max\{d_G(x, Sy), d_G(y, Sx), d_G(x, Sx)\}$$

$$\leq \lambda \max\{d_G(y, Sx), d_G(x, Sx), d_G(y, Sy)\} \quad \forall x, y \in X.$$

Since $0 \leq \lambda < 1$, then the existence and uniqueness of fixed point theorem follows from metric space $(X, d_G)$. However, if $(X,G)$ is not symmetric, then by definition of the metric $(X, d_G)$ and we have,

$$d_G(Sx, Sy) \leq \frac{2\lambda}{3} \max\{d_G(x, Sy), d_G(y, Sx), d_G(y, Sy)\} + \frac{2\lambda}{3} \max\{d_G(x, Sy), d_G(y, Sx), d_G(x, Sx)\}$$

$\forall\, x, y \in X$.

Then the metric space $(X, d_G)$ gives no information about the map since $\frac{4\lambda}{3}$ need not be less than 1. But we will prove it by $G$-metric.

Let $x_0 \in X$ be arbitrary point and define the sequence $\{x_n\}$ by $x_n = S^n(x_0)$.

Then by given condition and using $\lambda < 1$, we deduce that

$$G(x_n, x_{n+1}, x_{n+1}) \leq \lambda \max\{G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1})\}$$
$$= \lambda G(x_{n-1}, x_{n+1}, x_{n+1})$$

So, $G(x_n, x_{n+1}, x_{n+1}) \leq \lambda G(x_{n-1}, x_{n+1}, x_{n+1})$

and using,

$$G(x_{n-1}, x_{n+1}, x_{n+1}) \leq \lambda \max\{G(x_{n-2}, x_{n+1}, x_{n+1}), G(x_n, x_{n-1}, x_{n-1}), G(x_n, x_{n+1}, x_{n+1})\}$$

Then,

$$G(x_n, x_{n+1}, x_{n+1}) \leq \lambda^2 \max\{G(x_{n-2}, x_{n+1}, x_{n+1}), G(x_n, x_{n-1}, x_{n-1})\}$$

Continuing in this procedure, we will have

$$G(x_n, x_{n+1}, x_{n+1}) \leq \lambda^n G(x_0, x_1, x_1) \quad \forall n, m \in \mathbb{N}, n < m.$$

Then by triangle inequality, we have

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) + \ldots + G(x_{m-1}, x_m, x_m)$$
$$\leq (\lambda^n + \lambda^{n+1} + \ldots + \lambda^{m-1})\, G(x_0, x_1, x_1)$$
$$\leq \frac{\lambda^n}{1-\lambda} G(x_0, x_1, x_1)$$

This proves that $\lim_{n, m \to \infty} G(x_n, x_m, x_m) = 0$.

Thus $\{x_n\}$ is $G$-Cauchy sequence. Since $(X, G)$ is $G$-complete, then □ $u \in X$ such that $\{x_n\}$ is $G$-convergent to $u$.

Suppose that $S(u) \neq u$, then

$$G(x_n, Su, Su) \leq \lambda \max \{G(u, Su, Su), G(u, x_{n+1}, x_{n+1}), G(u, Su, Su)\}$$

Taking the limit as $n \to \infty$ and using the fact that the function $G$ is continuous, we get

$G(u, Su, Su) \leq \lambda\, G(u, Su, Su)$.

This contradiction implies that $u = Su$.

To prove uniqueness, suppose that $u \neq v$ such that $S(v) = v$.

Then by given condition we have

$$G(u, v, v) \leq \lambda \max\{G(u, v, v), G(v, u, u)\}.$$

This implies that $G(u, v, v) \leq \lambda G(v, u, u)$.

Again we will find $G(v, u, u) \leq \lambda\, G(u, v, v)$. So $G(u, v, v) \leq \lambda^2 G(u, v, v)$

Since $\lambda < 1$, implies that $u = v$. This completes the proof.

□

There is an another theorem, its proof follows from previous theorem.

**Theorem 3.3 :**

Let $(X, G)$ be a complete $G$-metric space, and let $S : X \to X$ be a mapping. Then $S$ has a unique fixed point if:

$$G(Sx, Sy, Sy) \leq \lambda \max\{ G(x, Sy, Sy), G(y, Sx, Sx)\}$$
$$\forall x, y \in X \text{ where } \lambda \in [0, \tfrac{1}{2}).$$

□

We overview some further fixed point theorems without proof for self mapping satisfying various contractive conditions in complete $G$ - metric spaces.

