International Journal of Mathematics Trends and Technology
ISSN: 2231 – 5373 /doi:10.14445/22315373/IJMTT-V67I6P513

Volume 67 Issue 6, 119-124, June, 2021

Ordinary Functional Differential Equations With Periodic Boundary Conditions Involving Caratheodory Condition

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Abstract: *In this paper, we proved an existence theorem for ordinary functional differential equations with periodic boundary conditions via a fixed point theorem in Banach algebras, some mixed generalized Lipschitz and caratheodory conditions.*

Keyword and Phrases: *Ordinary functional differential equation, periodic boundary conditions, fixed point theorem, Lipschitz and caratheodory condition*.

Subject classifications: 34K10

1. Introduction

In last few years, the study of nonlinear differential equations in Banach algebras is received the attention of several authors and at present, there is a considerable literature available in this way. See Dhage and O'Regan[5]. Dhage et.at. [1] and the references therein. In this article, we proved the existence results for first order ordinary functional differential equation in Banach algebras with periodic boundary condition under Lipschitz condition and caratheodory condition. We apply the fixed point theorem of Dhage[2,3.4] for proving existence results of our problem. The nonlinear differential equations as well as the existence results of this are new to the literature on the theory of ordinary differential equations. Our method of study is to converts the ordinary functional differential equation into equivalent integral equation and apply the fixed point theorem of Dhage[2,3.4] under suitable conditions on the nonlinearities f and g.

2. Statement of problem

Let $\mathbb{R}$ be the real line and $I_0 = [-\delta, 0]$ and $I = [0, T]$ be two closed and bounded intervals in $\mathbb{R}$. Let C be the space of continuous real valued functions on I_0. Given a function $\phi \in C$. We have studied the following periodic boundary value problem (In short PBVP) of first order ordinary functional differential equation

$$\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g\left(t, x_t, \int_0^t k(s,x_s)ds\right), \quad t \in I$$
$$x(0) = x(T), \;\; x_0 = \phi \qquad 2.1$$

Where $f: I \times \mathbb{R} \to \mathbb{R} - \{0\}$ is continuous and $k: I \times C \to \mathbb{R}$, $g: I \times C \times \mathbb{R} \to \mathbb{R}$,
$x_t = I_0 :\to C$ is continuous function defined by $x_t(\theta) = x(t+\theta)$ for all $\theta \in I_0$.

When $f(t,x) = 1$ on $I \times \mathbb{R}$. By a solution of the PBVP (2.1) we means a function $x \in AC(I,\mathbb{R})$ that satisfies

i. The function $t \mapsto \left(\frac{x(t)}{f(t,x(t))}\right)$ is absolutely continuous on I and
ii. x Satisfies the equation (2.1).

where $AC(I,\mathbb{R})$ is the space of continuous functions whose first derivatives exists and is absolutely continuous real valued function on I. The periodic boundary value problem (2.1) is quite general in the sense that it includes several known classes of periodic boundary value problem as special cases, for example, if $f(t,x) = 1$ on $I \times \mathbb{R}$ then PBVP (2.1) reduce to the PBVP

$$\frac{d}{dt}(x(t)) = g\left(t, x_t, \int_0^t k(s,x_s)ds\right), \;\; \text{a.e. } t \in I$$
$$x(0) = x(T) \qquad 2.2$$

which further, when $g(t, x_t, y) = g(t, x_t)$ on $I \times C \to \mathbb{R}$, includes the following PBVP studied in Nieto[1997,2002].

$$\frac{d}{dt}\big(x(t)\big) = g\big(t, x(t)\big), \text{ a. e. } t \in I$$
$$x(0) = x(T) \qquad 2.3$$

There is good deal of literature on the PBVP (2.3) for different aspects if the solutions. In this article, we discuss the PBVP (2.1) for existence theory only under suitable conditions on the nonlinearities f and g involved in it.

