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Fekete-Szego Inequality for Certain Classes of Analytic Functions using (p,q)-Ruscheweyh Derivative

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Abstract – *In the present work, we investigate the Fekete-Szego inequalities for certain classes of analytic functions by using (p,q)-Ruscheweyh derivative and also the estimates on the coefficients for second and third coefficients of these classes are discussed.*

Keywords — *Fekete-Szego inequality, Subordination, (p,q)-Ruscheweyh differential operator.*

I. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad , \quad a_n \geq 0 \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in C : |z| < 1\}$.

For any two analytic functions f and g in D, we say that f is subordinate to g in D i.e $f \prec g$, if there exists a Schwarz function w, which is analytic in D with $w(0) = 1$ and $|w(z)| < 1$ such that $f(z) = g\big(w(z)\big)$, $z \in D$ (see[4]).

For $0 < q < p \leq 1$, the (p, q)-integer number is defined as [6]

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}$$

and (p, q) −factorial of integer number n is given by [6]

$$[n]_{p,q}! = \begin{cases} [n]_{p,q}[n-1]_{p,q} \dots [1]_{p,q}, & n = 1, 2, \dots \\ 1 & n = 0. \end{cases}$$

The (p, q) −analogue of Jackson derivative of the function f is given by

$$D_{p,q} f(z) = \frac{f(pz) - f(qz)}{(p-q)z}, \ (z \neq 0, 0 < q < p \leq 1).$$

For $\delta \geq 0$, the Ruscheweyh type (p, q) −differential operator $R^{\delta}_{p,q} : A \to A$ is given by

$$R^{\,\delta}_{\,p,q} f(z) = z + \sum_{n=2}^{\infty} \frac{[n+\delta-1]_{p,q}!}{[\delta]_{p,q}![n-1]_{p,q}!} a_n z^n. \tag{1.2}$$

We note that $R^0_{p,q} f(z) = f(z)$, $\quad R^1_{p,q} f(z) = z D_{p,q} f(z)$, $\quad R^{\delta}_{p,q} f(z) = \frac{z D^{\delta}_{p,q}(z^{\delta-1} f(z))}{[\delta]_{p,q}!}$.

We observe that, for $p = 1$ and $q \to 1$, the (p, q) − integer number $[n]_{p,q}$ reduces to the ordinary number n and (p, q) −Ruscheweyh differential operator reduces to the the Ruscheweyh differential operator defined by Ruscheweyh in [9].

Using the concept of subordination and Ruscheweyh differential operator, we define the following subclasses of analytic functions

Definition 1.1. A function $f \in A$ is said to be in the class $S^*_{p,q}(\delta, \Phi)$ if it obeys the condition

$$\frac{z D_{p,q}(R^{\delta}_{p,q} f(z))}{R^{\delta}_{p,q} f(z)} \prec \Phi(z) \quad (0 < q < p \leq 1), \tag{1.3}$$

where $\Phi(z)$ is analytic in D with $Re\{\Phi(z)\} > 0, \Phi(0) = 1$ and $\Phi'(0) > 0$.

Definition 1.2. A function $f \in A$ is said to be in the class $C_{p,q}(\delta, \Phi)$ if it satisfies the condition

$$\frac{D_{p,q}\left(zD_{p,q}\left(R_{p,q}^{\delta}f(z)\right)\right)}{D_{p,q}\left(R_{p,q}^{\delta}f(z)\right)} \prec \Phi(z) \quad (0 < q < p \le 1), \tag{1.4}$$

where $\Phi(z)$ is analytic in D with $Re\{\Phi(z)\} > 0, \Phi(0) = 1$ and $\Phi'(0) > 0$.

The classes $S_{p,q}^{*}(\delta,\Phi)$ and $C_{p,q}(\delta,\Phi)$ contains many well- known classes of analytic functions such as :

i) $S_{p,q}^{*}(0,\Phi) = S_{p,q}^{*}(\Phi)$ and $C_{p,q}(0,\Phi) = C_{p,q}(\Phi)$, defined by Srivastava [2].

ii) $S_{1,q}^{*}(0,\Phi) = S_{,q}^{*}(\Phi)$ and $C_{1,q}(0,\Phi) = C_{q}(\Phi)$, introduced by Cetinkaya [1].

iii) $S_{1,1}^{*}(0,\Phi) = S^{*}(\Phi)$ and $C_{1,1}(0,\Phi) = C(\Phi)$, studied by Ma-Minda [8].

We require the following lemma to prove our main results:

***Lemma 1.3.*[8]** If $p(z) = 1 + c_1 z + c_2 z^2 + . \; . \; .$ is a function with $Re\{p(z)\} > 0$ and $\mu \in C$, then

$$|c_2 - \mu c_1^2| \le 2\max\{1, |2\mu - 1|.$$

The result is sharp for the functions $p(z)$ given by

$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z}.$$

II. MAIN RESULTS

In this section we obtain the Fekete-Szego inequalities for the classes $S_{p,q}^{*}(\delta,\Phi)$ and $C_{p,q}(\delta,\Phi)$.

