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Common Fixed Point And Coincidence Point Theorem For Expansive Mapping In b-Metric Spaces

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Abstract: *The purpose of this paper is to present coincidence point and common fixed point results for three and four self-expansive mappings in b- metric spaces. The results presented in this paper generalize and extend several well-known results in the literature.*

Keywords: *b- metric space, Common fixed point, Coincidence Point, Expansive Mapping.*

I. INTRODUCTION

It is well known that contractive type conditions play an important role in the study of fixed-point theory. The Banach contraction mapping [1] is one of the conclusive results of analysis. He established the existence and uniqueness theorem for a solution of an operator equation $Tx = x$, by using a contraction condition. It is a very popular tool in solving problems in different areas of mathematics.

The idea of b- metric space was initiated by the works of Bourbaki [16], Bakhtin [2] and Czerwik [3] and [4], which gave an axiom and proved the contraction mapping principle in B- metric spaces that generalized the famous Banach contraction mapping theorem. In 1981, Gillespie and Williams [5] introduced a new class of maps where the existing constant is greater than one. In the sequel, the concept of expansive mapping has been introduced by Daffer, P. Z. and Kaneko, H. [6] and obtain fixed point results. After that various authors have been studied fixed point theorems in metric spaces for expansive mappings see for instance [Daheriya, et al. [7] and Huang, X. et al. [8].

In 2015, Jain, R. et al. [9] obtained some results for fixed point and coincidence point for expansive mappings in b- metric space. Further, in [10] he is proved that fixed point and common fixed point theorems for expansive mappings in parametric space and parametric b- metric spaces with coincidence point.

In the same year, Mohanta, S.K. [11], obtained sufficient conditions for existence of points of coincidence and common fixed points for a pair of self mappings satisfying expansive type conditions in b- metric spaces. We prove the coincidence point theorem for expanding maps without assuming subjectivity of the maps there in metric spaces.

Our results extend and generalize of the results Mohanta, S.K. [11] for expansive mapping in b- metric spaces.

II. Preliminary Notes

In this section we need to recall some basic notations, definitions, and necessary results from existing literature.

Definition 2.1 [8]: Let X be a non-empty set and $s \geq 1$ be a given real number. A function $d: X \times X \to R^+$ is said to be b-metric on X, if the following condition hold:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(iii) $d(x,y) \le s[d(x,z)+d(z,y)]$, for all $x,y \in X$. Then the pair (X,d) is called a b- metric Space. Observe that, if $s=1$, then the ordinary triangle inequality in a metric space is satisfied, however it does not hold true when $s>1$.

Then the class of b- metric spaces is effectively larger than that of ordinary metric space. i.e.

Every metric space is a b- metric space but the converse need not be true. The following example illustrates the above remarks.

Example 2.2: Let $=\{-1,0,1\}$. Define $d: X \times X \to R^+$ by

$$d(x,y)=d(y,x),$$

for all $x,y \in X$, $d(x,x)=0, x \in X$ and $d(-1,0)=3, d(-1,1)=d(0,1)=1$. Then (X,d) is a b- metric space, but not a metric space, because the triangle inequality is not satisfied. Indeed, we have that

$$\begin{aligned} d(-1,1)+d(1,0) &= 1+1 \\ &= 2 \\ &< 3 \\ &= d(-1,0). \end{aligned}$$

It is easy to verify that

$$s = {}^3/_2.$$

Definition 2.3 [12]: Let (X,d) be a b -metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to x, if and only if $\lim_{n\to\infty} d(x_n,x)=0$, we denote this by

$$\lim_{n\to\infty} x_n = x \text{ or } x \to x(n\to\infty).$$

(ii) $\{x_n\}$ is a Cauchy sequence if and only if

$$\lim_{n,m\to\infty} d(x_n,x_m)=0.$$

(iii) (X,d) is complete, if and only if every Cauchy sequence in X is convergent.

Remark 2.4[12]: Let (X,d) be a b- metric space, then the following assertions hold:

(i) A convergent sequence has a unique limit.

(ii) Each convergent sequence is Cauchy.

(iii) In general, b -metric is not continuous.

Definition 2.5[11]: Let (X,d) be a metric space with the coefficient $s \ge 1$ and let $T: X \to X$ be a given mapping. We say that T is continuous at $x_0 \in X$, we have $x_n \to x_0$ as $n\to\infty$.If T is continuous at each point $x_0 \in X$, Then we say that $T$is continuous on X.

