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Degree of Approximation of Conjugate Fourier Series of Functions in Besov space by Riesz Mean

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Abstract: *The paper studies the degree of approximation of functions by their Conjugate Fourier series in the Besov space by Riesz mean and this generalizing many known results.*

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I. Introduction

Let f be a 2π periodic function and let $f \in L_p[0,2\pi], p \geq 1$. The fourier series of f at x is given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1.1}$$

The conjugate series of (1.1) is given by

$$\sum_{k=1}^{\infty}(a_k \sin kx - b_k \cos kx). \tag{1.2}$$

Let $\tilde{S}_n(x)$ be the nth partial sum of conjugate series (1.2) given by [Zygmund [6]]

$$\tilde{S}_n(x) = \frac{-1}{\pi}\int_0^{\pi}\psi_x(t)\tilde{D}_n(t)dt \tag{1.3}$$

where

$$\psi_x(t) = f(x+t) - f(x-t) \tag{1.4}$$

and the conjugate Dirichlet kernel is defined by

$$\tilde{D}_n(t) = \sum_{k=1}^{n}\sin kt = \frac{\cos\frac{t}{2} - \cos(n+\frac{1}{2})t}{2\sin\frac{1}{2}t} \tag{1.5}$$

Let $\tilde{t}_n(x)$ be the Riesz transform of the conjugate series (1.2); that is

$$\tilde{t}_n(x) = \frac{1}{B_n}\sum_{k=0}^{n} b_k \tilde{S}_k(x) \tag{1.6}$$

We write

$$\tilde{K}_n(t) = \frac{1}{B_n}\sum_{k=1}^{n} b_k \tilde{D}_k(t) \tag{1.7}$$

$$H_n(t) = \frac{1}{B_n}\sum_{k=1}^{n} b_k \frac{\cos(k+\frac{1}{2})t}{2\sin\frac{t}{2}} \tag{1.8}$$

$$\tilde{f}(x;\varepsilon) = -\frac{1}{\pi}\int_{\varepsilon}^{\pi}\psi_x(t)\frac{1}{2}\cot\frac{t}{2}dt, \varepsilon > 0 \tag{1.9}$$

$\tilde{f}(x) = \lim_{\varepsilon\to 0}\tilde{f}(x;\varepsilon)$, whenever the limit exists.

$$\tilde{T}_n(x) = \tilde{t}_n(x) - \tilde{f}\left(x;\frac{\pi}{n}\right) \tag{1.10}$$

It is known ([6]) that for any integrable f the function $\tilde{f}$ exists almost everywhere. It is easy to see that

$$\tilde{K}_n(t) = \frac{1}{2}\cot\frac{t}{2} - H_n(t) \tag{1.11}$$

Using (1.3) in (1.6) and there after making use of notations given in (1.7),(1.8), (1.9) and (1.11), we get

$$\tilde{t}_n(x) = -\frac{1}{\pi}\int_0^{\pi}\psi_x(t)\tilde{K}_n(t)dt$$

$$= -\frac{1}{\pi}\int_0^{\frac{\pi}{n}}\psi_x(t)\tilde{K}_n(t)dt - \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi}\psi_x(t)\left\{\frac{1}{2}\cot\frac{t}{2} - H_n(t)\right\}dt.$$

$$= -\frac{1}{\pi}\int_0^{\frac{\pi}{n}}\psi_x(t)\tilde{K}_n(t)dt + \tilde{f}(x;\frac{\pi}{n}) + \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi}\psi_x(t)H_n(t)dt.$$

Now

$$\tilde{T}_n(x) = -\frac{1}{\pi}\int_0^{\frac{\pi}{n}}\psi_x(u)\tilde{K}_n(u)du + \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi}\psi_x(u)H_n(u)du. \tag{1.12}$$

II. Definitions and Notations

Modulus of Continuity:

Let $A = R, R + [a,b] \subset R$ or T (which usually taken to be R with identification of points modulo 2π).

