International Journal of Mathematics Trends and Technology
ISSN: 2231 – 5373 /doi:10.14445/22315373/IJMTT-V67I6P512

Volume 67 Issue 6, 97-118, June, 2021
© 2021 Seventh Sense Research Group®

Four Formulations of Average Derived from Pythagorean Means

Dhritikesh Chakrabarty

Department of Statistics, Handique Girls' College, Gauhati University, Guwahati – 781001, Assam, India.

Abstract:

Four definitions / formulations of average termed here as Arithmetic-Geometric Mean (abbreviated as AGM), Arithmetic-Harmonic Mean (abbreviated as AHM), Geometric-Harmonic Mean (abbreviated as GHM) and Arithmetic-Geometric-Harmonic (abbreviated as Mean AGHM) respectively have been derived / developed from the three Pythagorean means namely arithmetic mean (abbreviated as Mean AM), geometric mean (abbreviated as Mean GM) and harmonic mean (abbreviated as Mean HM) with an objective of developing of more accurate measures of central tendency of data. The derivations of these four formulations of average, with numerical examples, have been presented in this paper.

Key Words:

AGM, AHM, GHM, AGHM, Formulation of Average.

1. Introduction

Average [1] is a concept which describes any characteristic of an aggregate / population / class of individuals overall but not of an individual in the aggregate / population / class in particular. It is used in most of the measures associated to data (or list of numerical values). Pythagoras [2] , [4] , one exponent of mathematics, is the pioneer of defining average. He defined the three most common averages namely arithmetic mean, geometric mean and harmonic mean which were given the name "Pythagorean Means" [4] , [16] , [35] , [37] as a mark of honor to him. Later on, a number of definitions / formulations of average had been derived due to necessity of handling different situations (different types of data) some of which are Quadratic Mean Square Root Mean , Cubic Mean , Cube Root Mean , Generalized p Mean & Generalized p^{th} Root Mean etc. in addition to Arithmetic Mean , Geometric Mean & Harmonic Mean [17] , [26] , [31] , [32] , [35] ,]37] .

The next trend was towards developing generalized definitions / formulations of average and general method of defining average. Kolmogorov [35] , [36] , [39] , one great mathematician, formulated one generalized definition of average known as Generalized f - Mean [22] , [23] . Recently, two generalized definitions of average have been derived among which one is termed as <u>Generalized f_H</u> – Mean [24)] and the other as <u>Generalized f_G</u> – Mean [25] , [28] . However, none of these three generalized definitions is complete i.e. none of them can describe/yield all types of averages. This leads to the necessity of searching for one general method/definition of average which describe/yield most of the definitions/formulations of average. Accordingly, in another study, an attempt has been made on searching for a generalized method of defining average of a set of values of a variable [29] and later on a generalized method of defining average of a function of a set (or of a list) of values [30]. In statistics, the three Pythagorean means are used in measuring the central tendency of numerical data [38] , [42] , [43]. However, the accuracy of the value of central tendency yielded by each of the three Pythagorean means is not known.

Recently, there have been a lot of studies on analysis of numerical data based on average in general and on Pythagorean means specially [5] – [15] , [18] – [21] , [27] . In the mean time, several attempts have been made on determining accurate value of central tendency of numerical data. However, still there is necessity of more accurate measure of the same. With an objective of finding out more accurate measure of central tendency of data, definitions / formulations of average termed here as Arithmetic-Geometric Mean (abbreviated as AGM), Arithmetic-Harmonic Mean (abbreviated as AHM), Geometric-Harmonic Mean (abbreviated as GHM) and Arithmetic-Geometric-Harmonic (abbreviated as Mean AGHM) respectively have been derived / developed from the three Pythagorean means namely arithmetic mean (abbreviated as Mean AM), geometric mean (abbreviated as Mean GM) and harmonic mean (abbreviated as Mean HM). The derivations of these four formulations of average, with numerical examples, have been presented in this paper. It is to be mentioned that Arithmetic-Geometric Mean had been defined by Gauss [3] , [33] as the point of convergence of two sequences. In a similar manner, Arithmetic-Harmonic Mean [34] and Geometric-Harmonic Mean were also been defined later on [41]. However, these three definitions were formulated for two numbers only. In the current study, the definitions of these three along with one more namely Arithmetic-Geometric-Harmonic have been attempted for a set of finite number of values. The derivations of these four have been presented in this paper.

2. Derivation of Four Formulations of Average

Let a_0 , g_0 & h_0 be respectively the AM , the GM & the HM of the N numbers (or values or observations)

$$x_1 \, , \; x_2 \, , \, \ldots\ldots\ldots \, , x_N$$

of N positive numbers or values or observations (not all equal or identical) which are strictly positive

i.e. $a_0 = AM\,(x_1 \, , \; x_2 \, , \, \ldots\ldots\ldots \, , x_N)$

$$= \frac{1}{N} \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{x_i} \, , \qquad (2.1)$$

$g_0 = GM\,(x_1 \, , \; x_2 \, , \, \ldots\ldots\ldots \, , x_N)$

$$= \left(\prod_{i=1}^{N} x_i \right)^{1/N} \qquad (2.2)$$

& $h_0 = HM\,(x_1 \, , \; x_2 \, , \, \ldots\ldots\ldots \, , x_N)$

$$= \left(\frac{1}{N} \sum_{i=1}^{N} x_{\mathrm{i}}^{\,-\mathbf{1}} \right)^{-\mathbf{1}} \qquad (2.3)$$

which satisfy the inequality

$$\text{Max}\left(x_1 \, , \; x_2 \, , \, \ldots\ldots\ldots \, , x_N\right) > a_0 > g_0$$

$$> h_0 > \text{Min}\left(x_1 \, , \; x_2 \, , \, \ldots\ldots\ldots \, , x_N\right) \qquad (2.4)$$

[4 , 16 , 35 , 37].

The following theorem in the mathematical field of real analysis known as monotone convergence theorem [40] will be used in deriving the formulations here.

"*If a sequence is increasing and bounded above by a supremum, then the sequence will converge to the supremum; in the same way, if a sequence is decreasing and is bounded below by an infimum, it will converge to the infimum.*"

2. 1. *Arithmetic-Geometric Mean (AGM)*

Let the two sequences $\{a_n\}$ & $\{g_n\}$ be defined by

$$a_{n+1} = ½ (a_n + g_n) \qquad (2.5)$$

$$\& \quad g_{n+1} = (a_n . g_n)^{1/2} \qquad (2.6)$$

respectively where the square root takes the principal value.

It follows from the inequality (2.4) that

$$g_n < a_n$$

Now, $g_{n+1} = (a_n . g_n)^{1/2}$

$\Rightarrow \quad g_{n+1} > (g_n . g_n)^{1/2} \quad \Rightarrow \quad g_{n+1} > g_n$

This means that the sequence $\{g_n\}$ is non-decreasing.

Moreover, the sequence $\{g_n\}$ is bounded above by (2.4).

Therefore, by the monotone convergence theorem, the sequence is convergent.

Therefore, there exists a finite number M_{AG} such that g_n converges to M_{AG} .