Definition 2.6: Let (X,d) be a metric space with the coefficient $s \ge 1$. A mapping let $T: X \to X$ is called expansive if there exists a real constant $k > s$ such that

$$d(Tx,Ty) \ge kd(x,y), \text{ for all } x,y \in X.$$

Definition 2.7[14]: Let T and S be self-mappings of a set X. If $y=Tx=Sx$, for some $x \in X$, then x is called a coincidence point of T and S. and y is called a point of coincidence of T and S.

Definition 2.8[14]: Let S and T be a weakly compatible self-mapping of a non-empty set X. If S and T have a unique point of coincidence $y=Tx=Sx$, for some $x \in X$, then y is the unique common fixed point of coin T and S.

Definition 2.9[15]: The mapping $f,g: X \to X$ are weakly compatible, if for every $x \in X$, the following condition holds:

$$f(gx)=g(fx), \text{ whenever } fx=gx.$$

Lemma 2.10: Suppose (X,d) be a b- metric space and $\{y_n]$ be a sequence in X such that

$$d(y_{n+1},y_{n+2}) \leq \lambda d(y_n,y_{n+1}),$$

Where, $= 0,1,2,3, \ldots$, and e $0 \leq \lambda < 1$.

Then the sequence $\{y_n\}$ is Cauchy sequence in X provided by $s\lambda < 1$.

3. Main Results

Theorem 3.1. Let (X,d) be a complete b- metric space with the coefficient $s \geq 1$.Suppose the mapping $f,g,h: X \to X$ satisfying the condition

$$d(fx,fy) \geq \alpha_1\, d(fx,hx) + \alpha_2\, d(gy,hy) + \alpha_3\, d(fx,hy) + \alpha_4\, d(gy,hx) + \alpha_5\, d(fx,gy) \quad \ldots (3.1.1)$$

for all $x,y \in X$ where $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5 \geq 0$ with $\alpha_1+\alpha_2+\alpha_3+\alpha_4,+\alpha_5 > s$. Assume the following hypothesis:

(1) $\alpha_2+\alpha_3 > s(1-\alpha_1) \geq \alpha_5)$ or

$\alpha_1+\alpha_2 > s(1-\alpha_3) \geq \alpha_4)$

(2) $f(X) \subseteq h(X)$ and $g(X) \subseteq h(X)$.

If $f(x)$ *or* $g(X)$ *or* $h(X)$ area complete subspace of X. Then f,g and h have a point of coincidence in X. Moreover, $\alpha_2+\alpha_4+\alpha_5 >1$ and f,g and h are weakly compatible. Then f,g and h have a unique common fixed point in X.

Proof: Let $x_0 \in X$, then we have $fx_0 \in f(X)$. Since $f(X) \subseteq h(X)$ there exists $x_1 \in X$ such that $fx_0 = hx_1$. From $x_1 \in X$ then $gx_1 \in g(X)$. Since $g(X) \subseteq h(X)$ there exists $x_2 \in X$such that $gx_1 = hx_2$. From $x_2 \in X$, then $fx_2 \in f(X)$. Since $f(X) \subseteq h(X)$, there exists $x_3 \in X$ such that $fx_1 = hx_3$.

Now from $x_3 \in X$, then $gx_3 \in g(X)$. Since $g(X) \subseteq h(X)$ there exists $x_4 \in X$ such that $gx-3 = hx_4$. Continuing this process for all having chosen $x_n \in X$, we obtain $x_{n+1} \in X$ such that

$$fx_{2n} = hx_{2n+1} \text{ and } gx_{2n+1} = hx_{2n+2},$$

for all $n = 0,1,2 \ldots$

In general, we can define a sequence $\{z_n\}$ we have

$$z_{2n} = fx_{2n} = hx_{2n+1} \quad \text{and}$$
$$z_{2n+1} = gx_{2n+1} = hx_{2n+2,} \ldots \qquad (3.1.2)$$

for all $n = 0,1,2, \ldots$

From (3.1.1) and (3.1.2), we have

$$\begin{aligned} d(z_{2n-1},z_{2n}) &= d(hx_{2n},hx_{2n+1}) \\ &\geq \alpha_1 d(Fx_{2n},hx_{2n}) \\ &+ \alpha_2 d(gx_{2n+1},hx_{2n+1}) \\ &+\alpha_3 d(fx_{2n},hx_{2n+1}) \\ &+\alpha_4 d(gx_{2n+1},hx_{2n}) \\ &+ \alpha_5 d(fx_{2n},gx_{2n+1}) \\ &=\alpha_1 d(z_{2n-1},z_{2n}) + \alpha_2 d(z_{2n+1},z_{2n}) \\ &+\alpha_3 d(z_{2n},z_{2n}) \\ &+ \alpha_4 d(z_{2n+1},z_{2n}) + \alpha_5 d(z_{2n-1},z_{2n+1}) \end{aligned}$$

$$\geq \alpha_1 d(z_{2n-1},z_{2n}) + \alpha_2 d(z_{2n+1},z_{2n}) \quad +\alpha_3 d(z_{2n},z_{2n}) + \alpha_4 d(z_{2n+1},z_{2n})$$
$$+ \frac{\alpha_5}{s}[d(z_{2n-1},z_{2n}) + d(z_{2n+1},z_{2n})],$$