Thereom 2.1. Let $\Phi(z) = 1 + B_1 z + B_2 z^{2+\cdot\;\cdot\;\cdot}$ with $B_1 \ne 0$. If f is given by (1.1) belongs to the class $S_{p,q}^{*}(\Phi, b)$, then

$$|a_3 - \mu a_2^2| \le \frac{[2]_{p,q}!|B_1|}{[\delta+1]_{p,q}[\delta+2]_{p,q}([3]_{p,q}-1)}\, max\left\{1, \;\; \left|\frac{B_2}{B_1} + \frac{B_1}{[2]_{p,q}-1}\left(1 - \frac{[\delta+2]_{p,q}([3]_{p,q}-1)}{[\delta+1]_{p,q}[2]_{p,q}([2]_{p,q}-1)}\,\mu\right)\right|\right\}, \tag{2.1}$$

Where $B_{1\,,}\; B_2 . \; . \; . \in R,\; \mu \in C \;\; and \;\; 0 < q < p \le 1$. The result is sharp.

Proof. Let $f \in S_{p,q}^{*}(\delta,\Phi)$, then by definition 1.1 and from subordination principle , there exists a Schwarz function w such that

$$\frac{zD_{p,q}(R_{p,q}^{\delta}f(z))}{R_{p,q}^{\delta}f(z)} = \Phi\big(w(z)\big). \tag{2.2}$$

Define the function

$$p(z) = 1 + c_1 z + c_2 z^2 + . \; . \; . \tag{2.3}$$

In terms of the function $w(z)$ as

$$p(z) = \frac{1+w(z)}{1-w(z)},$$

which gives

$$w(z) = \frac{p(z)-1}{p(z)+1} \tag{2.4}$$

From equations (2.3) and (2.4), we obtain

$$\Phi\big(w(z)\big) = \Phi\left(\frac{c_1 z + c_2 z^2 + . \; . \; .}{2 + c_1 z + c_2 z^2 + \; . \; . \; . \; .}\right) = \Phi\left(\frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^2}{4}\right)z^3 + \; . \; . \; . \;\right]\right). \tag{2.5}$$

Since $\Phi(z) = 1 + B_1 z + B_2 z^{2+\cdot\;\cdot\;\cdot}$, the equation (2.5) gives

$$\Phi\big(w(z)\big) = 1 + \frac{B_1 C_1}{2}\, z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 C_1^2}{4}\right] z^2 + . \; . \; . \; . \tag{2.6}$$

Thus the equation (2.2) becomes

$$\frac{zD_{p,q}(R_{p,q}^{\delta}f(z))}{R_{p,q}^{\delta}f(z)} = 1 + \frac{B_1 C_1}{2}\, z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 C_1^2}{4}\right] z^2 + . \; . \; .$$

Equationg the coefficients of z^2 and z^3 on both sides of the equations and on simplification, we get

$$a_2 = \frac{B_1 c_1}{2[\delta+1]_{p,q}([2]_{p,q}-1)} \tag{2.7}$$

and

$$a_3 = \frac{[2]_{p,q}!B_1}{2[\delta+1]_{p,q}[\delta+2]_{p,q}([3]_{p,q}-1)}\left(c_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \frac{B_1}{[2]_{p,q}-1}\right)c_1^2\right). \tag{2.8}$$

For $\mu \in C$ from the equations (2.7) and (2.8), we have

$$a_3-\mu a_2^2=\frac{[2]_{p,q}!B_1}{2[\delta+1]_{p,q}[\delta+2]_{p,q}([3]_{p,q}-1)}\left(c_2-\frac{1}{2}\left[1-\frac{B_2}{B_1}-\frac{B_1}{[2]_{p,q}-1}\left(1-\frac{[\delta+2]_{p,q}(3_{p,q}-1)}{[\delta+1]_{p,q}[2]_{p,q}([2]_{p,q}-1)}\mu\right)\right]c_1^2\right). \tag{2.9}$$

If we put

$$v=\frac{1}{2}\left[1-\frac{B_2}{B_1}-\frac{B_1}{[2]_{p,q}-1}\left(1-\frac{[\delta+2]_{p,q}(3_{p,q}-1)}{[\delta+1]_{p,q}[2]_{p,q}([2]_{p,q}-1)}\mu\right)\right], \tag{2.10}$$

Then the equation (2.9) becomes

$$|a_3-\mu a_2^2|=\frac{[2]_{p,q}!|B_1|}{2[\delta+1]_{p,q}[\delta+2]_{p,q}([3]_{p,q}-1)}|c_2-vc_1^2|. \tag{2.11}$$

Hence on application of Lemma 1.3 to the equation (2.11), we get the Fekete-Szego inequality given by the equation (2.1) for the class $S^*_{p,q}(\delta,\Phi)$.

Further the equality holds , when $p(z)=p_1(z)=\frac{1+z}{1-z}=1+2z+2z^2+...$ and the equation (2.2), gives

$$\frac{zD_{p,q}(R^{\delta}_{p,q}f(z))}{R^{\delta}_{p,q}f(z)}=\Phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right)=\Phi(z)=1+B_1z+B_2z^2+... \tag{2.12}$$

Comparing the equations (2.6) and (2.12), we get $c_1=2$ and $c_2=2$, then the equation (2.9) gives the equality sign in the place of inequality in (2.1).

Similarly, for $p(z)=p_2(z)=\frac{1+z^2}{1-z^2}=1+2z^2+...$, equation (2.2) gives

$$\frac{zD_{p,q}(R^{\delta}_{p,q}f(z))}{R^{\delta}_{p,q}f(z)}=\Phi\left(\frac{p_2(z)-1}{p_2(z)+1}\right)=\Phi(z^2)=1+B_1z^2+B_2z^4+... \tag{2.13}$$

So, we get $c_1=0$ and $c_2=2$ by comparing (2.6) and (2.13). Hence the equation (2.9) gives the equality sign in the place of inequality in (2.1) .

