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Solution of Generalized Fractional Kinetic Equation by Laplace and Kamal Transformation

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Abstract *- In this paper we consider a generalized fractional kinetic equation which contain generalized Mittag-Leffler function* $E_{\alpha,\beta}^{\gamma,q}[z]$. *The solution of this generalized fractional kinetic equation are obtained by the method of Laplace transform and Kamal transform. The study will also try to establish the relation existing between these new integral transform in particular, Some known results are also obtain in a special cases. Both the transformation gives same results.*

Keywords - *Fractional Calculus, Riemann-Liouville operator, Generalized Mittag-Leffler function, Laplace Transform, Inverse Laplace Transform, Kamal Transform, Inverse Kamal Transform.*

I. INTRODUCTION

Fractional calculus has been a fruitful field of research in science and engineering. We aim to introduce fractional kinetic equation, which has an important role in various fields. Fractional kinetic equation play an important role in astrophysical problems, mathematical physics, modelling of many physical and chemical processes, engineering real-world physical problems, the fluid dynamic, traffic model, measurement of viscoelastic material mathematics, dynamic system, control systems, biology, nuclear physics and atomic physics. This extended generalized fractional kinetic equation can be used to compute the particle reaction rate and may be utilized in other branch of mathematics. Many authors recommended Fractional differential equations concerning different types of fractional operators or derivatives as [8], [9], [13], [16].
The standard kinetic equation is

$$\frac{d}{dt} N_i(t) = -C_i N_i(t) , \qquad (C_i > 0) \dots \tag{1.1}$$

is studied in the following form [4]

$$N(t) - N_0 = -c^{\upsilon} \, {}_0D_t^{-\upsilon} N(t) \dots \tag{1.2}$$

Where ${}_0D_t^{-\upsilon} f(t)$ is the Riemann-Liouville operator defined as [10], [12], [17]:

$${}_0D_t^{-\upsilon} f(t) = \frac{1}{\Gamma(\upsilon)} \int_0^t (t-u)^{\upsilon-1} f(u) du , \qquad \upsilon > 0 \dots \tag{1.3}$$

with ${}_0D_t^{0} f(t) = f(t)$

The Laplace Transform of the operator defined in (1.3) is given by [2]

$$L[{}_0D_t^{-\upsilon} f(t); p] = p^{-\upsilon} F(p) \dots \tag{1.4}$$

This fractional kinetic eqn. (1.2) is generalized and studied in [18], [19]. They considered the free term containing Mittag-Leffler function and derived the solutions of such fractional kinetic equations in terms of generalized Mittag-Leffler function.The generalized Mittag-Leffler function is defined by Shukla and Prajapati [21] in 2007.

$$E_{\alpha,\beta}^{\gamma,q}[z] = \sum_0^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} \dots \tag{1.5}$$

where $\alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0 \; and \; q \in (0,1) \cup N$

where $(\gamma)_{qn} = \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$ denotes the generalized pochhammer symbol.

For $q = 1$ in (1.5) reduce to generalized Mittag-leffler function is defined by Prabakar [14]

$$E_{\alpha,\beta}^{\gamma}[z] = \sum_0^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} \qquad (\alpha, \beta, \gamma \in C; Re(\alpha) > 0) \ldots \tag{1.6}$$

For $\gamma = 1$ it reduces into

$$E_{\alpha,\beta}^{1}[z] = E_{\alpha,\beta}(z) = \sum_0^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \qquad (\alpha, \beta \in C; Re(\alpha) > 0) \ldots \tag{1.7}$$

And in particular for $\gamma = \beta = 1$

$$E_{\alpha,1}^{1}[z] = E_{\alpha}(z) = \sum_0^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} \qquad (\alpha \in C; Re(\alpha) > 0) \ldots \tag{1.8}$$

The function $E_{\alpha,\beta}(z)$ and $E_{\alpha}(z)$ are the Mittag-Leffler function [3]

The following Laplace transforms are also required in the sequel see in [5]

$$L[(1 + c^{\upsilon}\,{}_0D_t^{-\upsilon})^n N(t); p] = \sum_{r=0}^{n} \binom{n}{r} c^{\upsilon r} L\{[{}_0D_t^{-\upsilon} N(t)]; p\} \ldots \tag{1.9}$$

Where

$$\binom{n}{r} = \frac{n!}{(n-r)!\; r!} \ldots \tag{1.10}$$

Which in view of (1.4) gives

$$L[(1 + c^{\upsilon}\,{}_0D_t^{-\upsilon})^n N(t); p] = (1 + c^{\upsilon} p^{-\upsilon})^n N(p) \ldots \tag{1.11}$$

The Laplace Transform in (1.12) is obtained with the help of binomial expansion and turn by turn integration in view of (1.4).