**Theorem 3.4 (See [6]):-**

Let $(X, G)$ be a complete $G$-metric space, and let $S : X \to X$ be such that

$$G(Sx, Sy, Sz) \leq aG(x, y, z) + bG(x, Sx, Sx) + cG(y, Sy, Sy) + dG(z, Sz, Sz)$$
$$+ e \max G(x, Sy, Sy), G(y, Sx, Sx), G(y, Sz, Sz), G(z, Sy, Sy), G(z, Sx, Sx), G(x, Sx, Sx)\}$$

$\forall x, y, z \in X$ where $a, b, c, d, e \geq 0$ with $a + b + c + d + 2e < 1$.

Then $S$ has a unique fixed point (say $p$) in $X$ and $S$ is $G$-continuous at $p$.

□

There is another a theorem, its proof follows from previous theorem.

**Theorem 3.5.(See [6]):-**

Let $(X, G)$ be a complete $G$-metric space and let $S$: $X \to X$ be a mapping which satisfies the following condition:

$\forall x, y, z \in X$ and $\lambda \in [0, \frac{1}{3})$.

$$G(Sx, Sy, Sz) \leq \lambda \max\{G(x, Sx, Sx), G(y, Sy, Sy), G(,Sz, Sz), G(y, Sx, Sx), G(y, Sz, Sx), G(z, Sx, Sy),$$
$$G(x, y, Sz), G(y, z, Sx), G(x, Sy), G(x, y, z)\}.$$

Then $S$ has a unique fixed point (say $p$) in $X$ and is continuous at $p$. □

By similar arguments as in the proof in the metric space, we may prove that many contraction conditions for maps on *G*-metric spaces reduce to certain contraction conditions for maps on metric spaces.

**Theorem 3.6 (See [4]):-**

Let $(X, G)$ be a complete *G*-metric space and let: $S: X \to X$ be a mapping which satisfies the following conditions:

$$G(Sx, Sy, Sz) \leq \lambda \max \{G(x, y, z) + G(x, Sx, Sx), G(y,Sy,Sy)\ G(x,Sy,Sy), G(y, Sx, Sx), G(z, Sz, Sz)\}$$

$$\forall\ x, y, z \in X \text{ and } 0 \leq \lambda < 1.$$

Then $S$ has a unique fixed point (say $(p)$). □

## Conclusion

In this work, we have introduced some existing results in G-metric space as the the usual notion of a metric space. Besides this, we have studied some of the generalizations of some recent works in metric space. In fact, this work extends many other authors existing literature and can be used for further research work in fixed point theory in Metric space.

## Acknowledgements

The authors would like to thank for the unknown referee for his/her comments that helped us to improve this paper.

## References

[1] A. Gupta (2014). Fixed points of a new type of contractive mappings in G-metric spaces, International Journal of Advances in Mathematics, 1(1): 56-61

[2] K.S. Ha, Y.J. Cho and A. White (1988). Strictly convex and strictly 2-convex 2-normed spaces, Mathematica Japoni-ka, 33(3): 375-384

[3] P.N.Dutta and B.S.Chaudhary (2008). A generalization of contraction principle in metric spaces, Journal of fixed point theory and Applications, (2008): 1-8

[4] R. Chauh , T.Kadian , A. Rani and B.E.Rhoods (2010). Property P in G-metric space, Fixed Point Theory and Applications, (2010): 1-12.

[5] S. Gahler (1963). 2-metrishce Ráume and ihre topologische struktur, Mathematisische Nachrichten, 26: 115-148.

[6] S.K. Mohanta (2012). Some fixed point theorem in G-mettric spaces, An.st. Univ. ovidius constanta, 20(1): 285-306

[7] S. V. R. Naidu ., K. P. R. Rao and N.S. Raw (2005), On the concept of balls in D-metric spaces, Journal of Mathematics and Mathematical Sciences, (1): 133-141.

[8] W. Shantanwi (2010). Fixed Points Theory for contractive mappings satisfying ∅-maps in G-metric spaces, Fixed point theory and Applications , (2010): 1-9

[9] Z. Mustafa (2005). A new structure for genralized metric spaces with applications to fixed point theory, The University of Newcastle, Callaghan. Aurtralia, (Ph.D. Thesis)

[10] Z. Mustafa and B. Sims (2006). A new approach to generalized metric, Journal of Nonlinear and Convex Analysis, 7(2): 289-297

[11] Z. Mustafa and B. Sims (2009), Fixed point theorems for contractive mappings in complete G-metric spaces , Fixed Point Theory and Applications, (2009): 1-12

[12] Z. Mustafa , H. Obiedat and F. Awawdeh (2008). Some fixed point theorems for mapping on complete G-metric spaces, Fixed point Theory and Applications, (2008): 1-12

[13] Z. Mustafa , W. Shanta and M. Bataineh (2009). Existence of fixed piont result in G-metric spaces, International Journal of Mathematics and Mathematical Sciences, (2009): 1-10