$\geq \left(\alpha_2 + \alpha_4 + \frac{\alpha_5}{s}\right) d(z_{2n+1}, z_{2n})$

$+ \left(\alpha_1 + \frac{\alpha_5}{s}\right) d(z_{2n-1}, z_{2n})$.

Implies that,

$1\text{-}\left(\alpha_1 + \frac{\alpha_5}{s}\right) d(z_{2n-1}, z_{2n}) \geq \left(\alpha_2 + \alpha_4 + \frac{\alpha_5}{s}\right) d(z_{2n+1}, z_{2n})$

$\Rightarrow d(z_{2n+1}, z_{2n}) = \frac{1-\alpha_1-\alpha_5/s}{s(\alpha_2+\alpha_4+\alpha_5)} d(z_{2n-1}, z_{2n})$

Put $r = \frac{1-\alpha_1-\alpha_5/s}{s(\alpha_2+\alpha_4+\alpha_5)}$, then we have

$d(z_{2n+1}, z_{2n}) \leq rd(z_{2n_1}, z_{2n})$ …. (3.1.3)

By induction, we get

$d(z_{2n+1}, z_{2n}) \leq r^n d(z_0, z_1)$ …. (3.1.4)

For $m, n \in N$ with $m > n$, we have repeated us of (3.1.4)

$d(z_{2n}, z_{2m}) \leq s[d(z_{2n}, z_{2n+1}) + d(z_{2n+1,} z_{2n})]$

$\leq sd(z_{2n}, z_{2n+1}) + s^2 d(z_{2n+1}, z_{2n+2})$

$+ \cdots .. + s^{m-n}[d(z_{2n-2}, d(z_{2m-1})$

$+d(z_{2m-1}, z_{2m})]d(z_0, z_1)$

$d(z_{2n}, z_{2m}) \leq [sr^{2n} + s^2 r^{2n+1} + s^3 r^{2n+2} + \cdots + s^{2m-2n} r^{2m-2}$

$+ s^{2m-2n} + \cdots .. + s^{2m-1}]d(z_0, z_1)$

$= sr^{2n}[1 + sr + (sr)^2 + \cdots + (sr)^{2m-2n-2} + \cdots + (sr)^{2m-2n-1}]d(z_0, z_1)$

$sr^{2n}\left[\frac{1-(sr)^{2m-2n-1}}{1-sr}\right] d(z_0, z_1)$

Since $0 < sr < 1$, thus we get

$d(z_{2n}, z_{2m}) \leq \frac{sr^{2n}}{1-sr} d(z_0, z_1)$. Thus for $m, n \to \infty$. We get

$d(z_{2n}, z_{2m}) \to 0$.

Hence $\{z_{2n}\}$ is a Cauchy sequence in $h(X)$. Since $h(X)$ is a complete in X, so there exists $z^* \in h(X)$ such that $(z_{2n}, z^*) \to 0$ .... (3.1.5)

As $z^* \in h(X)$, then there exists $y^* \in X$ such that

$hy^* = z^*$.

We claim that $f, g,$ and h have a coincidence point in X. Before that, we will show that

$d(fy^*, z_{2n}) \to 0$.

From (3.1.1), we have

$d(y^*, z_{2n}) = d(hy^*, z_{2n})$

$= d(hy^*, hx_{2n+1})$

$\geq \alpha_1 d(fy^*, hy^*) + \alpha_2 d(gx_{2n+!}, hx_{2n+1})$

$+ \alpha_3 d(fy^*, hx_{2n+1})$

$= + \alpha_4 d(gx_{2n+1}, fy^*) + \alpha_5 d(fy^*, gx_{2n+1})$

$\geq \alpha_2 d(z_{2n}, z_{2n+1}) + \alpha_3 d(fy^*, z_{2n})$

$+\alpha_4 d(z_{2n}, fy^*) + \alpha_5 d(fy^*, z_{2n})$

$= \alpha_2 d(z_{2n}, z_{2n+1}) + (\alpha_3 + \alpha_4 + \alpha_5) d(fy^*, z_{2n})$