3. Auxiliary Results

Definition (3.1): Let X be a Banach algebra with norm$\|\cdot\|$. A mapping $A: X \to X$ is called $'\mathfrak{D} - Lipschitz'$if there exists a continuous nondecreasing function $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Ax - Ay\| \le \psi\|x - y\| \qquad 3.1$$

for all $x, y \in$ with $\psi(0) = 0$.In the special case when $\psi(r) = \alpha r\ (\alpha > 0)$. A is called a Lipschitz with a Lipschitz constant α. In particular, if $\alpha < 1$, A is called a contraction with a contraction constant α. Further, if $\psi(r) < r$ for all $r > 0$, then A is called a nonlinear contraction on X. We call the function ψ and $\mathfrak{D}$ function for our convenience.

Definition (3.2): An operator $T: X \to X$ is called compact if $\overline{T(S)}$ is compact subset of X for any $S \subset X$. Also $T: X \to X$ is called totally bounded if T maps a bounded subset of X into the relatively compact subset of X. Finally $T: X \to X$ is called completely continuous operator if it is continuous and totally bounded operator on X.

It is clearly that every compact operator is totally bounded, but the converse may not be true. The nonlinear alternative of Schaefer type recently proved by Dhage [3] is engroove in the following theorem.

Theorem (3.1) Dhage[3]: Let X be a Banach algebra and $A, B: X \to X$ be two operators satisfying

i. A is a $\mathfrak{D} - Lipschitz$ with a $\mathfrak{D}$ function ψ.
ii. B is compact and continuous, and
iii. $M\psi(r) < r$ Whenever $r > 0$, where $M = \|B(X)\| = sup\{\|Bx\|: x \in X\}$.

Then either

i. The equation $\lambda AxBx = x$ has a solution for $\lambda = 1$, or
ii. The set $\mathcal{E} = \{u \in X: \lambda AuBu = u, 0 < \lambda < 1\}$ is unbounded.

It is know that theorem (3.1) is useful for proving the existence theorem for the integral equations of mixed type. See[4] and the references therein. The method is commonly known as priori bound method for the nonlinear equations. See for example, Dugundji and Granas[7, i] and Zeidler[11,12] and the references therein.

An interesting corollary to theorem (3.1) in its applicable form is

Corollary (3.1): Let X be a Banach algebra and $A, B: X \to X$ be two operators satisfying

i. A is a $Lipschitz$ with a Lipschitz constant α.
ii. B is compact and continuous, and
iii. $\alpha M < 1$, where $M = \|B(X)\| = sup\{\|Bx\|: x \in X\}$.

Then either

i. The equation $\lambda AxBx = x$ has a solution for $\lambda = 1$, or
ii. The set $\mathcal{E} = \{u \in X: \lambda AuBu = u, 0 < \lambda < 1\}$ is unbounded.

Definition (3.3): A non-empty closed set K in a Banach algebra X is called a cone if
(i) $+K \subseteq K$, (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}, \lambda \ge 0$ and (iii) $-K \cap K = 0$, where '0' is zero element in X. A cone K is called to be positive if (iv) $KoK \subseteq K$, where 'o' is multiplication composition in X.

We introduce an order relation $\le$ in X as follows.
Let $x, y \in X$. Then $x \le y$ if and only if $y - x \in K$.

Definition (3.4): A cone K is called to be normal if the norm $\|\cdot\|$ is monotone increasing on K. If the cone K is normal in X, then every order bounded set in X is norm-bounded. The details of cones and their properties appear in Guo and Lakshmikantam [10].

Let $AC(I, \mathbb{R})$ be a space of absolutely continuous real-valued functions on X. We equip the space $AC(I, \mathbb{R})$ with the order relation $\le$ with the help of the cone defined by

$$K = \{x \in C(I, \mathbb{R}): x(t) \ge 0, \forall t \in I\} \qquad 3.2$$

where $C(I, \mathbb{R})$ is space of all continuous real-valued functions on I. It is well known that the cone K is positive and normal in $AC(I, \mathbb{R})$. As a result of positivity of the cone K in $AC(I, \mathbb{R})$, we have

Lemma(3.1) Dhage[2]: Let $u_1, u_2, v_1, v_2 \in K$ be such that $u_1 \le v_1$ and $u_2 \le v_2$. Then $u_1 v_1 \le u_2 v_2$. For any $a, b \in X = AC(I, \mathbb{R}), a \le b$ then

$$[a, b] = \{x \in X : a \le x \le b\} \tag{3.3}$$

We use the following fixed point theorem of Dhage$[3, i]$ for proving the existence of extremal solutions of the PBVP (2.1) under certain monotonicity conditions.