Thereom 2.2. Let $\Phi(z)=1+B_1z+B_2z^{2+...}$ with $B_1\neq 0$. If f is given by (1.1) belongs to the class $C_{p,q}(\delta,\Phi)$, then

$$|a_3-\mu a_2^2|\leq\frac{[2]_{p,q}!|B_1|}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}\,max\left\{1,\ \left|\frac{B_2}{B_1}+\frac{B_1}{[2]_{p,q}-1}\left(1-\frac{[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}{[\delta+1]_{p,q}[2]^3_{p,q}([2]_{p,q}-1)}\mu\right)\right|\right\}, \tag{2.14}$$

Where $B_1, B_2... \in R,\ \mu\in C\ and\ 0<q<p\leq 1$. The result is sharp.

Proof. Let $f\in C_{p,q}(\delta,\Phi)$, then by definition 1.2 and from subordination principle , there exists a Schwarz function w such that

$$\frac{D_{p,q}\left(zD_{p,q}\left(R^{\delta}_{p,q}f(z)\right)\right)}{D_{p,q}\left(R^{\delta}_{p,q}f(z)\right)}=\Phi(w(z)), \tag{2.15}$$

From the equations (2.15) and (2.6), we have

$$\frac{D_{p,q}\left(zD_{p,q}\left(R^{\delta}_{p,q}f(z)\right)\right)}{D_{p,q}\left(R^{\delta}_{p,q}f(z)\right)}=1+\frac{B_1C_1}{2}z+\left[\frac{B_1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_2C_1^2}{4}\right]z^2+....$$

Equating the coefficients of z and z^2 on both sides and on simplification, we get

$$a_2=\frac{B_1c_1}{2[\delta+1]_{p,q}[2]_{p,q}([2]_{p,q}-1)} \tag{2.16}$$

and

$$a_3=\frac{[2]_{p,q}!B_1}{2[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}\left(c_2-\frac{1}{2}\left(1-\frac{B_2}{B_1}-\frac{B_1}{[2]_{p,q}-1}\right)c_1^2\right). \tag{2.17}$$

For $\mu\in C$ from the equations (2.16) and (2.17), we have

$$a_3-\mu a_2^2=\frac{[2]_{p,q}!B_1}{2[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}\left(c_2-\frac{1}{2}\left[1-\frac{B_2}{B_1}-\frac{B_1}{[2]_{p,q}-1}\left(1-\frac{[\delta+2]_{p,q}[3]_{p,q}(3_{p,q}-1)}{[\delta+1]_{p,q}[2]^3_{p,q}([2]_{p,q}-1)}\mu\right)\right]c_1^2\right) \tag{2.18}$$

If we put

$$v=\frac{1}{2}\left[1-\frac{B_2}{B_1}-\frac{B_1}{[2]_{p,q}-1}\left(1-\frac{[\delta+2]_{p,q}[3]_{p,q}(3_{p,q}-1)}{[\delta+1]_{p,q}[2]^3_{p,q}([2]_{p,q}-1)}\mu\right)\right], \tag{2.19}$$

Then the equation (2.18) becomes,

$$|a_3-\mu a_2^2|=\frac{[2]_{p,q}!|B_1|}{2[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}|c_2-vc_1^2| \tag{2.20}$$

Hence on application of Lemma 1.3 to the equation (2.20), we get the Fekete-Szego inequality given by the equation

(2.14) for the class $C_{p,q}(\delta,\Phi)$.

Further the equality holds , when $p(z)=p_1(z)=\frac{1+z}{1-z}=1+2z+2z^2+\ldots$ and the equation (2.15), gives

$$\frac{D_{p,q}\left(zD_{p,q}\left(R^{\delta}_{p,q}f(z)\right)\right)}{D_{p,q}\left(R^{\delta}_{p,q}f(z)\right)}=\Phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right)=\Phi(z)=1+B_1z+B_2z^2+\ldots \tag{2.21}$$

Comparing the equations (2.6) and (2.21), we get $c_1=2$ and $c_2=2$, then the equation (2.18) gives the equality sign in the place of inequality in (2.14).

Similarly, for $p(z)=p_2(z)=\frac{1+z^2}{1-z^2}=1+2z^2+\ldots$, equation (2.15) gives

$$\frac{D_{p,q}\left(zD_{p,q}\left(R^{\delta}_{p,q}f(z)\right)\right)}{D_{p,q}\left(R^{\delta}_{p,q}f(z)\right)}=\Phi\left(\frac{p_2(z)-1}{p_2(z)+1}\right)=\Phi(z^2)=1+B_1z^2+B_2z^4+\ldots \tag{2.22}$$

So, we get $c_1=0$ and $c_2=2$ by comparing (2.6) and (2.22). Thus the equation (2.18) gives the equality sign in the place of inequality in (2.14) .

III. COEFFICIENT BOUNDS

In this section, we investigate the estimates on the coefficients for second and third coefficients of the classes $S^{*}_{p,q}(\delta,\Phi)$ and $C_{p,q}(\delta,\Phi)$.