Implies that

$\alpha_2 d(z_{2n}, z_{2n+1}) + (\alpha_3 + \alpha_4 + \alpha_5) d(fy^*, z_{2n})$

$\leq d(y^*, z_{2n}) \ldots$ (3.1.6)

Since $0 < r < 1$. Then for $n \to \infty$. We have $d(z_{2n}, z_{2n+1}) \to 0 \ldots$ (3.1.7)

From (3.1.5), we have

$d(z^*, z_{2n}) \to 0$, then by using (3.1.7) into (3.1.6), we get $d(fy^*, z_{2n}) \to 0$ ... (3.1.8)

Now, we will show that f, g and h have a coincidence point in X. By definition of b- metric space

$$d(fy^*, hy^*) \leq s[d(fy^*, z_{2n}) + d(z_{2n}, hy^*)]$$
$$= s[d(fy^*, z_{2n}) + d(z_{2n}, z^*)].$$

Thus, by using (3.1.5) and (3.1.8), we get

$$(fy^*, hy^*) = 0 \quad \ldots$$

From (3.1.1.) and (3.1.2), we have

$$d(z^*, z_{2n-1}) = d(hy^*, z_{2n-1})$$
$$= d(hy^*, hx_{2n})$$
$$d(z^*, z_{2n-1}) = \alpha_1 d(fy^*, hy^*) + \alpha_2 d(gx_{2n}, hx_{2n}) + \alpha_3 d(fy^*, hx_{2n}) + \alpha_4 d(gx_{2n}, hy^*) + \alpha_5 d(fy^*, gx_{2n})$$
$$d(z^*, z_{2n-1}) \geq \alpha_1 d(y^*, y^*) + \alpha_2 d(gx_{2n}, z_{2n-1}) + \alpha_3 d(y^*, z_{2n-1}) + \alpha_4 d(gx_{2n}, y^*) + \alpha_5 d(y^*, gx_{2n}).$$

Thus, we have

$$(\alpha_2 + \alpha_3)\, d(z_{2n-1},\, gx_{2n}) + (\alpha_4 + \alpha_5) d(gx_{2n}, y^*) \leq d(z^*, gx_{2n-1})$$

Then from (3.1.5) and (3.1.8), we have

$d(z_{2n-1},\, gx_{2n}) \to 0 \ldots$ (3.1.9)

Thus, by triangle inequality of b- metric spaces

$$d(gy^*, hy^*) \leq s[d(gy^*, z_{2n-1}) + d(z_{2n-1}, hy^*)]$$
$$= sd(gy^*, z_{2n-1}) + sd(z_{2n-1}, z^*)$$

Thus, by using (3.1.5) and (3.1.9), for $n \to \infty$, then we get

$$d(gy^*, hy^*) = 0.$$

Thus, we have

$$fy^* = hy^* = z^*$$

and

$$fy^* = gy^* = hy^* = z^*.$$

Hence f, g and h have a coincidence point.

Now, we prove the uniqueness of the coincidence point of f, g, and h.

Suppose there exists another coincidence point w^* of f , g and h such that

$fx = gx = hx = w^*$, for some $x \in X$.

From (3.1.1.) we have

$$d(z^*, w^*) = d(hy^*, hx)$$

$\geq \alpha_1 d(fy^*, hy^*) + \alpha_2 d(gx, hx)$

$+\alpha_3 d(fy^*, hx) + \alpha_4 d(w^*, z^*)$

$+ \alpha_5 d(z^*, w^*)$

$\geq (\alpha_3 + \alpha_4 + \alpha_5) d(z^*, w^*).$

Thus, we have

$1 - (\alpha_3 + \alpha_4 + \alpha_5) d(z^*, w^*) \leq 0.$

Since$(\alpha_3 + \alpha_4 + \alpha_5) > 0$. Then we have

$$d(z^*, w^*) = 0.$$

Hence $z^* = w^*$. Now, we claim that f, g and h are common fixed point in X, then by using (3.1.1.), we get