Again, a_n can be expressed as

$$a_n = {g_{n+1}}^2 / g_n$$

This implies that the limiting value of a_n as n approaches infinity is M_{AG}.

Therefore, a_n also converges to M_{AG} .

Thus, the two sequences $\{a_n\}$ & $\{g_n\}$ converge to a common point M_{AG}.

This common point M_{AG} can be termed as the Arithmetic-Geometric Mean of $x_1 , x_2 , \ldots\ldots\ldots , x_N$.

Accordingly, Arithmetic-Geometric Mean can be defined as follows:

Definition of Arithmetic-Geometric Mean (AGM)

The Arithmetic-Geometric Mean (abbreviated by *AGM*) of the N positive real numbers

$$x_1 , x_2 , \ldots\ldots\ldots , x_N$$

denoted by $AGM (x_1 , x_2 , \ldots\ldots\ldots , x_N)$

can be defined to be is the common point of convergence M_{AG} of the two sequences $\{a_n\}$ & $\{g_n\}$ defined respectively by

$a_{n+1} = ½ (a_n + g_n)$ & $g_{n+1} = (a_n . g_n)^{1/2}$

where the square root takes the principal value

with a_0 & g_0 defined by (2.1) & (2.2) respectively.

Note:

(1) It is clear that each of $\{a_n\}$ & $\{g_n\}$ converges to $\mathrm{M_{AG}}$

Thus, the converging point of either $\{a_n\}$ or $\{g_n\}$ can be taken as the value of $\mathrm{M_{AG}}$.

(2) Since $g_n < g_{n+1}$

i.e. $g_0 < g_1 < g_2 < g_3$, ……………

Therefore, $g_0 < \mathrm{M_{AG}}$

Again ,

$a_n = {g_{n+1}}^2 / g_n$ & $a_{n+1} = {g_{n+2}}^2 / g_{n+1}$

which implies, $a_n > a_{n+1}$

i.e. $a_0 > a_1 > a_2 > a_3 >$, …………

Therefore, $a_0 > \mathrm{M_{AG}}$

Hence, the following inequality is obtained:

$$a_0 > \mathrm{M_{AG}} > g_0$$

i.e. $AM > AGM > GM$ (2.7)

One Important Property of AGM

For positive real constant c ,

$\mathrm{a}_0(cx_1 , cx_2 , \ldots\ldots\ldots , cx_N)$

$= \frac{1}{\mathrm{N}} \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{cx_i} = c \frac{1}{N} \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{x_i}$

i.e. $a_0(cx_1 , cx_2 , \ldots\ldots\ldots , cx_N)$

$= c.\mathrm{a}_0(x_1 , x_2 , \ldots\ldots\ldots , x_N)$

Also,

$g_0(cx_1 , cx_2 , \ldots\ldots\ldots , cx_N)$

$= (\prod_{i=1}^{N} cx_i)^{1/N} = c. (\prod_{i=1}^{N} x_i)^{1/N}$

i.e. $g_0(cx_1 , cx_2 , \ldots\ldots\ldots , cx_N)$

$= c.g_0(x_1 , x_2 , \ldots\ldots\ldots , x_N)$

Therefore,

$a_1(cx_1 , cx_2 , \ldots\ldots\ldots , cx_N)$

$= ½ \{a_0(cx_1 , cx_2 , \ldots\ldots\ldots , cx_N)$

$+ \ g_0(cx_1 , \quad cx_2 , \ldots\ldots\ldots , cx_N)\}$

$= c.½ \ \{a_0(x_1 , \ x_2 , \ldots\ldots\ldots , x_N)$

$+ \ g_0(x_1 , \ x_2 , \ldots\ldots\ldots , x_N)\}$

i.e. $a_1(cx_1 , \ cx_2 , \ldots\ldots\ldots , cx_N)$

$= \ c. \ a_1(x_1 , \ x_2 , \ldots\ldots\ldots , x_N)$

and

$g_1(cx_1 , \ cx_2 , \ldots\ldots\ldots , cx_N)$

$= \{a_0(cx_1 , \ cx_2 , \ldots\ldots\ldots , cx_N) \ .$

$g_0(cx_1 , \ cx_2 , \quad \ldots\ldots\ldots , cx_N)\}^{1/2}$

$= \ c.\{a_0(x_1 , \ x_2 , \ldots\ldots\ldots , x_N) \ .$

$g_0(x_1 , \ x_2 , \quad \ldots\ldots\ldots , x_N)\}^{1/2}$

i.e. $g_1(cx_1 , \ cx_2 , \ldots\ldots\ldots , cx_N)$

$= c.g_0(x_1 , x_2 , \ldots\ldots\ldots , x_N)$

In general,

$a_{n+1}(cx_1 , \ cx_2 , \ldots\ldots\ldots , cx_N)$

$= \ ½ \ \{a_n(cx_1 , \ cx_2 , \ldots\ldots\ldots , cx_N)$

$+ \ g_n(cx_1 , \ cx_2 , \ldots\ldots\ldots , cx_N)\}$

$= c.½ \ \{a_n(x_1 , \ x_2 , \ldots\ldots\ldots , x_N)$

$+ \ g_n(x_1 , \ x_2 , \ldots\ldots\ldots , x_N)\}$

i.e. $a_{n+1}(cx_1 , \ cx_2 , \ldots\ldots\ldots , cx_N)$

$= \ c.a_{n+1}(x_1 , \ x_2 , \ldots\ldots\ldots , x_N)$

Now since $a_n(cx_1 , \ cx_2 , \ldots\ldots\ldots , cx_N)$ converges to M_{AG},

therefore, $a_n(cx_1 , \ cx_2 , \ldots\ldots\ldots , cx_N)$ converges to $c.M_{AG}$.

Hence, *AGM* satisfies the following property:

Property (2.1.1): For positive real constant c,

$AGM \ (cx_1 , \ cx_2 , \ldots\ldots\ldots , cx_N)$

$= \ c. \ AGM \ (x_1 , \ x_2 , \ldots\ldots\ldots , x_N)$ (2.8)

2.2. *Arithmetic-Harmonic Mean (AHM)*

Let $\{a'_n\}$ & $\{h'_n\}$ be two sequences defined by

$$a'_{n+1} = ½(a'_n + h'_n) \quad (2.9)$$

$$\& \ h'_{n+1} = \{½(a'^{-1}_n + h'^{-1}_n)\}^{-1} \quad (2.10)$$

respectively where . $a'_0 = a_0$ & $h'_0 = h_0$.

By inequality (2.1), $h'_n < a'_n$

Now, $a'_{n+1} = ½ (a'_n + h'_n)$

$\Rightarrow \ a'_{n+1} < ½(a'_n + a'_n)$

$\Rightarrow \ a'_{n+1} < a'_n$

This means that the sequence $\{a'_n\}$ is non-increasing.

Moreover, the sequence $\{a'_n\}$ is bounded below by the inequality (2.4).

Hence, by the monotone convergence theorem, the sequence is convergent.

Therefore, there exists a finite number M_{AH} such that a'_n converges to $\mathrm{M_{AH}}$.