In order to prove our results we need the following lemma

Lemma 3.1.[8] If $p(z)=1+c_1z+c_2z^2+\ldots$ is a function with $Re\{p(z)\}>0$, then

$$|c_2-vc_1^2|\le\begin{cases}-4v+2, & if\ \ v\le 0\,;\\ 2, & if\ \ 0\le v\le 1\,;\\ 4v-2 & if\ \ v\ge 1.\end{cases} \tag{3.1}$$

Also, the upper bound is sharp, and it can be improved as follows when $0<v<1$;

$$|c_2-vc_1^2|+v|c_1|^2\le 2 \quad \left(0<v\le\frac{1}{2}\right) \tag{3.2}$$

and

$$|c_2-vc_1^2|+(1-v)|c_1|^2\le 2 \quad \left(\frac{1}{2}\le v<1\right). \tag{3.3}$$

Theorem 3.2. Let $\Phi(z)=1+B_1z+B_2z^{2+\cdots}$ with $B_1>0$ and $B_2\ge 0$. Let

$$\eta_1=\frac{[\delta+1]_{p,q}[2]_{p,q}([2]_{p,q}-1)[B_1^2+([2]_{p,q}-1)(B_2-B_1)]}{[\delta+2]_{p,q}([3]_{p,q}-1)B_1^2}, \tag{3.4}$$

$$\eta_2=\frac{[\delta+1]_{p,q}[2]_{p,q}([2]_{p,q}-1)[B_1^2+([2]_{p,q}-1)(B_2+B_1)]}{[\delta+2]_{p,q}([3]_{p,q}-1)B_1^2} \tag{3.5}$$

$$\eta_3=\frac{[\delta+1]_{p,q}[2]_{p,q}([2]_{p,q}-1)[B_1^2+([2]_{p,q}-1)B_2]}{[\delta+2]_{p,q}([3]_{p,q}-1)B_1^2} \tag{3.6}$$

If f as in (1.1) belongs to the class $S^{*}_{p,q}(\delta,\Phi)$, then

$$|a_3-\mu a_2^2|\le\begin{cases}\frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}([3]_{p,q}-1)}\left[\frac{B_2}{B_1}+\frac{B_1}{([2]_{p,q}-1)}\left(1-\frac{[\delta+2]_{p,q}([3]_{p,q}-1)}{[\delta+1]_{p,q}[2]_{p,q}([2]_{p,q}-1)}\mu\right)\right] & if\ \ \mu\le\eta_1\\ \frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}([3]_{p,q}-1)} & if\ \ \eta_1\le\mu\le\eta_2\\ \frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}([3]_{p,q}-1)}\left[-\frac{B_2}{B_1}-\frac{B_1}{([2]_{p,q}-1)}\left(1-\frac{[\delta+2]_{p,q}([3]_{p,q}-1)}{[\delta+1]_{p,q}[2]_{p,q}([2]_{p,q}-1)}\mu\right)\right] & if\ \ \mu\ge\eta_2.\end{cases} \tag{3.7}$$

Further, if $\eta_1<\mu\le\eta_3$, then

$$|a_3-\mu a_2^2|+\frac{[\delta+1]_{p,q}[2]_{p,q}!([2]_{p,q}-1)^2}{[\delta+2]_{p,q}([3]_{p,q}-1)B_1^2}\left[B_1-B_2-\frac{B_1^2}{([2]_{p,q}-1)}\left(1-\frac{[\delta+2]_{p,q}([3]_{p,q}-1)}{[\delta+1]_{p,q}[2]_{p,q}([2]_{p,q}-1)}\mu\right)\right]|a_2|^2\le\frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}([3]_{p,q}-1)}, \tag{3.8}$$

and if $\eta_3\le\mu<\eta_2$, then

$$|a_3-\mu a_2^2|+\frac{[\delta+1]_{p,q}[2]_{p,q}!\left([2]_{p,q}-1\right)^2}{[\delta+2]_{p,q}\left([3]_{p,q}-1\right)B_1^2}\left[B_1+B_2+\frac{B_1^2}{\left([2]_{p,q}-1\right)}\left(1-\frac{[\delta+2]_{p,q}\left([3]_{p,q}-1\right)}{[\delta+1]_{p,q}[2]_{p,q}\left([2]_{p,q}-1\right)}\mu\right)\right]|a_2|^2\leq\frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}\left([3]_{p,q}-1\right)}. \quad (3.9)$$

Proof. For $v\leq 0$, equation (2.10), gives

$$\mu\leq\frac{[\delta+1]_{p,q}[2]_{p,q}\left([2]_{p,q}-1\right)\left[B_1^2+\left([2]_{p,q}-1\right)(B_2-B_1)\right]}{[\delta+2]_{p,q}\left([3]_{p,q}-1\right)B_1^2}.$$

Let $\eta_1=\frac{[\delta+1]_{p,q}[2]_{p,q}\left([2]_{p,q}-1\right)\left[B_1^2+\left([2]_{p,q}-1\right)(B_2-B_1)\right]}{[\delta+2]_{p,q}\left([3]_{p,q}-1\right)B_1^2}$, then the above expression becomes $\mu\leq\eta_1$.

Let $p(z)$ given by (2.3) with $Re\{p(z)\}>0$ and $f(z)$ given by (1.1) belongs to the class $S_{p,q}^*(\delta,\Phi)$, then equation (2.11) holds. Thus apply Lemma 3.1 for $v\leq 0$, to the equation (2.11), we get

$$|a_3-\mu a_2^2|\leq\frac{[2]_{p,q}!B_1}{2\,[\delta+1]_{p,q}[\delta+2]_{p,q}\left([3]_{p,q}-1\right)}\,(-4v+2),$$

from equation (2.10), we have

$$|a_3-\mu a_2^2|\leq\frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}\left([3]_{p,q}-1\right)}\left[\frac{B_2}{B_1}+\frac{B_1}{\left([2]_{p,q}-1\right)}\left(1-\frac{[\delta+2]_{p,q}\left([3]_{p,q}-1\right)}{[\delta+1]_{p,q}[2]_{p,q}\left([2]_{p,q}-1\right)}\mu\right)\right], \quad (3.10)$$

where $\mu\leq\eta_1$.