Theorem (3.2) Dhage[2]: Let K be cone in a Banach algebra X and let $a, b \in X$. Suppose that $A, B: [a, b] \to K$ are two operators such that

(a) A is Lipschitz with a Lipschitz constant α
(b) B is completely continuous,
(c) $AxBx \in [a, b]$ for each $x \in [a, b]$, and
(d) $A\&B$ are nondecreasing.

Further if the cone K is positive and normal, then the operator $AxBx = x$ has a least and a greatest positive solution in$[a, b]$, whenever $\alpha M < 1$, where $M = \|B([a, b])\| = sup\{\|Bx\| : x \in [a, b]\}$.

Theorem (3.3) Dhage[3]: Let K be cone in a Banach algebra X and let $a, b \in X$. Suppose that $A, B: [a, b] \to K$ are two operators such that

(a) A is Lipschitz with a Lipschitz constant α
(b) B is totally bounded,
(c) $AxBy \in [a, b]$ for each $x, y \in [a, b]$, and
(d) $B$is nondecreasing.

Further if the cone K is positive and normal, then the operator $AxBx = x$ has a least and a greatest positive solution in$[a, b]$, whenever $\alpha M < 1$, where $M = \|B([a, b])\| = sup\{\|Bx\| : x \in [a, b]\}$.

Remark (3.1): Note that hypothesis (c) of theorem (3.2) and (3.3) holds if the operators A and B are monotone increasing and there exist $a, b \in X$ such that $a \le AaBa$ and $AbBb \le b$.

4. Existence Results

Let $M(I, \mathbb{R}), B(I, \mathbb{R})$ and $C(I, \mathbb{R})$ denote the spaces of measurable, bounded and continuous real-valued functions on I respectively. Define a norm $\|\cdot\|$ in $C(I, \mathbb{R})$ by

$$\|x\| = \sup_{t \in I} |x(t)|$$

Clearly $C(I, \mathbb{R})$ becomes a Banach algebra with this norm and the multiplication '$\cdot$'. Defined by $(x \cdot y) = x(t) \cdot y(t)$ for all $t \in I$. By $L'(I, \mathbb{R})$ we denote the set of Lebesque Integrable functions on I and the norm $\|\cdot\|$ in $L'(I, \mathbb{R})$ is defined by

$$\|x\|_{L'} = \int_0^t |x(t)| ds$$

Lemma (4.1): If $h \in L'(I, \mathbb{R})$, then x is a solution of differential equation

$$\frac{d}{dt}\left[\frac{x(t)}{f(t, x(t))}\right] = \int_0^t h(s) ds \quad \text{a.e. } t \in I \tag{4.1}$$
$$x(0) = x_0$$

If and only if is solution of the integral equation

$$x(t) = f(t, x(t))\left[\phi(0) + \int_0^t (t - s) h(s) ds\right], \qquad t \in I \tag{4.2}$$

We need the following definition in the sequel.

Definition (4.1): A mapping $\beta: I \times \mathbb{R} \to \mathbb{R}$ is said to be caratheodory if

i. $t \to \beta(t, x)$ is measurable for each $x \in \mathbb{R}$ and
ii. $x \to \beta(t, x)$ is continuous almost everywhere for $t \in I$.

Again a caratheodory function $\beta(t, x)$ is called L'- caratheodory if

iii. For each real number $r > 0$ there exists a function $h_r \in L'(I, \mathbb{R})$ such that $|\beta(t, x)| \le h_r(t)$ a.e. $t \in I$ for all $x \in \mathbb{R}$ with $|x| \le r$.

Finally a caratheodory function $\beta(t, x)$ is called $L'_X -$ caratheodory if

iv. There exists a function $h \in L'(I, \mathbb{R})$ such that $|\beta(t, x)| \le h(t)$ a.e. $t \in I$ for a,ll $x \in \mathbb{R}$.