Again, h'_n can be expressed as

$$h'_n = 2\, a'_{n+1} - a'_n$$

This implies that the limiting value of h'_n as n approaches infinity is $\mathrm{M_{AH}}$.

Therefore, h'_n converges to $\mathrm{M_{AH}}$.

Thus, the two sequences $\{a'_n\}$ & $\{h'_n\}$ converge to the same point $\mathrm{M_{AH}}$.

This common point $\mathrm{M_{AH}}$ can be termed as the Arithmetic-Harmonic Mean of $x_1 , x_2 , \ldots\ldots\ldots , x_N$.

Accordingly, Arithmetic-Harmonic Mean can be defined as follows:

Definition of Arithmetic-Harmonic Mean (AHM)

The Arithmetic-Harmonic Mean (abbreviated by *AHM*) of the *N* positive real numbers

$$x_1 , x_2 , \ldots\ldots\ldots , x_N$$

denoted by $AHM\,(x_1 , x_2 , \ldots\ldots\ldots , x_N)$

can be defined to be the common point of convergence $\mathrm{M_{AH}}$ of the two sequences $\{a'_n\}$ & $\{h'_n\}$ defined respectively by

$$a'_{n+1} = ½(a'_n + h'_n)$$

$$\& \ h'_{n+1} = \{½(a'^{-1}_n + h'^{-1}_n)\}^{-1}$$

respectively where $a'_0 = a_0$ & $h'_0 = h_0$

with a_0 & g_0 defined by (2.1) & (2.3) respectively.

Note

(1) It is clear that each of $\{a'_n\}$ & $\{h'_n\}$ converges to $\mathrm{M_{AH}}$.

Thus, the converging point of either $\{a'_n\}$ or $\{h'_n\}$ can be taken as the value of M_{AH} .

(2) Since $a'_{n+1} < a'_n$,

i.e. $a_0 > a'_1 > a'_2 > a'_3 > \ldots\ldots\ldots\ldots\ldots$

Therefore, $a_0 > M_{AH}$

Again , $h'_n = 2\,a'_{n+1} - a'_n$

& $h'_{n+1} = 2a'_{n+2} - a'_{n+1}$

which implies, $h'_n < h'_{n+1}$

i.e. $h_0 < h'_1 < h'_2 < h'_3 < \ldots\ldots\ldots\ldots$

Therefore, $h_0 < M_{AH}$

Hence, the following inequality is obtained:

$$a_0 > M_{AH} > h_0$$

i.e. $AM > AHM > HM$ (2.11)

One Important Property of AHM

For positive real constant c, it has been shown that

$$a_0(cx_1 ,\ cx_2 , \ldots\ldots\ldots , cx_N)$$

$$= c.a_0(x_1 ,\ x_2 , \ldots\ldots\ldots , x_N)$$

i.e. $a'_0(cx_1 ,\ cx_2 , \ldots\ldots\ldots , cx_N)$

$$= c.\ a'_0\ (x_1 ,\ x_2 , \ldots\ldots\ldots , x_N)$$

Also,

$$h'_0\ (cx_1 ,\ cx_2 , \ldots\ldots\ldots , cx_N)$$

$= h_0(cx_1 ,\ cx_2 , \ldots\ldots\ldots , cx_N)$

$[\{(cx_1)^{-1} + (cx_2)^{-1} + \ldots\ldots\ldots + (cx_n)^{-1}\} / n\}]^{-1}$

= c. $\{(x_1^{-1} + x_2^{-1} + \ldots\ldots\ldots + x_n^{-1}) / n\}^{-1}$

i.e. $h'_0\ (cx_1 ,\ cx_2 , \ldots\ldots\ldots , cx_N)$

$= c.\ h'_0\,(x_1\ ,\ x_2\ ,\ \dots\dots\dots\ ,\ x_N)$

Similarly it can be found that

$a'_1\,(cx_1\ ,\ cx_2\ ,\ \dots\dots\dots\ ,\ cx_N)$

$= c.\ a'_1\,(x_1\ ,\ x_2\ ,\ \dots\dots\dots\ ,\ x_N)$

& $h'_1\,(cx_1\ ,\ cx_2\ ,\ \dots\dots\dots\ ,\ cx_N)$

$= c.\ h'_1\ (x_1\ ,\ x_2\ ,\ \dots\dots\dots\ ,\ x_N)$

In general,

$a'_{n+1}\,(cx_1\ ,\ cx_2\ ,\ \dots\dots\dots\ ,\ cx_N)$

$= ½\ \{\ a'_n\,(cx_1\ ,\ cx_2\ ,\ \dots\dots\dots\ ,\ cx_N)$

$+\ h'_n\,(cx_1\ ,\ cx_2\ ,\ \dots\dots\dots\ ,\ cx_N)\}$

$= c.\ ½\ \{\ a'_n\,(x_1\ ,\ x_2\ ,\ \dots\dots\dots\ ,\ x_N)$

$+\ h'_n\ (x_1\ ,\ x_2\ ,\ \dots\dots\dots\ ,\ x_N)$

i.e. $a'_{n+1}\,(cx_1\ ,\ cx_2\ ,\ \dots\dots\dots\ ,\ cx_N)$

$= c.\ a'_{n+1}\,(x_1\ ,\ x_2\ ,\ \dots\dots\dots\ ,\ x_N)$

Now since $a'_{n+1}\,(x_1\ ,\ x_2\ ,\ \dots\dots\dots\ ,\ x_N)$ converges to M_{AH} ,

therefore, $a'_n\,(cx_1\ ,\ cx_2\ ,\ \dots\dots\dots\ ,\ cx_N)$ converges to $c.M_{AH}$.

Hence, *AHM* satisfies the following property:

Property (2.2.1): For positive real constant c,

$AHM\ (cx_1\ ,\ cx_2\ ,\ \dots\dots\dots\ ,\ cx_N)$

$= c.\ AHM\ (x_1\ ,\ x_2\ ,\ \dots\dots\dots\ ,\ x_N)$ (2.12)

2. 3. *Geometric-Harmonic Mean (GHM)*

Let $\{g''_n\}$ & $\{h''_n\}$ be two sequences defined respectively by

$g''_{n+1}\ =\ (g''_n\ .\ h''_n)^{1/2}$ (2.13)

& $h''_{n+1}\ =\ \{½(\ g''_n{}^{-1}\ +\ h''_n{}^{-1}\ )\}^{-1}$ (2.14)

where $g''_0 = g_0$ & $h''_0 = h_0$

and the square cube takes the principal value.

By inequality (2.1),

$h''_n\ <\ g''_n$

Therefore,

$g''_{n+1}\ =\ (h''_n\ .\ g''_n)^{1/2}$

$\Rightarrow \; g''_{n+1} < (g''_n \,.\, g''_n)^{1/2}$

i.e. $\; g''_{n+1} < g''_n$

This means that the sequence $\{g''_n\}$ is non-increasing.

Moreover, the sequence $\{g''_n\}$ is bounded below by (2.4).