Further, for $0\leq v\leq 1$, equation(2.10) gives,

$$\eta_1\leq\mu\leq\frac{[\delta+1]_{p,q}[2]_{p,q}\left([2]_{p,q}-1\right)\left[B_1^2+\left([2]_{p,q}-1\right)(B_2+B_1)\right]}{[\delta+2]_{p,q}\left([3]_{p,q}-1\right)B_1^2},$$

where η_1 given by (3.4).

Let $\eta_2=\frac{[\delta+1]_{p,q}[2]_{p,q}\left([2]_{p,q}-1\right)\left[B_1^2+\left([2]_{p,q}-1\right)(B_2+B_1)\right]}{[\delta+2]_{p,q}\left([3]_{p,q}-1\right)B_1^2}$, then the above relation becomes $\eta_1\leq\mu\leq\eta_2$.

Now apply Lemma 3.1 for $0\leq v\leq 1$ to the equation (2.11), we obtain

$$|a_3-\mu a_2^2|\leq\frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}\left([3]_{p,q}-1\right)},$$

which gives the second inequality of Assertion (3.7).

Further for $v\geq 1$, equation (2.10), gives that $\mu\geq\eta_2$. Now apply Lemma 3.1 for $v\geq 1$ to the equation(2.11), we have

$$|a_3-\mu a_2^2|\leq\frac{[2]_{p,q}!B_1}{2\,[\delta+1]_{p,q}[\delta+2]_{p,q}\left([3]_{p,q}-1\right)}\,(4v-2),$$

from the equation (2.10), we have

$$|a_3-\mu a_2^2|\leq\frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}\left([3]_{p,q}-1\right)}\left[-\frac{B_2}{B_1}-\frac{B_1}{\left([2]_{p,q}-1\right)}\left(1-\frac{[\delta+2]_{p,q}\left([3]_{p,q}-1\right)}{[\delta+1]_{p,q}[2]_{p,q}\left([2]_{p,q}-1\right)}\mu\right)\right], \quad (3.11)$$

which gives the third inequality of Assertion (3.7).

Further if $0<v\leq\frac{1}{2}$, then from equation (2.10), we have

$$0<\frac{1}{2}\left[1-\frac{B_2}{B_1}-\frac{B_1}{[2]_{p,q}-1}\left(1-\frac{[\delta+2]_{p,q}\left(3_{p,q}-1\right)}{[\delta+1]_{p,q}[2]_{p,q}\left([2]_{p,q}-1\right)}\mu\right)\right]\leq\frac{1}{2},$$

which on simplification, we get

$$\eta_1<\mu\leq\frac{[\delta+1]_{p,q}[2]_{p,q}\left([2]_{p,q}-1\right)\left[B_1^2+\left([2]_{p,q}-1\right)B_2\right]}{[\delta+2]_{p,q}\left([3]_{p,q}-1\right)B_1^2} \quad (3.12)$$

where η_1 given by (3.4).

Let $\eta_3=\frac{[\delta+1]_{p,q}[2]_{p,q}\left([2]_{p,q}-1\right)\left[B_1^2+\left([2]_{p,q}-1\right)B_2\right]}{[\delta+2]_{p,q}\left([3]_{p,q}-1\right)B_1^2}$, then from relation (3.12), we have $\eta_1<\mu\leq\eta_3$.

Now using equations (2.7), (2.8) and (3.4), we have

$$|a_3-\mu a_2^2|+(\mu-\eta_1)|a_2|^2=|a_3-\mu a_2^2|+\left(\mu-\frac{[\delta+1]_{p,q}[2]_{p,q}\left([2]_{p,q}-1\right)\left[B_1^2+\left([2]_{p,q}-1\right)(B_2-B_1)\right]}{[\delta+2]_{p,q}\left([3]_{p,q}-1\right)B_1^2}\right)\frac{B_1^2|c_1|^2}{4[\delta+1]_{p,q}^2\left([2]_{p,q}-1\right)^2},$$

On using equation (2.11), we get

$$|a_3-\mu a_2^2|+(\mu-\eta_1)|a_2|^2=\frac{[2]_{p,q}!B_1}{2[\delta+1]_{p,q}[\delta+2]_{p,q}\left([3]_{p,q}-1\right)}\left(|c_2-vc_1^2|+\frac{1}{2}\left[1-\frac{B_2}{B_1}-\frac{B_1}{[2]_{p,q}-1}\left(1-\frac{[\delta+2]_{p,q}\left([3]_{p,q}-1\right)}{[\delta+1]_{p,q}[2]_{p,q}\left([2]_{p,q}-1\right)}\mu\right)\right]|c_1|^2\right). \quad (3.14)$$

Using the equation (2.10) in (3.14), we get

$$|a_3 - \mu a_2^2| + (\mu - \eta_1)|a_2|^2 = \frac{[2]_{p,q}!\,B_1}{2[\delta+1]_{p,q}[\delta+2]_{p,q}\left([3]_{p,q}-1\right)}\left(|c_2 - vc_1^2| + v|c_1|^2\right),$$

from the inequality (3.2) we have

$$|a_3 - \mu a_2^2| + (\mu - \eta_1)|a_2|^2 \le \frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}\left([3]_{p,q}-1\right)}\,. \tag{3.15}$$