For convenience, the function h is referred to as a bound function of β.

We will need the following hypotheses in the sequel.

(A_1). The function $f: I \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $\ell \in B(I, \mathbb{R})$ such that $\ell(t) > 0$ a.e. $t \in I$ and

$\left|f\left(t,x(t)\right)-f\left(t,y(t)\right)\right| \leq \ell(t)|x-y|$ a.e. $t \in I$ for all $x \in \mathbb{R}$.
(A_2). The function g is $L_X'-$ carathèodory with bounded function h.
(A_3). There exists a continuous and nondecreasing function $\Omega: [0,\infty) \to (0,\infty)$ and a functio $\gamma \in L'(I,\mathbb{R})$ such that $\gamma(t) > 0$

a.e. $t \in I$ and $\left|g\left(t,x_t,\int_0^t k(s,x_s)ds\right)\right| \leq \gamma(t)\Omega(|x|)$, a.e. $t \in I$ for all $x \in \mathbb{R}$.

Theorem (4.1): Assume that the hypotheses (A_1) - (A_3) hold. Suppose that

$$\int_{c_1}^{\infty} \frac{ds}{\Omega(s)} > c_2\|\gamma\|_{L'} \qquad 4.3$$

where $c_1 = \frac{F|\phi(0)|}{1-\|K\|[|\phi(0)|+T\|h\|_{L'}]}$, $c_2 = \frac{FT}{1-\|K\|[|\phi(0)|+T\|h\|_{L'}]}$, $\|K\|[|\phi(0)|+T\|h\|_{L'}] < 1$,

$F = \max_{t\in I}|f(t,o)|$, and $\|K\| = \max_{t\in I}|k(t)|$. Then the PBVP (2.1) has a solution on I.

Proof: The PBVP (2.1) is convert to an equivalent integral equation

$$x(t) = \left[f\left(t,x(t)\right)\right]\left(\phi(0) + \int_0^t (t-s)g\left(s,x_s,\int_0^t k(s,x_s)ds\right)ds\right), \text{ a.e. } t \in I \qquad 4.4$$

Let set $X = C(I,\mathbb{R})$. Define the operators A and B on X by

$$Ax(t) = f\left(t,x(t)\right), \quad t \in I \qquad 4.5$$

and
$$Bx(t) = \phi(0) + \int_0^t (t-s)g\left(s,x_s,\int_0^t k(s,x_s)ds\right)ds, \quad t \in I \qquad 4.6$$

Obviously A and B define the operators $A,B: X \to X$. Then the PBVP (2.1) is equivalent to the equation

$$Ax(t)Bx(t) = x(t), \quad t \in I \qquad 4.7$$

We shall show that the operators A and B satisfy all the hypotheses of corollary (3.1).

We first show that A is a Lipschitz on X. Let $x,y \in X$, then by condition (i),

$$\begin{aligned} |Ax(t)-Ay(t)| &\leq \left|f\left(t,x(t)\right)-f\left(t,y(t)\right)\right| \\ &\leq \ell(t)|x(t)-y(t)| \\ &\leq \|\ell\|\|x-y\| \end{aligned}$$

for all $t \in I$. Taking the supremum over t, we obtain

$$\|Ax-Ay\| \leq \|\ell\|\|x-y\|, \text{ for all } x,y \in X.$$

So A is a Lipschitz on X with a Lipschitz constant $\|\ell\|$. Next we show that B is completely continuous on X, using the standard arguments as in Granas et.at.[9], it is show that B is continuous operator on X. Let S be a bounded set in X. We shall show that $B(X)$ is a uniformly bounded and equicontinous set in X. Since $g\left(t,x_t,\int_0^t k(s,x_s)ds\right)$ is L_X'-carathèodory,
we have

$$\begin{aligned} |Bx(t)| &\leq |\phi(0)| + \int_0^t |t-s|\left|g\left(s,x_s,\int_0^t k(s,x_s)ds\right)\right|ds \\ &\leq |\phi(0)| + T\int_0^t h(s)ds \\ &\leq |\phi(0)| + T\|h\|_{L'}. \end{aligned}$$

Taking the supremum over t, we obtain $\|Bx\| \leq M$ for all $x \in S$, where $M = |\phi(0)| + \|h\|_{L'}$. This shows that $B(X)$ is a uniformly bounded set in X. Now we show that $B(X)$ is equicontinous set.