The modulus of continuity $w(f,t) = w(t)$ of a function f on A can be defined as

$$w(t) = w(f,t) = \sup_{\substack{|x-y|\le t,\\ x,y\in A}} |f(x) - f(y)|, t \ge 0.$$

Modulus of Smoothness:

The k^{th} order modulus of smoothness [2] of a function $f: A \to R$ is defined by

$$w_k(f,t) = \sup_{0<h\le t} \{\sup |\Delta_h^k(f,x)| : x, x+kh \in A\}, t \ge 0 \quad (2.1)$$

where

$$\Delta_h^k(f,x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x+ih), k \in N. \quad (2.2)$$

For $k = 1, w_1(f,t)$ is called the modulus of continuity of f. The function w is continuous at $t = 0$ if and only if f is uniformly continuous on A, that is $f \in \tilde{c}(A)$. The k^{th} order modulus of smoothness of $f \in L_p(A), 0 < p < \infty$ or of $f \in \tilde{c}(A), if p = \infty$ is defined by

$$w_k(f,t)_p = \sup_{0<h\le t} \| \Delta_h^k(f,\cdot) \|_p, t \ge 0 \quad (2.3)$$

if $p \ge 1, k = 1$, then $w_1(f,t)_p = w(f,t)_p$ is a modulus of continuity (or integral modulus of continuity). If $p = \infty, k = 1$ and f is continuous then $w_k(f,t)_p$ reduces to modulus of continuity $w_1(f,t)$ or $w(f,t)$.

Lipschitz Space:

If $f \in \tilde{c}(A)$ and

$$w(f,t) = O(t^{\alpha}), 0 < \alpha \le 1 \quad (2.4)$$

then we write $f \in Lip\alpha$. If $w(f,t) = O(t)$ as $t \to 0+$ (in particular (1.9) holds for $\alpha > 1$) then f reduces to a constant.

If $f \in L_p(A), 0 < p < \infty$ and

$$w(f,t)_p = O(t^{\alpha}), 0 < \alpha \le 1 \quad (2.5)$$

then we write $f \in Lip(\alpha, p), 0 < p < \infty, 0 < \alpha \le 1$.

The case $\alpha > 1$ is of no interest as the function reduces to a constant, whenever

$$w(f,t)_p = O(t) \text{ as } t \to 0+ \quad (2.6)$$

We note that if $p = \infty$ and $f \in c(A)$, then $Lip(\alpha, p)$ class reduces to $Lip\ \alpha$ class.

Generalized Lipschitz Space:

Let $\alpha > 0$ and suppose that $k = [\alpha] + 1$. For $f \in L_p(A), 0 < p < \infty$, if

$$w_k(f,t) = O(t^{\alpha}), t > 0 \quad (2.7)$$

then we write

$$f \in Lip^*(\alpha, p), \alpha > 0, 0 < p \le \infty \quad (2.8)$$

and say that f belongs to generalized Lipschitz space. The seminorm is then

$$|f|_{Lip^*(\alpha, L_p)} = \sup_{t>0} (t^{-\alpha} w_k(f,t)_p).$$

It is known ([2], p-52) that the space $Lip^*(\alpha, L_p)$ contains $Lip(\alpha, L_p)$. For $0 < \alpha < 1$ the spaces coincide, (for $p = \infty$, it is necessary to replace L_∞ by $\tilde{c}$ of uniformly continuous function on A). For $0 < \alpha < 1$

and $p=1$ the space $Lip^*(\alpha, L_p)$ coincide with $Lip\,\alpha$.

For $\alpha=1, p=\infty$, we have

$$Lip(1,\tilde{c}) = Lip\,1 \tag{2.9}$$

but

$$Lip^*(1,\tilde{c}) = z \tag{2.10}$$

is the Zygmund space [5] which is characterized by (2.7) with $k=2$.

Holder (H_α) Space:

For $0<\alpha\le 1$, let

$$H_\alpha = \{ f \in C_{2\pi} : w(f,t) = O(t^\alpha)\}. \tag{2.11}$$

It is known [3] that H_α is a Banach Space with the norm $\|\cdot\|_\alpha$ defined by

$$\| f \|_\alpha = \| f \|_c + \sup_{t>0} t^{-\alpha} w(t),\ 0<\alpha\le 1 \tag{2.12}$$

$$\| f \|_0 = \| f \|_c$$

and

$$H_\alpha \subseteq H_\beta \subseteq C_{2\pi},\ 0<\beta\le\alpha\le 1 \tag{2.13}$$