The Laplace transform of power function is defined as

$$L\{t^{\mu-1}\} = \frac{\Gamma(\mu)}{p^{\mu}} \ldots \tag{1.12}$$

Then the following inverse Laplace transforms in view of (1.13) are required.

$$L^{-1}\{p^{-\mu}\} = \frac{t^{\mu-1}}{\Gamma(\mu)} \ldots \tag{1.13}$$

Laplace of $$L\left\{t^{\beta-1}E_{\alpha,\beta}^{\gamma,q}(-c^{v}t^{v})\right\} = p^{-\beta}(1+c^{v}p^{-v})^{-\gamma,q} \ldots \quad (1.14)$$

Where $$(1-zs^{-\alpha})^{-\gamma,q} = \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{(n)!}z^{n} \ \ldots \quad (1.15)$$

is a slightly different notation of binomial theorem see in [15].

Kamal Transform:

The Kamal transform is a new integral transform similar to Laplace transform (see [11]) we can take set A the function is defined in the form:

$$A = \left\{f: |f(t)| < qe^{\frac{|t|}{\eta}}\ if\ t \in (-1)^{j}.\ [0,\infty], j = 1,2; (q, \eta_1, \eta_2 > 0)\right\}$$

Where q is constant and η_1, η_2 can be finite or infinite and the constant q must be finite [7]. Then the integral equation is

$$G(p) = \text{ꝁ}f(t) = \int_0^\infty e^{\frac{-t}{p}}f(t)dt,\ \ p \in (\eta_1, \eta_2) \ldots \quad (1.16)$$

If we apply convolution theorem for Kamal transform [5], [6], we observe that (1.3) gives us the following identity:

$$K\{{}_0D_t^{-\upsilon}f(t);p\} = K\left\{\frac{t^{\upsilon-1}}{\Gamma\upsilon}\right\}.K\{f(t)\} = p^{\upsilon}\mathrm{F}(p) \ \ldots \quad (1.17)$$

Relation between Kamal and Laplace transform:

Let $f(t) \in A$ with Kamal transform $G(p)$ and Laplace transform $F(p)$ see [11], then

$$G(p) = F\left(\frac{1}{p}\right)\ldots \quad (1.18)$$

and $$F(p) = G\left(\frac{1}{p}\right)\ldots \quad (1.19)$$

The following Kamal transform are also required in the sequel:

$$K[(1+c^{\upsilon}\,{}_0D_t^{-\upsilon})^{n}N(t);p] = \sum_{r=0}^{n}\binom{n}{r}c^{\upsilon r}K[\{{}_0D_t^{-\upsilon r}N(t)\};p] \ldots \quad (1.20)$$

Where $\binom{n}{r} = \frac{n!}{(n-r)!}$, This in view of (1.17) gives

$$K[(1+c^{\upsilon}\,{}_0D_t^{-\upsilon})^{n}N(t);p] = [1+c^{\upsilon}p^{\upsilon}]^{n}N(p) \ldots \quad (1.21)$$

The Kamal transform in (1.21) is obtained with the help of binomial expansion and turn by turn integration in view of (1.17). The Kamal transform of power function is defined by:

$$K[t^{\mu-1}] = (\mu-1)!\, p^{\mu} \ \ldots \tag{1.22}$$

Then the following inverse Kamal transforms in (1.22)

$$K^{-1}[p^{\mu}] = \frac{t^{\mu-1}}{(\mu-1)!} \quad , \Gamma\mu = (\mu-1)\ \Gamma(\mu-1 = (\mu-1)!\ \ldots \tag{1.23}$$

Kamal of

$$K\left\{t^{\beta-1} E_{\alpha,\beta}^{\gamma,q}(-c^{v}t^{v})\right\} = p^{\beta}(1+c^{v}p^{v})^{-\gamma,q} \ldots \tag{1.24}$$

In section -2 deals with fractional kinetic equation in the form of theorem. The solutions of these kinetic equation obtained in term of Generalized Mittag-Leffler function. In section -3, some known fractional kinetic equations are given as special cases.

II. MAIN RESULTS

Theorem 1: If $c > 0, v > 0, \delta > 0$ then the solution of the equation

$$N(t) - N_0\ t^{\beta-1} E_{\propto,\beta}^{\gamma,q}(-c^{v}t^{v}) = -\left\{\sum_{r=1}^{n} \binom{n}{r} (c^{v})^{r}\ {}_0D_t^{-vr}\right\} N(t) \ldots \tag{2.1}$$

$$\text{i.e.} \qquad (1+c^{v}{}_0D_t^{-v})^{n}\ N(t) = N_0 t^{\beta-1}\ E_{\propto,\beta}^{\gamma,q}\ [-c^{v}t^{v}] \ldots \tag{2.2}$$

is given by

$$N(t) = N_0 t^{\beta-1} E_{v,\beta}^{(\gamma+n),q}\ [-c^{v}t^{v}] \ldots \tag{2.3}$$