$d(z^*, hz^*) = d(hy^*, hz^*)$

$\geq \alpha_1 d(fz^*, hy^*) + \alpha_2 d(gz^*, hz^*)$

$+\alpha_3 d(fz^*, hz^*) + \alpha_4 d(gz^*, hy^*)$

$+ \alpha_5 d(fz^*, gz^*)$

$= \alpha_1 d(z^*, hz^*) + \alpha_2 d(gz^*, gz^*)$

$+\alpha_3 d(z^*, hz^*) + \alpha_4 d(g`z^*, hy^*)$

$+ \alpha_5 d(z^*, gz^*)$

$d(z^*, hz^*) \geq \alpha_1 d(z^*, hz^*) + \alpha_3 d(z^*, hz^*)$

$+\alpha_4 d(z^*, hz^*)$

$= (\alpha_1 + \alpha_3 + \alpha_4) d(z^*, hz^*).$

Thus, we get

$1 - (\alpha_1 + \alpha_3 + \alpha_4) d(z^*, hz^*) \leq 0.$

Since $1 - (\alpha_1 + \alpha_3 + \alpha_4) > 0$, then we obtain

$$d(z^*, hz^*) = 0 \Rightarrow z^* = hz^*.$$

Since $hz^* = fz^*$ and $hz^* = gz^*$, so we have $z^* = fz^* = gz^* = hz^*$.

Hence f, g and h are common fixed point.

Now we shall show that f, g and h have unique fixed point.

Suppose $x^* \in X$ is another common fixed point of f, g and h. It means that $fx^* = gx^* = hx^* = x^*$. By using (3.1.1), we have

$d(z^*, x^*) = d(hz^*, hx^*)$

$\geq \alpha_1 d(fz^*, hz^*) + \alpha_2 d(gx^*, hx^*)$

$+\alpha_3 d(fz^*, hx^*) + \alpha_4 d(gx^*, hz^*)$

$+ \alpha_5 d(fz^*, gx^*)$

$= \alpha_1 d(z^*, z^*) + \alpha_2 d(gx^*, hx^*)$

$+\alpha_3 d(z^*, x^*) + \alpha_4 d(gx^*, hz^*)$

$+\alpha_5 d(z^*, gx^*$

$\geq (\alpha_3 + \alpha_4 + \alpha_5) d(z^*, x^*)$

Thus, we get

1-$(\alpha_3 + \alpha_4 + \alpha_5) d(z^*, x^*) \leq 0.$

Since $(\alpha_3 + \alpha_4 + \alpha_5) > 0$, therefore, we get

$d(z^*, x^*) = 0$. Thus, we have $z^* = x^*$.

Hence f , g and h have unique common fixe point in X.

Example 3.2: Let $X = [0,1)$ and $p > 1$ be a constant. We define $d: X \times X \to \mathbb{R}^+$ as $d(x,y) = |x-y|^p$, for all $x, y \in X$. Then (X,d) is a b- metric space with the coefficient $s = 2^{p-1.}$ Let us define $f, g, h: X \to X$ defined by $fx = \frac{x}{8}$, $gx = \frac{x}{12}$ and $hx = \frac{x}{2}$.

Clearly, $f(X) \subseteq h(X)$and $g(X) \subseteq h(X)$, for $x, y \in X$.

Now, $d(fx,fy) = \left|\frac{x}{8} - \frac{y}{8}\right|^p$, $d(fx,hx) = \left|\frac{x}{8} - \frac{y}{2}\right|^p = \left|\frac{3x}{8}\right|^p$, $d(gy,hy) = \left|\frac{y}{12} - \frac{y}{2}\right|^p = \left|\frac{5y}{12}\right|^p$

$d(fx,hy) = \left|\frac{x}{8} - \frac{y}{2}\right|^p$, $d(gy,hx) = \left|\frac{y}{12} - \frac{x}{2}\right|^p$

And

$d(fx,gy) = \left|\frac{x}{8} - \frac{y}{12}\right|^p$.

So, $d(fx,fy) = \frac{1}{8}\left|\frac{x}{2} - \frac{y}{2}\right|^p$

$$\geq \frac{1}{6}\left|\frac{3x}{8}\right|^p + \frac{1}{6}\left|\frac{5y}{12}\right|^p + \frac{1}{6}\left|\frac{x}{8} - \frac{y}{2}\right|^p$$
$$+ \frac{1}{6}\left|\frac{y}{12} - \frac{x}{2}\right|^p + \frac{1}{6}\left|\frac{x}{8} - \frac{y}{12}\right|^p$$
$$\geq \alpha_1 d(fx,hx) + \alpha_2 d(gy,hy)$$
$$+ \alpha_3 d(fx,hy) + \alpha_4 d(gy,hx)$$
$$+\alpha_5 d(fx,gy).$$

Thus, all conditions of theorem 3.1. are satisfies $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, +\alpha_5 > \frac{5s}{6}$. Note that, 0 is unique common fixed point of the mapping f, g and h.