Hence, by the monotone convergence theorem, the sequence is convergent.

Therefore, there exists a finite number $\mathrm{M_{GH}}$ such that g''_n converges to $\mathrm{M_{GH}}$.

Again, $\; h''_n = {g''_{n+1}}^2 / g''_n$

This implies that the limiting value of h''_n as n approaches infinity is $\mathrm{M_{GH}}$.

Therefore, h''_n converges to $\mathrm{M_{GH}}$.

Thus the two sequences $\{g''_n\}$ & $\{h''_n\}$ converge to the same point$\mathrm{M_{GH}}$.

This common limit $\mathrm{M_{GH}}$ can be termed as the Geometric-Harmonic Mean of $x_1 , \; x_2 , \ldots\ldots\ldots , x_N$.

Accordingly, Geometric-Harmonic Mean can be defined as follows:

Definition of Geometric-Harmonic Mean (GHM)

The Arithmetic-Harmonic Mean (abbreviated by *GHM*) of the *N* positive real numbers

$$x_1 , \; x_2 , \ldots\ldots\ldots , x_N$$

denoted by $GHM\,(x_1 , \; x_2 , \ldots\ldots\ldots , x_N)$

is the common limit (or equivalently the common point of convergence) $\mathrm{M_{GH}}$ of the two sequences $\{g''_n\}$ & $\{h''_n\}$

defined respectively by

$$g''_{n+1} = (g''_n \,.\, h''_n)^{1/2}$$

$$\& \;\; h''_{n+1} = \{½(\, {g''_n}^{-1} + {h''_n}^{-1})\}^{-1}$$

where $g''_0 = g_0$ & $h''_0 = h_0$

with g_0 & h_0 defined by (2.2) & (2.3) respectively

and the square root takes the principal value .

Note

(1) It is clear that each of $\{g''_n\}$ & $\{h''_n\}$ converges to $\mathrm{M_{GH}}$.

Thus, the converging point of either $\{g''_n\}$ & $\{h''_n\}$ can be taken as the value of $\mathrm{M_{GH}}$.

(2) Since $\; g''_{n+1} < g''_n$,

i.e. $g_0 > g''_1 > g''_2 > g''_3 > \ldots\ldots\ldots\ldots$

Therefore, $g_0 > M_{GH}$

Again , $h''_n = g''_{n+1}{}^2 / g''_n$

$\Rightarrow h''_n < h''_{n+1}$

i.e. $h_0 < h''_1 < h''_2 < h''_3 < \ldots\ldots\ldots\ldots$

Therefore, $h_0 < M_{GH}$

Hence, the following inequality is obtained:

$$g_0 > M_{GH} > h_0$$

i.e. $GM > GHM > HM$ (2.15)

One Important Property of GHM

For positive real constant c, it has been shown that

$g''_0 (cx_1 , cx_2 , \ldots\ldots , cx_N)$

$= c.\, g''_0 (x_1 , x_2 , \ldots\ldots , x_N)$

& $h''_0 (cx_1 , cx_2 , \ldots\ldots , cx_N)$

$= c.\, h''_0 (x_1 , x_2 , \ldots\ldots , x_N)$

Similarly it can be found that

$g''_1 (cx_1 , cx_2 , \ldots\ldots , cx_N)$

$= c.\, g''_1 (x_1 , x_2 , \ldots\ldots , x_N)$

& $h''_1 (cx_1 , cx_2 , \ldots\ldots , cx_N)$

$= c.\, h''_1 (x_1 , x_2 , \ldots\ldots , x_N)$

In general,

$g''_{n+1}(cx_1 , cx_2 , \ldots\ldots , cx_N)$

$= \{ g''_n (cx_1 , cx_2 , \ldots\ldots , cx_N).$

$h''_n (cx_1 , cx_2 , \ldots\ldots , cx_N)\}^{1/2}$

$= c.\{ g''_n (x_1 , x_2 , \ldots\ldots , x_N).$

$h''_n (x_1 , x_2 , \ldots\ldots , x_N)\}^{1/2}$

i.e. $g''_{n+1}(cx_1 , cx_2 , \ldots\ldots , cx_N)$

$= c.g^{//}{}_{n+1}(x_1 , x_2 , \ldots\ldots\ldots , x_N)$

Now since $g^{//}{}_n (x_1 , x_2 , \ldots\ldots\ldots , x_N)$ converges to $\mathrm{M_{GH}}$,

therefore, $g^{//}{}_n (cx_1 , cx_2 , \ldots\ldots\ldots , cx_N)$ converges to $c.\mathrm{M_{GH}}$.

Hence, *GHM* satisfies the following property:

Property (2.3.1): For positive real constant c ,

$GHM (cx_1 , cx_2 , \ldots\ldots\ldots , cx_N)$

$= c.GHM (x_1 , x_2 , \ldots\ldots\ldots , x_N)$ (2.16)

2.4. Arithmetic-Geometric-Harmonic Mean (AGHM)

Let the three sequences $\{a^{///}{}_n\}$, $\{ g^{///}{}_n\}$ & $\{h^{///}{}_n\}$ be respectively defined by

$a^{///}{}_{n+1} = 1/3 \, (a^{///}{}_n + g^{///}{}_n + h^{///}{}_n)$, (2.17)

$g^{///}{}_{n+1} = (a^{///}{}_n \, g^{///}{}_n \, h^{///}{}_n)^{1/3}$ (2.18) & $h^{///}{}_{n+1} = \{1/3 \, (a^{///}{}_n{}^{-1} + g^{///}{}_n{}^{-1} + h^{///}{}_n{}^{-1})\}^{-1}$ (2.19)

where $a^{//}{}_0 = a_0$, $g^{//}{}_0 = g_0$ & $h^{//}{}_0 = h_0$

and the cube root takes the principal value.

By inequality (2.4),

$$h^{///}{}_n < g^{///}{}_n < a^{///}{}_n$$

Therefore,

$a^{///}{}_{n+1} < 1/3 \, (a^{///}{}_n + a^{///}{}_n + a^{///}{}_n)$

i.e. $a^{///}{}_{n+1} < a^{///}{}_n$

This means that the sequence $\{a^{///}{}_n\}$ is non-increasing.

Moreover, the sequence $\{a^{///}{}_n\}$ is bounded below by (2.4).

Hence, by the monotone convergence theorem, the sequence $\{a^{///}{}_n\}$ is convergent.

Therefore, there exists a finite number c such that $a^{///}{}_n$ converges to c .

Again, $g^{///}{}_n < a^{///}{}_n$,

and the sequence $\{a^{///}{}_n\}$ is convergent.

Therefore, the sequence $\{ g^{///}{}_n \}$ is also convergent.

Similarly, $h^{///}{}_n < a^{///}{}_n$,

and the sequence $\{a^{///}{}_n\}$ is convergent.

Therefore, the sequence $\{a^{///}{}_n\}$ is also convergent.

Suppose that the two sequences $\{ g'''_n \}$ and $\{h'''_n\}$ converge to g and h respectively.