Let $t,\tau \in I$. Then for any $x \in X$, we have

$$\begin{aligned} |Bx(t)-Bx(\tau)| &\leq \left|\phi(0) + \int_0^t (t-s)g\left(s,x_s,\int_0^t k(s,x_s)ds\right)ds - \phi(0) - \int_0^\tau (\tau-s)g\left(s,x_s,\int_0^t k(s,x_s)ds\right)ds\right| \\ &\leq \left|\int_0^t (t-s)g\left(s,x_s,\int_0^t k(s,x_s)ds\right)ds - \int_0^\tau (\tau-s)g\left(s,x_s,\int_0^t k(s,x_s)ds\right)ds\right| \\ &\leq \left|\int_0^t (t-s)g\left(s,x_s,\int_0^t k(s,x_s)ds\right)ds - \int_0^t (\tau-s)g\left(s,x_s,\int_0^t k(s,x_s)ds\right)ds\right| \\ &\quad + \left|\int_0^t (\tau-s)g\left(s,x_s,\int_0^t k(s,x_s)ds\right)ds - \int_0^\tau (\tau-s)g\left(s,x_s,\int_0^t k(s,x_s)ds\right)ds\right| \\ &\leq \left|\int_0^t (t-\tau)g\left(s,x_s,\int_0^t k(s,x_s)ds\right)ds\right| + \left|\int_\tau^t (\tau-s)g\left(s,x_s,\int_0^t k(s,x_s)ds\right)ds\right| \\ &\leq \int_0^T |t-\tau|\left|g\left(s,x_s,\int_0^t k(s,x_s)ds\right)\right|ds + T\left|\int_\tau^t \left|g\left(s,x_s,\int_0^t k(s,x_s)ds\right)\right|ds\right| \\ &\leq \int_0^T |t-\tau|h(s)ds + T\left|\int_\tau^t h(s)ds\right| \end{aligned}$$

$\leq |t-\tau| \|h\|_{L'} + |p(t)-p(\tau)|$

where $p(t) = T\int_0^t h(s)ds$. Therefore $|Bx(t) - Bx(\tau)| \to 0$ as $t \to \tau$. Hence $B(X)$ is an equicontinous set and consequently $B(X)$ is relatively compact by Arzela-Ascoli theorem. As a result B is compact and continuous operator on X, thus all conditions of theorem (3.1) are satisfied and a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible. Let $x \in X$ be any solution to PBVP (2.1). Then we have for any $\lambda \in (0,1)$,

$$x(t) = \lambda\big[f\big(t, x(t)\big)\big]\left(\phi(0) + \int_0^t (t-s) g\left(s, x_s, \int_0^t k(s, x_s)ds\right)ds\right), \quad \text{for } t \in I,$$

Therefore,

$$|x(t)| \leq \lambda\big|f\big(t,x(t)\big)\big|\left(|\phi(0)| + \left|\int_0^t (t-s) g\left(s,x_s,\int_0^t k(s,x_s)ds\right)ds\right|\right)$$

$$\leq \lambda\big(\big|f\big(t,x(t)\big) - f(t,0)\big| + |f(t,0)|\big)\left(|\phi(0)| + \int_0^t |t-s|\left|g\left(s,x_s,\int_0^t k(s,x_s)ds\right)\right|ds\right)$$

$$\leq [\ell(t)|x(t)| + F]\left(|\phi(0)| + \int_0^t |t-s|\left|g\left(s,x_s,\int_0^t k(s,x_s)ds\right)\right|ds\right)$$

$$\leq \ell(t)|x(t)|\left(|\phi(0)| + \int_0^t |t-s|\left|g\left(s,x_s,\int_0^t k(s,x_s)ds\right)\right|ds\right)$$

$$+ F\left(|\phi(0)| + \int_0^t |t-s|\left|g\left(s,x_s,\int_0^t k(s,x_s)ds\right)\right|ds\right)$$

$$\leq \|\ell\||x(t)|(|\phi(0)| + \|h\|_{L'}) + F|\phi(0)| + FT\int_0^t \gamma(s)\Omega(|x(t)|)ds \qquad 4.8$$