Corollary 3.3. Let (X,d) be a complete b- metric space with the coefficient $s \geq 1$.Suppose the mapping $f, g, : X \to X$ satisfying the condition

$$d(fx,fy) \geq \alpha_1 d(fx,gx) + \alpha_2 d(fy,gy)$$
$$+\alpha_3 d(fx,gy) + \alpha_4 d(gx,gy)$$
$$+\alpha_5 d(fy,gx) \qquad (3.3.1)$$

for all $x, y \in X$ where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \geq 0$ with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, +\alpha_5 > s$.

Assume the following hypothesis:

(1) $\alpha_2 + \alpha_3 > s(1 - \alpha_1) \geq \alpha_5)$

or $\alpha_1 + \alpha_2 > s(1 - \alpha_3) \geq \alpha_4)$

(2) $f(X) \subseteq h(X)$

(3) If $f(x)$ or $g(X)$ or $h(X)$ area complete subspace of X. Then f, g and h have a point of coincidence in X. Moreover, $\alpha_2 + \alpha_4 + \alpha_5$ >1 and f, g and h are weakly compatible. Then f, g and h have a unique common fixed point in X.

Proof: Setting $f = g = h$ in the theorem 3.3. the required above result.

Corollary 3.4: Let (X,d) be a complete b- metric space with the coefficient $s \geq 1$.Suppose the mapping $f, g: X \to X$ satisfy the condition

$$d(fx,fy) \geq \alpha_1 d(fx,gx) + \alpha_2 d(fy,gy)$$
$$+\alpha_3 d(gx,gy)\ldots \qquad (3.4.1)$$

for all $x, y \in X$ where $\alpha_1, \alpha_2, \alpha_3 \geq 0$ with $\alpha_1 + \alpha_2 + \alpha_3 > s$. Suppose the following hypothesis are also satisfy

(1) $\alpha_1 + \alpha_2 > s(1-\alpha_2)$ or $\alpha_1 + \alpha_2 > s(1-\alpha_3)$.

(2) $g(X) \subseteq f(X)$.

(3) If $f(x)$ or $g(X)$ are a complete subspace of X. Then f and g have a point of coincidence in X. Moreover, α_1 >1 and f and g are weakly compatible. Then f and g have a unique common fixed point in X.

Proof: It follows by taking $\alpha_3 = \alpha_5 = 0$ and $\alpha_4 = \alpha_3$ in Corollary 3.3, then we get the result of Mohanta, S. K. (2016).

Corollary 3.5: Let (X,d) be a complete b- metric space with the coefficient $s \geq 1$.Suppose the mapping $f,g: X \to X$ satisfy the condition

$$d(fx,fy) \geq \alpha_1 d(gx,gy) \ldots\ldots \qquad (3.5.1)$$

for all $x,y \in X$ where $\alpha_1 \geq 0$. Suppose the following hypothesis are also satisfy

(1). $g(X) \subseteq f(X)$.

(2). If $f(x)$ or $g(X)$ are a complete subspace of X. Then f and g have a point of coincidence in X. Moreover, α_1>1 and f and g are weakly compatible. Then f and g have a unique common fixed point in X.

Proof: It follows by taking $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = \alpha_1$ in Corollary 3.4, then we get the above result, which is corollary of Mohanta, S. K. (2016).

Corollary 3.6: Let (X,d) be a complete b- metric space with the coefficient $s \geq 1$.Suppose the mapping $g: X \to X$ satisfy the condition

$$d(gx,gy) \leq kd(x,y) \qquad (3.6.1)$$

for all $x,y \in X$ where $k \geq (0,\frac{1}{s})$ .Then g has a unique fixed point in X. Furthermore, the iterative sequence $\{g^n x\}$ converges to the fixed point

Proof: Setting $\frac{1}{\alpha_1} = k$ and $f = I$, the identity mapping on X, in 3.5, which is b- metric version of Banach contraction principle.

Example 3.7: Let $X = \mathbb{R}^+$and $d: X \times X \to \mathbb{R}^+$ as $d(x,y) = \{\max(x,y)\}^2$, for all $x,y \in X$. Then (X,d) is a b- metric space with the coefficient $s = 2$. Define $f,g,h: X \to X$ defined by $fx = \frac{x}{2}$, $gx = \frac{x}{5}$ for all $x,y \in X$, we have $d(fx,fy) \geq 6d(gx,gy)$. $i.e.$ the condition of corollary 3.3 holds for $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$ and α_5= 6. Therefore, we have all hypothesis of corollary 3.3. and 0 is the coincidence point of f and g.