Then, $g'''_n \rightarrow g$ & $h'''_n \rightarrow h$ as $n \rightarrow \infty$

Equation (2.17) $\Rightarrow g'''_n = 3\, a'''_{n+1} - a'''_n - h'''_n$

& Equation (2.18) $\Rightarrow h'''_n = {g'''_{n+1}}^{1/3} / (a'''_n\, g'''_n)$

Therefore,

$g'''_n + {g'''_{n+1}}^3 / (a'''_n\, g'''_n) = 3\, a'''_{n+1} - a'''_n$

Taking limit as $n \rightarrow \infty$ it is found that

$$g + \frac{g3}{c\, g} = 3c - c$$

$$\Rightarrow g^2 + c\, g - a'''_n = 0$$

This is a quadratic equation in g.

Solving this equation for g, it is obtained that

$$g = c \quad \& \ -2c$$

Since g cannot be negative, therefore we must have

$$g = c$$

Again from (2.1),

$$h'''_n = 3\, a'''_{n+1} - a'''_n - g'''_n$$

Taking limit as $n \rightarrow \infty$,

$$h = 3c - c - g$$

i.e. $h = 3c - c - c$, since $g = c$

which implies $h = c$

Thus it is obtained that $g = h = c$

Therefore, $\{a'''_n\}$, $\{ g'''_n \}$ & $\{h'''_n\}$ converge to the same limit c.

This common limit can be termed as the Geometric-Harmonic Mean of $x_1, x_2, \ldots\ldots\ldots, x_N$.

Accordingly, Arithmetic-Geometric-Harmonic Mean can be defined as follows:

Definition of Arithmetic-Geometric-Harmonic Mean (AGHM)

The Arithmetic-Geometric-Harmonic Mean (abbreviated by *AGHM*) of the N positive real numbers

$$x_1, x_2, \ldots\ldots\ldots, x_N$$

denoted by $AGHM\,(x_1, x_2, \ldots\ldots\ldots, x_N)$

is the common point of convergence M_{AGH} of the three sequences $\{a'''_n\}$, $\{ g'''_n \}$ & $\{h'''_n\}$ defined by

$$a'''_{n+1} = 1/3\,(a'''_n + g'''_n + h'''_n)\ ,$$

$$g'''_{n+1} = (a'''_n\, g'''_n\, h'''_n)^{1/3}$$

$$\&\ h'''_{n+1} = \{1/3\,(a'''^{-1}_n + g'''^{-1}_n + h'''^{-1}_n)\}^{-1}$$

respectively where

$$a'''_0 = a_0\ ,\ g'''_0 = g_0\ \&\ h'''_0 = h_0$$

with a_0 , g_0 & h_0 defined by (2.1) , (2.2) & (2.3) respectivelyand the cube root takes the principal value .

One Important Property of AGHM)

For positive real constant c , it has been shown that

$$a'''_0(cx_1\ ,\ cx_2\ ,\ \ldots\ldots\ldots\ ,\ cx_N)$$
$$= c.\,a'''_0(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ ,\ x_N)\ ,$$
$$g'''_0(cx_1\ ,\ cx_2\ ,\ \ldots\ldots\ldots\ ,\ cx_N)$$
$$= c.\,g'''_0(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ ,\ x_N)$$
$$\&\quad h'''_0(cx_1\ ,\ cx_2\ ,\ \ldots\ldots\ldots\ ,\ cx_N)$$
$$= c.\,h'''_0(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ ,\ x_N)$$

Similarly it can be found that

$$a'''_1(cx_1\ ,\ cx_2\ ,\ \ldots\ldots\ldots\ ,\ cx_N)$$
$$= c.\ a'''_1(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ ,\ x_N)\ ,$$
$$g'''_1(cx_1\ ,\ cx_2\ ,\ \ldots\ldots\ldots\ ,\ cx_N)$$
$$= c.\,g'''_1(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ ,\ x_N)$$
$$\&\quad h'''_1(cx_1\ ,\ cx_2\ ,\ \ldots\ldots\ldots\ ,\ cx_N)$$
$$= c.\,h'''_1(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ ,\ x_N)$$

In general,

$$a'''_{n+1}\,(cx_1\ ,\ cx_2\ ,\ \ldots\ldots\ldots\ ,\ cx_N)$$
$$= 1/3\{\ a'''_n\,(cx_1\ ,\ cx_2\ ,\ \ldots\ldots\ldots\ ,\ cx_N)$$
$$+\ g'''_n\,(cx_1\ ,\ cx_2\ ,\ \ldots\ldots\ldots\ ,\ cx_N)$$
$$+\ h'''_n\,(cx_1\ ,\ cx_2\ ,\ \ldots\ldots\ldots\ ,\ cx_N)\}$$
$$= c.\ 1/3\{\ a'''_n\,(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ ,\ x_N)$$
$$+\ g'''_n\,(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ ,\ x_N)$$
$$+\ h'''_n\,(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ ,\ x_N)\}$$

i.e. $a'''_{n+1}\,(cx_1\ ,\ cx_2\ ,\ \ldots\ldots\ldots\ ,\ cx_N)$

$= c.\, a'''_{n+1}(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ ,\ x_N)$

Now since $a'''_n(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ ,\ x_N)$ converges to $\mathrm{M_{AGH}}$,

therefore, $a'''_n(cx_1\ ,\ cx_2\ ,\ \ldots\ldots\ldots\ ,\ cx_N)$ converges to $c.\mathrm{M_{AGH}}$.

Hence, *AGHM* satisfies the following property:

Property (2.4.1): For positive real constant c,

$AGHM\ (cx_1\ ,\ cx_2\ ,\ \ldots\ldots\ldots\ ,\ cx_N)$

$= c.\ AGHM\ (x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ ,\ x_N)$ (2.20)

3. Example:

Example (3.1): Let us take the two numbers **1 & 2** .

Here,

a_0 = *AM* (1 , 2) = 1.5 ,

g_0 = *GM* (1 , 2) = 1.4142135623730950488016887242097 ,

h_0 = *HM* (1 , 2) = 1.4142135623730950488016887242097

***Calculation of AGM* (1 , 2)**

In order to compute the AGM, it is required to compute the values of a_n & g_n for integral values of n .So, let us construct the following table (**Table – 3.1.1)**:

Table – 3.1.1

(Computed values of a_n & g_n**)**

n	Term of sequence	Value of sequence a_n & g_n
1	a_1	1.4571067811865475244008443621049
	g_1	1.4564753151219702608511618824733
2	a_2	1.4567910481542588926260031222891
	g_2	1.4567910139395549461941753969818
3	a_3	1.4567910310469069194100892596355
	g_3	1.4567910310469068189627755068948
4	a_4	1.4567910310469068691864323832652
	g_4	1.4567910310469068691864323832651
5	a_5	1.4567910310469068691864323832652
	g_5	1.4567910310469068691864323832652

The digits in a_n & g_n , which are agreed, have been underlined in the above table.