Hence

$$|a_3 - \mu a_2^2| + \left(\mu - \frac{[\delta+1]_{p,q}[2]_{p,q}\left([2]_{p,q}-1\right)\left[B_1^2+\left([2]_{p,q}-1\right)(B_2-B_1)\right]}{[\delta+2]_{p,q}\left([3]_{p,q}-1\right)B_1^2}\right)|a_2|^2 \le \frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}\left([3]_{p,q}-1\right)},$$

where $\eta_1 < \mu \le \eta_3$. Thus on simplification, we obtain the Assertion (3.8).

Similarly, if $\frac{1}{2} \le v < 1$, from equation (2.10), we have $\eta_3 \le \mu < \eta_2$. Where η_2 and η_3 are given by equations (3.5) and (3.6) respectively.

Now, using equations (2.7) and (3.5), we get

$$|a_3 - \mu a_2^2| + (\eta_2 - \mu)|a_2|^2 = |a_3 - \mu a_2^2| + \left(\frac{[\delta+1]_{p,q}[2]_{p,q}\left([2]_{p,q}-1\right)\left[B_1^2+\left([2]_{p,q}-1\right)(B_2+B_1)\right]}{[\delta+2]_{p,q}\left([3]_{p,q}-1\right)B_1^2} - \mu\right)\frac{B_1^2|c_1|^2}{4[\delta+1]_{p,q}^2\left([2]_{p,q}-1\right)^2}. \tag{3.16}$$

Using equations (2.11) in (3.16), we get

$$|a_3 - \mu a_2^2| + (\eta_2 - \mu)|a_2|^2 = \frac{[2]_{p,q}!B_1}{2[\delta+1]_{p,q}[\delta+2]_{p,q}\left([3]_{p,q}-1\right)}\left(|c_2 - vc_1^2| + \frac{1}{2}\left[1 + \frac{B_2}{B_1} + \frac{B_1}{[2]_{p,q}-1}\left(1 - \frac{[\delta+2]_{p,q}\left([3]_{p,q}-1\right)}{[\delta+1]_{p,q}[2]_{p,q}\left([2]_{p,q}-1\right)}\mu\right)\right]|c_1|^2\right), \tag{3.17}$$

On using equation (2.10), we have

$$|a_3 - \mu a_2^2| + (\eta_2 - \mu)|a_2|^2 = \frac{[2]_{p,q}!B_1}{2[\delta+1]_{p,q}[\delta+2]_{p,q}\left([3]_{p,q}-1\right)}\left(|c_2 - vc_1^2| + (1-v)|c_1|^2\right), \tag{3.18}$$

Since $\frac{1}{2} \le v < 1$, from the inequality (3.3) of Lemma 3.1 , we have

$$|a_3 - \mu a_2^2| + (\eta_2 - \mu)|a_2|^2 \le \frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}\left([3]_{p,q}-1\right)}. \tag{3.19}$$

Thus

$$|a_3 - \mu a_2^2| + \left(\frac{[\delta+1]_{p,q}[2]_{p,q}\left([2]_{p,q}-1\right)\left[B_1^2+\left([2]_{p,q}-1\right)(B_2+B_1)\right]}{[\delta+2]_{p,q}\left([3]_{p,q}-1\right)B_1^2} - \mu\right)|a_2|^2 \le \frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}\left([3]_{p,q}-1\right)}.$$

Where $\eta_3 \le \mu < \eta_2$. Finally on simplifying the above inequality , we obtain the Assertion (3.9).

Theorem 3.3. Let $\Phi(z) = 1 + B_1 z + B_2 z^{2+} \cdots$ with $B_1 > 0$ and $B_2 \ge 0$. Let

$$\chi_1 = \frac{[\delta+1]_{p,q}[2]_{p,q}^3\left([2]_{p,q}-1\right)\left[B_1^2+\left([2]_{p,q}-1\right)(B_2-B_1)\right]}{[\delta+2]_{p,q}[3]_{p,q}\left([3]_{p,q}-1\right)B_1^2}, \tag{3.20}$$

$$\chi_2 = \frac{[\delta+1]_{p,q}[2]_{p,q}^3\left([2]_{p,q}-1\right)\left[B_1^2+\left([2]_{p,q}-1\right)(B_2+B_1)\right]}{[\delta+2]_{p,q}[3]_{p,q}\left([3]_{p,q}-1\right)B_1^2} \tag{3.21}$$

$$\chi_3 = \frac{[\delta+1]_{p,q}[2]_{p,q}^3\left([2]_{p,q}-1\right)\left[B_1^2+\left([2]_{p,q}-1\right)B_2\right]}{[\delta+2]_{p,q}[3]_{p,q}\left([3]_{p,q}-1\right)B_1^2} \tag{3.22}$$