Theorem 3.8: Let P,Q,R and S are four surjective mappings of a complete $b-$ metric space(X,d) with coefficient $s \geq 1$. Satisfying the following inequalities:

$$d(P(Qx),Qx) + \frac{k}{s} d(P(Qx),x) \geq \alpha d(Qx,x) \ldots \quad (3.8.1)$$

$$d(Q(Px),Px) + \frac{k}{s} d(Q(Px),x) \geq \beta d(Px,x) \ldots (3.8.2)$$

and

$$d(R(Sx),Sx) + \frac{k}{s} d(R(Sx),x) \geq \gamma d(Sx,x) \ldots\ldots \quad (3.8.3)$$

$$d(S(Rx),Rx) + \frac{k}{s} d(S(Rx),x) \geq \alpha d(Rx,x) \ldots \quad (3.8.4)$$

for $x \in X$, where α,β,γ,k are non-negative real numbers with $\alpha > s + (1+s)k, \beta > s + (1+s)k$, $\gamma > s + (1+s)k$ and $\lambda > s + (1+s)k$. If P,Q,R and S are continuous, then P,Q,R and S have a common fixed point in X.

Proof: Let w_0 be an arbitrary point in X. Since P is surjective there exists $w_1 \in X$ such that

$$w_1 = Pw_0.$$

Again, Q is surjective there exists $w_2 \in X$ such that

$$w_2 = Q\, w_1.$$

And R is surjective there exists $w_3 \in X$ such that

$$w_3 = Rw_2.$$

Also, S be a surjective there exists $w_4 \in X$ such that

$$w_4 = Sw_3.$$

Continuing this process. We can construct a sequence $\{w_n\}$ in X such that

$$w_{2n} = Pw_{2n+1} \ldots \qquad (3.8.5)$$

And

$$w_{2n+1} = Qw_{2n+2} \qquad (3.8.6)$$

also

$$w_{2n+2} = Rw_{2n+3} \qquad (3.8.7)$$

$$w_{2n+3} = Sw_{2n+4} \qquad (3.8.8)$$

Now from (3.8.1), we have, for $n \in \mathbb{N} \cup \{0\}$

$$d(P(Qw_{2n+2}), Qw_{2n+2}) + \frac{k}{s} d(P(Qw_{2n+2}), w_{2n+2})$$

$$\geq \ \alpha d(\, Qw_{2n+2}, w_{2n+2}),$$

which implies that

$$d(w_{2n}, w_{2n+1}) + \frac{k}{s} d(w_{2n}, w_{2n+2})$$

$$\geq \alpha d(\, w_{2n+1}, w_{2n+2})$$

Hence, we have

$$\alpha d(\, w_{2n+1}, w_{2n+2}) \ \leq d(w_{2n}, w_{2n+1})$$

$$+sk[d(w_{2n}, w_{2n+1})$$

$$+d(w_{2n+1}, w_{2n+1})$$

Therefore,

$$d(\, w_{2n+1}, w_{2n+2}) \ \leq \ \frac{1+sk}{\alpha - sk} \ d(w_{2n}, w_{2n+1}) \ldots \ (3.8.9)$$

Now, we using (3.8.2) by argument similar to that used above, we obtain that

$$d(w_{2n}, w_{2n+1}) \leq \frac{1+sk}{\beta - sk} \ d(w_{2n-1}, w_{2n}) \ldots \ (3.8.10)$$

On the other hand, we have by (3.8.3)

$$d(R(Sw_{2n+4}), Sw_{2n+4}) + \frac{k}{s} d(R(Sw_{2n+4}), w_{2n+4})$$

$$\geq \gamma d(\, Sw_{2n+4}, w_{2n+4}),$$

which implies that

$$d(w_{2n+2}, w_{2n+3}) + \frac{k}{s} d(w_{2n+2}, w_{2n+4})$$

$$\geq \gamma d(\, w_{2n+2}, w_{2n+4})$$

Hence, we have

$$\gamma d(\, w_{2n+2}, w_{2n+4}) \ \leq d(w_{2n+2}, w_{2n+3})$$

$$+sk[d(w_{2n+2}, w_{2n+3})$$

$+d(w_{2n+3}, w_{2n+4})$

Therefore,

$d(w_{2n+3}, w_{2n+4}) \leq \frac{1+sk}{\gamma-sk} d(w_{2n+2}, w_{2n+3})$ .... (3.8.11)

..