The common point of convergence of $\{a_n\}$ & $\{g_n\}$ is found to be

1.4567910310469068691864323832652

Therefore,

AGM (1 , 2) = 1.4567910310469068691864323832652

***Calculation of AHM* (1 , 2)**

In order to compute the AHM, it is required to compute the values of a'_n & h'_n for integral values of n .So, let us construct the following table (**Table – 3.1.2)**:

Table – 3.1.2

(Computed values of a'_n & h'_n **)**

n	Term of sequence	Value of sequence a'_n & h'_n
1	a'_1	1.41666666666666666666666666666667
	h'_1	1.41176470588235294117647058823 53
2	a'_2	1.4142156862745098039215686274 51
	h'_2	1.41421143847487001733102253032 93
3	a'_3	1.41421356237468991062629557889 01
	h'_3	1.41421356237150018697708366811 49
4	a'_4	1.41421356237309504880168962350 25
	h'_4	1.41421356237309504880168782491 68
5	a'_5	1.41421356237309504880168872420 97
	h'_5	1.41421356237309504880168872420 97

The digits in a'_n & h'_n , which are agreed, have been underlined in the above table.

The common point of convergence of $\{a'_n\}$ & $\{h'_n\}$ } is found to be

1.4142135623730950488016887242097

Therefore,

AHM (1 , 2) = 1.4142135623730950488016887242097

***Calculation of GHM* (1 , 2)**

In order to compute the GHM, it is required to compute the values of g''_n & h''_n for integral values of n .

Therefore, the following table (**Table – 3.1.3)** has been prepared for the computed values of g''_n & h''_n :

The digits in $g^{//}_n$ & $h^{//}_n$, which are agreed, have been underlined in the table.

Table – 3.1.3

(Computed values of $g^{//}_n$ & $h^{//}_n$ **)**

n	Term of sequence	Value of sequence $g^{//}_n$ & $h^{//}_n$
1	$g^{//}_1$	1.3731780959380785038868515034673
	$h^{//}_1$	1.3725830020304792191729804126448
2	$g^{//}_2$	1.3728805167403262392057405319496
	$h^{//}_2$	1.3728804844963743741743109895869
3	$g^{//}_3$	1.3728805006183502120284336369279
	$h^{//}_3$	1.3728805006183501173668415130876
4	$g^{//}_4$	1.3728805006183501646976375750077
	$h^{//}_4$	1.3728805006183501646976375750078
5	$g^{//}_5$	1.37288050061835016469763757500774
	$h^{//}_5$	1.37288050061835016469763757500774

The common point of convergence of $\{g^{//}_n\}$ & $\{h^{//}_n\}$ is found to be 1.37288050061835016469763757500774 .

Therefore,

GHM (1 , 2) = 1.37288050061835016469763757500774

***Calculation of AGHM* (1 , 2)**

In order to compute the AGHM, it is required to compute the values of $a^{///}_n$, $g^{///}_n$ & $h^{///}_n$ for integral values of n .

Therefore, the following table (**Table – 3.1.4)** has been prepared for the computed values of $a^{///}_n$, $g^{///}_n$ & $h^{///}_n$:

The digits in $a^{///}_n$, $g^{///}_n$ & $h^{///}_n$, which are agreed, have been underlined in the table.

The common point of convergence of $a^{///}_n$, $g^{///}_n$ & $h^{///}_n$ is found to be 1.4142135623730950488016887242097 .

Therefore,

AGHM (1 , 2) = 1.4142135623730950488016887242097

Table – 3.1.4

(Computed values of $a^{///}_n$, $g^{///}_n$ & $h^{///}_n$)

n	Term of sequence	Value of sequence $a^{///}_n$, $g^{///}_n$ & $h^{///}_n$
1	$a^{///}_1$	1.4158489652354761273783406858477
	$g^{///}_1$	1.4142135623730950488016887242097

	$h^{///}_1$	1.412580048513416909948549472676
2	$a^{///}_2$	1.4142141920406626953761929609111
	$g^{///}_2$	1.4142135623730950488016887242097
	$h^{///}_2$	1.4142129327058077566798391059762
3	$a^{///}_3$	1.4142135623731885002859069303657
	$g^{///}_3$	1.4142135623730950488016887242097
	$h^{///}_3$	1.414213562373001597317470524229
4	$a^{///}_4$	1.4142135623730950488016887262681
	$g^{//}_4$	1.4142135623730950488016887242097
	$h^{///}_4$	1.4142135623730950488016887221513
5	$a^{///}_5$	1.4142135623730950488016887242097
	$g^{///}_5$	1.4142135623730950488016887242097
	$h^{///}_5$	1.4142135623730950488016887242097

Example (3.2): Let us take the three numbers **1 , 2 , 3**

In this case,

$a_0 = AM(1 , 2 , 3) = 2$,

$g_0 = GM(1 , 2 , 3)$

$= 1.8171205928321396588912117563273$

& $h_0 = HM(1 , 2 , 3)$

$= 1.6363636363636363636363636363636$

***Calculation of AGM* (1 , 2 , 3)**

As earlier, in order to compute the AGM, the following table (**Table – 3.2.1)** has been prepared for the computed values of a_n & g_n , in this case also, for integral values of n .

Table – 3.2.1

(Computed values of a_n & g_n**)**

n	Term of sequence	Value of sequence a_n & g_n
1	a_1	1.9085602964160698294456058781637
	g_1	1.9063685859938731475148179907989

2	a_2	1.9074644412049714884802119344813
	g_2	1.9074641264156845057408620234234
3	a_3	1.9074642838103279971105369789524
	g_3	1.9074642838103215033916217579023
4	a_4	1.9074642838103247502510793684274
	g_4	1.9074642838103247502510793684246
5	a_5	1.907464283810324750251079368426
	g_5	1.907464283810324750251079368426

The digits in a_n & g_n , which are agreed, have been underlined in the above table.

The common point of convergence of $\{a_n\}$ & $\{g_n\}$ is found to be

1.907464283810324750251079368426

Therefore,

$AGM\,(1\ ,\ 2\ ,\ 3)\ =\ 1.907464283810324750251079368426$

***Calculation of AHM* (1 , 2 , 3)**

In this case, in order to compute the AHM, the following table (**Table – 3.2.2)** has been prepared for the computed values of a'_n & h'_n for integral values of n :

The digits in a'_n & h'_n , which are agreed, have been underlined in the table.

Table – 3.2.2

(Computed values of a'_n & h'_n **)**

n	Term of sequence	Value of sequence a'_n & h'_n
1	a'_1	1.8181818181818181818181818181818
	h'_1	1.8
2	a'_2	1.8090909090909090909090909090909
	h'_2	1.8090452261306532663316582914573
3	a'_3	1.8090680676107811786203746002741
	h'_3	1.8090680673223822931528642315122
4	a'_4	1.8090680674665817358866194158932
	h'_4	1.8090680674665817358751253877476
5	a'_5	1.8090680674665817358808724018204
	h'_5	1.8090680674665817358808724018204

It is observed that the common point of convergence of $\{a'_n\}$ & $\{h'_n\}$ } is

1.8090680674665817358808724018204

Therefore,

AGM (1 , 2 , 3) = .8090680674665817358808724018204

***Calculation of GHM* (1 , 2 , 3)**

In this case , in order to compute the GHM, it is required to compute the values of g''_n & h''_n for integral values of n .