Put $u(t) = \sup_{t\in[0,t]} |x(s)|$ for $t \in I$. Then we have $|x(t) \leq u(t)|$ for all $t \in I$.and so, there is point $t^* \in [0,T]$ such that $u(t) = |x(t^*)|$, it follows that

$$u(t) = |x(t^*)| \leq \|k\||x(t^*)|(|\phi(0)| + T\|h\|_{L'}) + F\left(\phi(0) + T\int_0^{t^*} \gamma(s)\Omega(|x(t)|)ds\right)$$

$$\leq \|\ell\|u(t)(|\phi(0)| + T\|h\|_{L'}) + F\left(\phi(0) + T\int_0^t \gamma(s)\Omega\big(u(s)\big)ds\right)$$

$$= c_1 + c_2\int_0^t \gamma(s)\Omega\big(u(s)\big)ds \qquad 4.9$$

where $c_1 = \frac{F|\phi(0)|}{1-\|K\|[|\phi(0)|+T\|h\|_{L'}]}$ and $c_2 = \frac{FT}{1-\|K\|[|\phi(0)|+T\|h\|_{L'}]}$.

Let $\omega(t) = c_1 + c_2\int_0^t \gamma(s)\Omega\big(u(s)\big)ds$ then $u(t) \leq \omega(t)$ and direct differentiation of yields

$$\omega'(t) \leq c_2\,\gamma(t)\Omega\big(u(t)\big) \text{ and } \omega(0) = c_1 \qquad 4.10$$

that is

$$\int_0^t \frac{\omega'(s)}{\Omega(\omega(s))}ds \leq c_2\int_0^t \gamma(s)ds \leq c_2\|\gamma\|_{L'}$$

A change of variables in the above integral gives that

$$\int_{c_1}^{\omega(t)} \frac{ds}{\Omega(s)} \leq c_2\|\gamma\|_{L'} < \int_{c_1}^{\infty} \frac{ds}{\Omega(s)}.$$

Now, an application of mean value theorem yields that there is a constant $M > 0$ such that $\omega(t) \leq M$ for all $t \in I$. This further implies that

$$|x(t)| \leq u(t) \leq \omega(t) \leq M \text{ for all } t \in I.$$

Thus the conclusion (ii) of corollary (3.1) does not hold. Therefore the operator equation $AxBx = x$ and consequently the PBVP (2.1) has a solution on I. This completes the proof.

5. An Example

Given the closed and bounded interval $I = [0.1]$ in $\mathbb{R}$.

Conside PBVP $\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = \frac{p(t)x_t}{1+\int_0^t k(s,x_s)ds}$ a.e. $t \in I$. 5.1

where $p \in L'(I,\mathbb{R})$ and $f: I \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f\big(t,x(t)\big) = \frac{1}{2}\left[1 + \alpha\int_0^t k(s,x_s)ds\right], \ \alpha > 0 \text{ for all } t \in I.$$

Obviously $f: I \times \mathbb{R} \to \mathbb{R}^+ - \{0\}$.

Define $g: I \times C \times \mathbb{R} \to \mathbb{R}$ by $g\left(t, x_t, \int_0^t k(s,x_s)ds\right) = \frac{p(t)x_t}{1+\int_0^t k(s,x_s)ds}$.

It is easy to verify that f is continuous and Lipschitz on $I \times \mathbb{R}$ with a Lipschitz constant α. Further $g\left(t, x_t, \int_0^t k(s, x_s)ds\right)$ is L_X' - carathèodory with the bound function $h(t) = p(t)$ on I. Therefore if $\alpha(1 + \|p\|_{L'}) < 1$, then by theorem (4.1) the PBVP (5.1) has a solution on I, because the function $\Omega(r) = 1$ for all $r \in \mathbb{R}^+$.

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