Proof: Taking Laplace Transform of (2.2) in views of (1.11) and (1.14), we have

$$L[(1+c^{v}{}_0D_t^{-v})^{n}\ N(t)] = N_0 L\left[t^{\beta-1}\ E_{\propto,\beta}^{\gamma,q}\ (-c^{v}t^{v})\right]$$

$$(1+c^{v}p^{-v})^{n}\ N(p) = N_0\ p^{-\beta}(1+\ c^{v}p^{-v})^{-\gamma,q}$$

i.e.

$$N(p) = N_0\ p^{-\beta}(1+\ c^{v}p^{-v})^{-(\gamma+n),q}$$

Taking Inverse Laplace Transform we have,

$$N(t) = N_0 L^{-1}\left\{p^{-\beta}(1+c^{v}p^{-v})^{-(\gamma+n),q}\right\} = N_0 \sum_{r=0}^{\infty} \frac{(\gamma+n)_{rq}}{r!}\ (-c^{v})^{r}\ \ L^{-1}\left\{p^{-\beta-vr}\right\}$$

which in view of (1.15) gives

$$N(t) = N_0 t^{\beta-1} \sum_{r=0}^{\infty} \frac{(\gamma+n)_{rq}}{r!} \; \frac{(-c^v t^v)^r}{\Gamma(\beta+vr)}$$

On interpreting the resulting series with the help of (1.5) we at once arrive at the solution in (2.3).

Theorem 2: If $c > 0, v > 0, \delta > 0$ then the solution of the equation

$$N(t) - N_0 \; t^{\beta-1} E_{\propto,\beta}^{\gamma,q}(-c^v t^v) = -\left\{\sum_{r=1}^{n} \binom{n}{r} (c^v)^r \, {}_0D_t^{-vr}\right\} N(t) \ldots \quad (2.4)$$

i.e.

$$(1 + c^v{}_0 D_t^{-v})^n \; N(t) = N_0 t^{\beta-1} \; E_{\propto,\beta}^{\gamma,q} \; (-c^v t^v) \ldots \quad (2.5)$$

is given by

$$N(t) = N_0 t^{\beta-1} E_{v,\beta}^{(\gamma+n),q} [-c^v t^v] \ldots \quad (2.6)$$

Proof: Taking Kamal transform of (2.5) in views of (1.21) and (1.24), we have

$$K[(1 + c^v{}_0 D_t^{-v})^n \; N(t)] = N_0 K\left[t^{\beta-1} \; E_{\propto,\beta}^{\gamma,q} \; (-c^v t^v)\right]$$

$$(1 + c^v p^v)^n \; N(p) = N_0 \, p^{\beta} (1 + c^v p^v)^{-\gamma,q}$$

i.e.

$$N(p) = N_0 \; p^{\beta} (1 + c^v p^v)^{-(\gamma+n),q}$$

Taking Inverse Kamal transform we have,

$$K^{-1}[N(p)] = N_0 K^{-1}\left\{p^{\beta}(1 + c^v p^v)^{-(\gamma+n),q}\right\}$$

$$N(t) = N_0 \sum_{r=0}^{\infty} \frac{(\gamma+n)_{rq}}{r!} \; (-c^v)^r \; k^{-1}\left\{p^{\beta+vr}\right\}$$

$$N(t) = N_0 \sum_{r=0}^{\infty} \frac{(\gamma+n)_{rq}}{r!} \; (-c^v)^r \; \frac{t^{\beta+vr-1}}{(\beta+vr-1)!}$$

Where $(\beta + vr - 1)! = \Gamma(\beta + vr)$, Which in view of (1.15) gives

$$N(t) = N_0 t^{\beta-1} \sum_{r=0}^{\infty} \frac{(\gamma+n)_{rq}}{r!} \; \frac{(-c^v t^v)^r}{\Gamma(\beta+vr)}$$

On interpreting the resulting series with the help of (1.5) we at once arrive at the solution in (2.6).

III. SPECIAL CASESS

(i) If we take $q = 1$ and $\beta \rightarrow \delta$ in theorem – 1we get the known result of ([5], eqns. (2.10)-(2.11), p.231)

(ii) If we take $n = 1, \beta = \mu$ then these reduce to known result ([19], eqns.(27)-(28), p.661) which in turn at $\gamma = 1$ provide the known results ([19], eqns. (33)-(34), p.662) respectively.

(iii) If we replace $(\beta + vr - 1)! = \Gamma(\beta + vr)$ in theorem-1 then we get the same result of theorem-2.

(iv) If we take $n = 1, q = 1$ and $\beta = \mu - 1$ in equation (2.3) and (2.6) we get the known result of ([20], eqns. (30), p.326).

IV. CONCLUSIONS

The fractional kinetic equations are studied involving generalized Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,q}[z]$. The solution of this generalized fractional kinetic equation are obtained by the Laplace transform and Kamal transform. We observed that Laplace transform and Kamal transform solves fractional kinetic equation with a few applications and derived solution for same. It is found that the same results of theorem-1 and theorem-2. Other researcher can do more works using different integral transform and others special functions we can easily obtain various known and new fractional kinetic equations.

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