Therefore, the following table (**Table – 3.2.3)** has been prepared for the computed values of g''_n & h''_n :

The digits in g''_n & h''_n , which are agreed, have been underlined in the table.

Table – 3.2.3

(Computed values of g''_n & h''_n **)**

n	Term of sequence	Value of sequence g''_n & h''_n
1	g''_1	1.7243752668714680664498115720054
	h''_1	1.7220116633863929641517494706201
2	g''_2	1.7231930598768357397574174548246
	h''_2	1.7231926546248362693922850161974
3	g''_3	1.7231928572508240914289732446021
	h''_3	1.7231928572508121782830952537755
4	g''_4	1.7231928572508181348560342491785
	h''_4	1.7231928572508181348560342491682
5	g''_5	1.7231928572508181348560342491733
	h''_5	1.7231928572508181348560342491733

The common point of convergence of $\{g''_n\}$ & $\{h''_n\}$ is found to be 1.7231928572508181348560342491733.

Therefore,

GHM (1 , 2 , 3) = 1.7231928572508181348560342491733

***Calculation of AGHM* (1 , 2 , 3)**

In order to compute the AGHM, it is required to compute the values of a'''_n , g'''_n & h'''_n for integral values of n .

Therefore, the following table (**Table – 3.2.4)** has been prepared for the computed values of a'''_n , g'''_n & h'''_n :

The digits in a'''_n , g'''_n & h'''_n , which are agreed, have been underlined in the table.

The common point of convergence of a'''_n , g'''_n & h'''_n is found to be 1.8117422857304566825260427457086 .

Therefore,

AGHM (1 , 2 , 3) = 1.8117422857304566825260427457086

Table – 3.2.4

(Computed values of $a^{///}_n$, $g^{///}_n$ & $h^{///}_n$)

n	Term of sequence	Value of sequence $a^{///}_n$, $g^{///}_n$ & $h^{///}_n$
1	$a^{///}_1$	1.8178280763985920075091917975636
	$g^{///}_1$	1.8117482698069516271224852651028
	$h^{///}_1$	1.8056709052920116796224963481548
2	$a^{///}_2$	1.8117490838325184380847244702737
	$g^{///}_2$	1.8117422857379412045324993765718
	$h^{///}_2$	1.8117354876464184860385275271165
3	$a^{///}_3$	1.8117422857389593762185837913207
	$g^{///}_3$	1.8117422857304566825260544542536
	$h^{///}_3$	1.8117422857219539888335298955714
4	$a^{///}_4$	1.8117422857304566825260560470486
	$g^{//}_4$	1.8117422857304566825260427457086
	$h^{///}_4$	1.8117422857304566825260294443687
5	$a^{///}_5$	1.8117422857304566825260427457086
	$g^{///}_5$	1.8117422857304566825260427457086
	$h^{///}_5$	1.8117422857304566825260427457086

4. Conclusion:

From the description, presented above, one can arrive at the following conclusions:

(1) In addition to AM, GM, & HM, each of the four formulations namely AGM, AHM, GHM & AGHM can be regarded as a measure of average of a set of numbers.

(2) AGM, AHM, GHM & AGHM are defined only when the associated numbers are strictly positive. For numbers other than strictly positive, there is necessity of searching for technique of finding these types of average.

(3) It can be assumed that assumed that each of AGM, AHM, GHM & AGHM can be accepted as measure of central tendency of data. However, it is to be established with justification.

(4) From the computed results, it has been found that

AHM (1 , 2) & *GM* (1 , 2) are equal

but *AHM* (1 , 2 , 3) & *GM* (1 , 2 , 3) are not equal.

This may give an indication that AHM of two positive numbers can be equal to their GM. However, it is to be established theoretically.