Next, we using (3.8.4) by an argument similar to that used above, we obtain that,

$d(w_{2n+2}, w_{2n+3}) \leq \frac{1+sk}{\lambda-sk} d(w_{2n+2}, w_{2n+3})$ .... (3.8.12)

Let $\delta = \max\left\{\frac{1+sk}{\alpha-sk}, \frac{1+sk}{\beta-sk}, \frac{1+sk}{\gamma-sk} \,\&\, \frac{1+sk}{\lambda-sk}\right\} < \frac{1}{s}$

Combining (3.8.9), (3.8.10), (3.8.11) and (3.8.12), we get

$d(x_{2n}, x_{2n+1}) \leq \delta d(x_{2n-1}, x_{2n})$

&

$d(x_{2n+1}, x_{2n+2}) \leq \delta d(x_{2n}, x_{2n+1})$

Where $\delta \in [0, \frac{1}{s})$ for all $n \in \mathbb{N} \cup \{0\}$. By repeating this process, we have

$d(x_{2n+1}, x_{2n+2}) \leq \delta^n d(x_0, x_1)$.

Then by Lemma 2.10, $\{x_{2n}\}$ is a Cauchy sequence in complete b-metric space. Then there exists $w^* \in X$ such that

$w_{2n} \to w^*$ as $n \to \infty$.

Therefore, $w_{2n+1} \to w^*$ and $w_{2n+2} \to w^*$ as $n \to \infty$. The continuity of P, Q, R and S.

implies that, $Pw_{2n+1} \to Pw^*$. But $Pw_{2n+1} = w_{2n} \to w^*$ as $n \to \infty$. Thus $Pw^* = w^*$.

Now, since Q is continuous, then $Qw_{2n+1} \to Qw^*$as $n \to \infty$. But $Qw_{2n+1} = w_{2n+1} = w^*$.

Thus, $Qw^* = w^*$.

Similarly, R and S are continuous, so, $Rw_{2n+3} \to Rw^*$ and $Sw_{2n+4} \to Sw^*$ as $n \to \infty$. $i.e.$ $w_{2n} \to Pw^*, w_{2n+1} \to Qw^*, w_{2n+2} \to Rw^*$ and $w_{2n+3} \to Sw^*$ as $n \to \infty$. The uniqueness of limit yields that

$w^* = Pw^* = Qw^* = Rw^* = Sw^*$.

Hence, w^* is a common fixed point of P, Q, R and S.

Corollary 3.9: Let P and R are two surjective mappings of a complete $b-$metric space(X, d) with coefficient $s \geq 1$. Satisfying the following inequalities:

$d(P^2x, Px) + \frac{k}{s} d(P^2, x) \geq \alpha d(Px, x)$ ... (3.9.1)

and

$d(R^2x, Rx) + \frac{k}{s} d(R^2x, x) \geq \beta d(Rx, x)$...... (3.9.2)

For $x \in X$, where $\alpha, \beta, k > 0$ with $\alpha > s(1+s)k$,

$\beta > s(1+s)k$. Then P and R have a common fixed point in X.

Proof: It follows from theorem 6.2.8 by taking $Q = P, S = R$ and $\beta = \alpha$ and $\lambda = \gamma = \beta$. Then we get the above result.

Corollary 3.10: Let $P: X \to X$ be a surjective mappings on a complete $b-$ metric space(X, d) with coefficient $s \geq 1$. Satisfying the following inequalities:

$d(P^2x, Px) + \frac{k}{s} d(d(P^2x, x) \geq \alpha d(Px, x)$.... (3.10.1)

For $x \in X$, where $\alpha, k > 0$ with $\alpha > s(1+s)k$ .Then P has a unique fixed point in X.

Example 3.11: Let $X = [0, \infty)$ and define $d: X \times X \to \mathbb{R}^+$ by $d(x, y) = |x - y|^2$, for all $x, y \in X$. Then (X, d) is a complete b – metric space with $s = 2$. Define $P: X \to X$ by $Px = 2x$. Now

$d(P^2x, Px) + \frac{k}{s} d(d(P^2x, x) = d(4x, 2x) + d(4x, x)$

$= |4x - 2x|^2 + |4x - x|^2$

$= 4x^2 + 9x^2$

$= 13x^2$

$\geq 12x^2$

$= 12|2x - x|^2$

$= 12d(Px, x).$

For all $x \in$ X. Hence $k = 1$ and $\alpha = 12$.

Clearly, 12 = $\alpha > s(1 + s)k$

= 6.

Thus, P satisfies all the hypothesis of corollary 6.2.10 and 0 is the unique fixed point of P

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