If f as in (1.1) belongs to the class $C_{p,q}(\delta,\Phi)$, then

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}\left([3]_{p,q}-1\right)}\left[\frac{B_2}{B_1} + \frac{B_1}{\left([2]_{p,q}-1\right)}\left(1 - \frac{[\delta+2]_{p,q}[3]_{p,q}\left([3]_{p,q}-1\right)}{[\delta+1]_{p,q}[2]_{p,q}^3\left([2]_{p,q}-1\right)}\mu\right)\right] & if \quad \mu \le \chi_1 \\ \frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}\left([3]_{p,q}-1\right)} & if \quad \chi_1 \le \mu \le \chi_2 \\ \frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}\left([3]_{p,q}-1\right)}\left[-\frac{B_2}{B_1} - \frac{B_1}{\left([2]_{p,q}-1\right)}\left(1 - \frac{[\delta+2]_{p,q}[3]_{p,q}\left([3]_{p,q}-1\right)}{[\delta+1]_{p,q}[2]_{p,q}^3\left([2]_{p,q}-1\right)}\mu\right)\right] & if \quad \mu \ge \chi_2. \end{cases} \tag{3.23}$$

Further, if $\chi_1 < \mu \le \chi_3$, then

$$|a_3-\mu a_2^2|+\frac{[\delta+1]_{p,q}[2]_{p,q}^3([2]_{p,q}-1)^2}{[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)B_1^2}\left[B_1-B_2-\frac{B_1^2}{([2]_{p,q}-1)}\left(1-\frac{[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}{[\delta+1]_{p,q}[2]_{p,q}^3([2]_{p,q}-1)}\mu\right)\right]|a_2|^2\leq\frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}. \tag{3.24}$$

If $\chi_3\leq\mu<\chi_2$, then

$$|a_3-\mu a_2^2|+\frac{[\delta+1]_{p,q}[2]_{p,q}^3([2]_{p,q}-1)^2}{[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)B_1^2}\left[B_1+B_2+\frac{B_1^2}{([2]_{p,q}-1)}\left(1-\frac{[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}{[\delta+1]_{p,q}[2]_{p,q}^3([2]_{p,q}-1)}\mu\right)\right]|a_2|^2\leq\frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}. \tag{3.25}$$

Proof . For $v\leq 0$, equation (2.19) gives

$$\mu\leq\frac{[\delta+1]_{p,q}[2]_{p,q}^3([2]_{p,q}-1)[B_1^2+([2]_{p,q}-1)(B_2-B_1)]}{[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)B_1^2},$$

then the above expression becomes $\mu\leq\chi_1$.

Let $p(z)$ be a function given by (2.3) with $Re\{p(z)\}>0$ and $f(z)$ given by (1.1) belongs to the class $C_{p,q}(\delta,\Phi)$, then equation (2.20) holds. Thus using Lemma 3.1 for $v\leq 0$, to the equation (2.20), we get

$$|a_3-\mu a_2^2|\leq\frac{[2]_{p,q}!B_1}{2\,[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}\,(-4v+2),$$

now by using (2.19), we have

$$|a_3-\mu a_2^2|\leq\frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}\left[\frac{B_2}{B_1}+\frac{B_1}{([2]_{p,q}-1)}\left(1-\frac{[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}{[\delta+1]_{p,q}[2]_{p,a}^3([2]_{p,q}-1)}\mu\right)\right], \tag{3.26}$$

where $\mu\leq\chi_1$. Which gives the first inequality of Assertion (3.23).

For, $0\leq v\leq 1$, equation(2.19) gives,

$$\chi_1\leq\mu\leq\frac{[\delta+1]_{p,q}[2]_{p,q}^3([2]_{p,q}-1)[B_1^2+([2]_{p,q}-1)(B_2+B_1)]}{[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)B_1^2}$$

Let $\chi_2=\frac{[\delta+1]_{p,q}[2]_{p,q}^3([2]_{p,q}-1)[B_1^2+([2]_{p,q}-1)(B_2+B_1)]}{[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)B_1^2}$, then the above relation becomes $\chi_1\leq\mu\leq\chi_2$, where χ_1 is given by (3.20).

Now ,Using Lemma 3.1 for $0\leq v\leq 1$ to the equation (2.20), we get

$$|a_3-\mu a_2^2|\leq\frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)},$$

which gives the second inequality of Assertion (3.23).

Next, if we take $v\geq 1$, then (2.19) gives $\mu\geq\chi_2$. Now using Lemma 3.1 for $v\geq 1$ in (2.19), we get

$$|a_3-\mu a_2^2|\leq\frac{[2]_{p,q}!B_1}{2\,[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}\,(4v-2),$$

from the equation (2.19), we have

$$|a_3-\mu a_2^2|\leq\frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}\left[-\frac{B_2}{B_1}-\frac{B_1}{([2]_{p,q}-1)}\left(1-\frac{[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}{[\delta+1]_{p,q}[2]_{p,q}^3([2]_{p,q}-1)}\mu\right)\right], \tag{3.27}$$

where $\mu\geq\chi_2$ which gives the third inequality of Assertion (3.23).