References

[1] Bakker Arthur., The early history of average values and implications for education, Journal of Statistics Education, 11(1)(2003) 17 – 26.
[2] Christoph Riedweg ., Pythagoras: his life, teaching, and influence (translated by Steven Rendall in collaboration with Christoph Riedweg and Andreas Schatzmann, Ithaca), ISBN 0-8014-4240-0, Cornell University Press., (2005).
[3] David A. Cox., The Arithmetic-Geometric Mean of Gauss", In J.L. Berggren; Jonathan M. Borwein; Peter Borwein (eds.). Pi: A Source Book. Springer. 481. ISBN 978-0-387-20571-7 , (first published in L'Enseignement Mathématique, t. 30(1984)(2004) 275 – 330).
[4] David W. Cantrell (……): "Pythagorean Means", Math World.
[5] Dhritikesh Chakrabarty., Determination of Parameter from Observations Composed of Itself and Errors, International Journal of Engineering Science and Innovative Technology, 3(2)(2014) 304 – 311. Available at https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
[6] Dhritikesh Chakrabarty., Analysis of Errors Associated to Observations of Measurement Type, International Journal of Electronics and Applied Research, 1(1)(2014)15 – 28. Available at https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
[7] Dhritikesh Chakrabarty ., Observation Composed of a Parameter and Random Error: An Analytical Method of Determining the Parameter", International Journal of Electronics and Applied Research, 1(2)(2014) 20 – 38. Available at http://eses.net.in/online_journal.html .
[8] Dhritikesh Chakrabarty., Observation Consisting of Parameter and Error: Determination of Parameter, Proceedings of the World Congress on Engineering, 2015, ISBN: 978-988-14047-0-1, ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online) (2015). Available at https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
[9] Dhritikesh Chakrabarty., Observation Consisting of Parameter and Error: Determination of Parameter," Lecture Notes in Engineering and Computer Science (ISBN: 978-988-14047-0-1), London, 2015, (2015) 680 – 684. Available at https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats.
[10] Dhritikesh Chakrabarty., Central Tendency of Annual Extremum of Surface Air Temperature at Guwahati, J. Chem. Bio. Phy. Sci. (E- ISSN : 2249 – 1929), Sec. C, 5(3) (2015) 2863 – 2877. Available at: www.jcbsc.org.
[11] Dhritikesh Chakrabarty., Central Tendency of Annual Extremum of Surface Air Temperature at Guwahati Based on Midrange and Median, J. Chem. Bio. Phy. Sci. (E- ISSN : 2249 –1929), Sec. D, 5(3)(2015) 3193 – 3204. Available at: www.jcbsc.org.
[12] Dhritikesh Chakrabarty., Observation Composed of a Parameter and Random Error: Determining the Parameter as Stable Mid Range, International Journal of Electronics and Applied Research (ISSN : 2395 – 0064), 2(1)(2015) 35 – 47. Available at http://eses.net.in/online_journal.html .
[13] Dhritikesh Chakrabarty., A Method of Finding Appropriate value of Parameter from Observation Containing Itself and Random Error", Indian Journal of Scientific Research and Technology, (E-ISSN: 2321-9262), 3(4)(2015) 14 – 21. Available at https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats
[14] Dhritikesh Chakrabarty., Theoretical Model Modified For Observed Data: Error Estimation Associated To Parameter, International Journal of Electronics and Applied Research (ISSN : 2395 – 0064), 2(2)(2015) 29 – 45. Available at http://eses.net.in/online_journal.html .
[15] Dhritikesh Chakrabarty., Impact of Error Contained in Observed Data on Theoretical Model: Study of Some Important Situations, International Journal of Advanced Research in Science, Engineering and Technology, (ISSN : 2350 – 0328), 3(1)(2016) 1255 – 1265. Available at https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
[16] Dhritikesh Chakrabarty ., Pythagorean Mean: Concept behind the Averages and Lot of Measures of Characteristics of Data, NaSAEAST- 2016, Abstract ID: CMAST_NaSAEAST (Inv)-1601)., (2016) Available at https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
[17] Dhritikesh Chakrabarty., Objectives and Philosophy behind the Construction of Different Types of Measures of Average, NaSAEAST- 2017, Abstract ID: (2017) CMAST_NaSAEAST (Inv)- 1701), Available at https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
[18] Dhritikesh Chakrabarty., Theoretical Model and Model Satisfied by Observed Data: One Pair of Related Variables", International Journal of Advanced Research in Science, Engineering and Technology, 3(2)(2017) 1527 – 1534, Available at www.ijarset.com .
[19] Dhritikesh Chakrabarty., Variable(s) Connected by Theoretical Model and Model for Respective Observed Data, FSDM2017, Abstract ID: FSDM2220, (2017). Available at https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats.
[20] Dhritikesh Chakrabarty, Numerical Data Containing One Parameter and Random Error: Evaluation of the Parameter by Convergence of Statistic, International Journal of Electronics and Applied Research, 4(2)(2017) 59 – 73. Available at http://eses.net.in/online_journal.html .
[21] Dhritikesh Chakrabarty., Observed Data Containing One Parameter and Random Error: Evaluation of the Parameter Applying Pythagorean Mean, International Journal of Electronics and Applied Research, 5(1)(2018) 32 – 45. Available at http://eses.net.in/online_journal.html .
[22] Dhritikesh Chakrabarty ., Derivation of Some Formulations of Average from One Technique of Construction of Mean, American Journal of Mathematical and Computational Sciences, 3(3)(2018) 62 – 68. Available at http://www.aascit.org/journal/ajmcs.
[23] Dhritikesh Chakrabarty., One Generalized Definition of Average: Derivation of Formulations of Various Means, Journal of Environmental Science, Computer Science and Engineering & Technology, Section C, (E-ISSN: 2278 – 179 X), 7(3)(2018) 212 – 225. Available at www.jecet.org.
[24] Dhritikesh Chakrabarty., f_H -Mean: One Generalized Definition of Average", Journal of Environmental Science, Computer Science and Engineering & Technology, Section C, 7(4)(2018) 301 – 314. Available at www.jecet.org .
[25] Dhritikesh Chakrabarty., Generalized f_G - Mean: Derivation of Various Formulations of Average, American Journal of Computation, Communication and Control, 5(3) (2018) 101 – 108. Available at http://www.aascit.org/journal/ajmcs .
[26] Dhritikesh Chakrabarty., General Technique of Defining Average, NaSAEAST- 2018, Abstract ID: CMAST_NaSAEAST -1801 (I)(2018), Available at https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
[27] Dhritikesh Chakrabarty., Observed Data Containing One Parameter and Random Error: Probabilistic Evaluation of Parameter by Pythagorean Mean, International Journal of Electronics and Applied Research, 6(1)(2019) 24 – 40. Available at http://eses.net.in/online_journal.html .
[28] Dhritikesh Chakrabarty., One Definition of Generalized f_G - Mean: Derivation of Various Formulations of Average, Journal of Environmental Science, Computer Science and Engineering & Technology, Section C, (E- ISSN : 2278 – 179 X), 8(2)(2019) 051 – 066. Available at www.jecet.org.
[29] Dhritikesh Chakrabarty ., One General Method of Defining Average: Derivation of Definitions/Formulations of Various Means, Journal of Environmental Science, Computer Science and Engineering & Technology, Section C, (E-ISSN : 2278 – 179 X), 8(4)(2019) 327 – 338. Available at www.jecet.org .
[30] Dhritikesh Chakrabarty., A General Method of Defining Average of Function of a Set of Values, Aryabhatta Journal of Mathematics & Informatics {ISSN (Print) : 0975-7139, ISSN (Online) : 2394-9309}, 11(2)(2019) 269 – 284. Available at www.abjni.com .
[31] Dhritikesh Chakrabarty., Pythagorean Geometric Mean: Measure of Relative Change in a Group of Variables, NaSAEAST- 2019, Abstract ID: CMAST_NaSAEAST -1902 (I). Available at (2019) , https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
[32] Dhritikesh Chakrabarty ., Definition / Formulation of Average from First Principle, Journal of Environmental Science, Computer Science and Engineering & Technology, Section C, (E-ISSN : 2278 – 179 X), 9(2)(2020) 151 – 163. Available at www.jecet.org .
[33] Hazewinkel, Michiel ed., Arithmetic–geometric mean process, Encyclopedia of Mathematics, Springer Science+Business Media B.V. / Kluwer Academic Publishers, ISBN 978-1-55608-010-4 ., (2001).

[34] Foster D. M. E. and Phillips G. M., The Arithmetic-Harmonic Mean, Journal of American Mathematical Society, 42(165)183-191.
[35] Kolmogorov Andrey., On the Notion of Mean, in "Mathematics and Mechanics., (Kluwer 1991), (1930)144 – 146.
[36] Kolmogorov Andrey., Grundbegriffe der Wahrscheinlichkeitsrechnung (in German), Berlin: Julius Springer., (1933).
[37] Miguel de Carvalho., Mean, what do you Mean?", The American Statistician, 70(2016) 764 – 776.
[38] Plackett, R. L., Studies in the History of Probability and Statistics: VII. The Principle of the Arithmetic Mean., Biometrika. **45** (1/2)(1958) 130–135. doi:10.2307/2333051. JSTOR 2333051.
[39] Youschkevitch A. P., A. N. Kolmogorov: Historian and philosopher of mathematics on the occasion of his 80th birfhday, Historia Mathematica, 10(4)(1983) 383 – 395.
[40] Weir, Alan J.,The Convergence Theorems, Lebesgue Integration and Measure, Cambridge: Cambridge University Press. 93–118. ISBN 0-521-08728-7.
[41] Weisstein, Eric W. ( )., Harmonic-Geometric Mean., From MathWorld--A Wolfram Web Resource. https://mathworld.wolfram.com/Harmonic-GeometricMean.html.
[42] Weisberg H. F., Central Tendency and Variability, Sage University Paper Series on Quantitative Applications in the Social Sciences, ISBN 0-8039-4007-6 (1992) 2.
[43] Williams R. B. G., Measures of Central Tendency, Introduction to Statistics for Geographers and Earth Scientist**,** Softcover ISBN978-0-333-35275-5, eBook ISBN978-1-349-06815-9 , Palgrave, London, (1984) 51 – 60.