Further $0<v\leq\frac{1}{2}$, we get

$$0<\frac{1}{2}\left[1-\frac{B_2}{B_1}-\frac{B_1}{[2]_{p,q}-1}\left(1-\frac{[\delta+2]_{p,q}[3]_{p,q}(3_{p,q}-1)}{[\delta+1]_{p,q}[2]_{p,q}^3([2]_{p,q}-1)}\mu\right)\right]\leq\frac{1}{2},$$

Which implies that

$$\chi_1<\mu\leq\frac{[\delta+1]_{p,q}[2]_{p,q}^3([2]_{p,q}-1)[B_1^2+([2]_{p,q}-1)B_2]}{[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)B_1^2} \tag{3.28}$$

Let $\chi_3=\frac{[\delta+1]_{p,q}[2]_{p,q}^3([2]_{p,q}-1)[B_1^2+([2]_{p,q}-1)B_2]}{[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)B_1^2}$, then from above expression , we have $\chi_1<\mu\leq\chi_3$, where χ_1 is given by (3.20). Now using (2.16) and (3.20), we get

$$|a_3-\mu a_2^2|+(\mu-\chi_1)|a_2|^2=|a_3-\mu a_2^2|+\left(\mu-\frac{[\delta+1]_{p,q}[2]_{p,q}^3([2]_{p,q}-1)[B_1^2+([2]_{p,q}-1)(B_2-B_1)]}{[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)B_1^2}\right)\frac{B_1^2|c_1|^2}{4[\delta+1]_{p,q}^2[2]_{p,q}^2([2]_{p,q}-1)^2}, \tag{3.29}$$

From (2.20), we have

$$|a_3-\mu a_2^2|+(\mu-\ \chi_1)|a_2|^2=\frac{[2]_{p,q}!B_1}{2[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}\left(|c_2-vc_1^2|+\frac{1}{2}\left[1-\frac{B_2}{B_1}-\frac{B_1}{[2]_{p,q}-1}\left(1-\frac{[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}{[\delta+1]_{p,q}[2]_{p,q}^3([2]_{p,q}-1)}\mu\right)\right]|c_1|^2\right). \quad (3.30)$$

Using (2.19) in above equation, we get

$$|a_3-\mu a_2^2|+(\mu-\ \chi_1)|a_2|^2=\frac{[2]_{p,q}!\,B_1}{2[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}(|c_2-vc_1^2|+v|c_1|^2),$$

from the inequality (3.2) we have

$$|a_3-\mu a_2^2|+(\mu-\ \chi_1)|a_2|^2\le\frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}. \quad (3.31)$$

Thus,

$$|a_3-\mu a_2^2|+\left(\mu-\frac{[\delta+1]_{p,q}[2]_{p,q}^3([2]_{p,q}-1)[B_1^2+([2]_{p,q}-1)(B_2-B_1)]}{[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)B_1^2}\right)\ |a_2|^2\le\frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)},$$

Simplifying the above inequality, we obtain the Assertion (3.24).

Similarly, for $\frac{1}{2}\le v<1$, from (2.19), we have $\chi_3\le\mu<\chi_2$

Using (2.16) and (3.21), we have

$$|a_3-\mu a_2^2|+(\ \chi_2-\mu)|a_2|^2=|a_3-\mu a_2^2|+\left(\frac{[\delta+1]_{p,q}[2]_{p,q}^3([2]_{p,q}-1)[B_1^2+([2]_{p,q}-1)(B_2+B_1)]}{[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)B_1^2}-\mu\right)\frac{B_1^2|c_1|^2}{4[\delta+1]_{p,q}^2[2]_{p,q}^2([2]_{p,q}-1)^2}. \quad (3.32)$$

Using equations (2.20) in (3.32) and on simplification we get

$$|a_3-\mu a_2^2|+(\ \chi_2-\mu)|a_2|^2=\frac{[2]_{p,q}!B_1}{2[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}\left(|c_2-vc_1^2|+\frac{1}{2}\left[1+\frac{B_2}{B_1}+\frac{B_1}{[2]_{p,q}-1}\left(1-\frac{[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}{[\delta+1]_{p,q}[2]_{p,q}^3([2]_{p,q}-1)}\mu\right)\right]|c_1|^2\right), \quad (3.33)$$

which on using equation (2.19), gives

$$|a_3-\mu a_2^2|+(\ \chi_2-\mu)|a_2|^2=\frac{[2]_{p,q}!B_1}{2[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}(|c_2-vc_1^2|+(1-v)|c_1|^2). \quad (3.34)$$

Now, since $\frac{1}{2}\le v<1$, from the inequality (3.3) of Lemma 3.1 , we have

$$|a_3-\mu a_2^2|+(\chi_2-\mu)|a_2|^2\le\frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}.$$

(3.35)

Thus,

$$|a_3-\mu a_2^2|+\left(\frac{[\delta+1]_{p,q}[2]_{p,q}^3([2]_{p,q}-1)[B_1^2+([2]_{p,q}-1)(B_2+B_1)]}{[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)B_1^2}-\mu\right)|a_2|^2\le\frac{[2]_{p,q}!B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}([3]_{p,q}-1)}.$$

Where $\chi_3\le\mu<\chi_2$ Thus on simplification, we obtain the Assertion (3.25).

Remark 3.4.

i) Setting $\delta=0$ in Theorem 2.1, Theorem 2.2, Theorem 3.2 and Theorem 3.3 respectively, we obtain the corresponding results for the classes $S_{p,q}^*(\delta,\Phi)$ and $C_{p,q}(\delta,\Phi)$ defined by Srivastava [2].

ii) Setting $p=1$ and $\delta=0$, in Theorem 2.1, Theorem 2.2, Theorem 3.2 and Theorem 3.3 respectively, we get the corresponding results for the classes $S_q^*(\Phi)$ and $C_q(\Phi)$ introduced by Cetinkays and Polatoglu [1].

iii) Letting $p=1, q\to 1^-$ and $\delta=0$ in Theorem 2.1, Theorem 2.2, Theorem 3.2 and Theorem 3.3 respectively, we obtain the corresponding results for the classes $S^*(\Phi)$ and $C(\Phi)$ studied by Minda [8].

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