$H_{(\alpha,p)}$ Space:

For $0<\alpha\le 1$, let

$$H_{(\alpha,p)} = \{ f \in L_p[0,2\pi] : 0<p\le\infty, w(f,t)_p = O(t^\alpha)\} \tag{2.14}$$

and introduce the norm $\|\cdot\|_{(\alpha,p)}$ as follows

$$\| f \|_{(\alpha,p)} = \| f \|_p + \sup_{t>0} t^{-\alpha} w(f,t)_p,\ 0<\alpha\le 1. \tag{2.15}$$

$$\| f \|_{(0,p)} = \| f \|_p\ .$$

It is known [1] that $H_{(\alpha,p)}$ is a Banach space for $p\ge 1$ and a complete p-normed space for $0<p<1$.

Also

$$H_{(\alpha,p)} \subseteq H_{(\beta,p)} \subseteq L_p, 0<\beta\le\alpha\le 1. \tag{2.16}$$

Note that $H_{(\alpha,\infty)}$ is the space H_α defined above.

For study of degree of approximation problems the natural way to proceed to consider with some restrictions on some modulus of smoothness as prescribed in H_α and $H_{(\alpha,p)}$ spaces. As we have seen above only a constant function satisfies Lipschitz condition for $\alpha>1$. However for generalized Lipschitz class there is no such restriction on α. We required a finer scale of smoothness than is provided by Lipschitz class. For each $\alpha>0$ Besov developed a remarkable technique for restricting modulus of smoothness by introducing a third parameter q (in addition to p on α) and applying $\alpha\cdot q$ norms (rather than α,∞ norms) to the modulus of smoothness $w_k(f,\cdot)_p$ of f.

Besov space:

Let $\alpha > 0$ be given and let $k = [\alpha]+1$. For $0 < p,q \le \infty$, the Besov space ([2], p-54) $B_q^\alpha(L_p)$ is defined as follows:

$$B_q^\alpha(L_p) = \{ f \in L_p : | f |_{B_q^\alpha(L_p)} = \| w_k(f,\cdot) \|_{(\alpha,q)} \ is \ finite\}$$

where

$$\| w_k(f,\cdot)\|_{(\alpha,q)} = \begin{cases} \int_0^\infty (t^{-\alpha} w_k(f,t)_p)^q \frac{dt}{t})^{\frac{1}{q}}, & 0 < q < \infty \\ \sup_{t>0} t^{-\alpha} w_k(f,t)_p, & q = \infty. \end{cases}$$

It is known ([2], p-55) that $\| w_k(f,\cdot)\|_{(\alpha,q)}$ is a seminorm if $1 \le p,q \le \infty$ and a quasi-seminorm in other cases.

The Besov norm for $B_q^\alpha(L_p)$ is

$$\| f \|_{B_q^\alpha(L_p)} = \| f \|_p + \| w_k(f,\cdot)\|_{(\alpha,q)} \qquad (2.17)$$

It is known ([4], p-237) that for 2π-periodic function f, the integral $\int_0^\infty (t^{-\alpha} w_k(f,t)_p)^q \frac{dt}{t})^{\frac{1}{q}}$ is replaced by $\int_0^\pi (t^{-\alpha} w_k(f,t)_p)^q \frac{dt}{t})^{\frac{1}{q}}$.

We know ([2], p-56, [4], p-236) the following inclusion relations.

For fixed α and p

$$B_q^\alpha(L_p) \subset B_{q_1}^\alpha(L_p), q < q_1.$$

For fixed p and q

$$B_q^\alpha(L_p) \subset B_q^\beta(L_p), \beta < \alpha.$$

For fixed α and q

$$B_q^\alpha(L_p) \subset B_q^\alpha(L_{p_1}), p_1 < p.$$

Special cases of Besov space:

For $q = \infty, B_\infty^\alpha(L_p), \alpha > 0, p \ge 1$ is same as $Lip^*(\alpha, L_p)$ the generalized Lipschitz space and the corresponding norm $\| \cdot \|_{B_\infty^\alpha(L_p)}$ is given by

$$\| f \|_{B_\infty^\alpha(L_p)} = \| f \|_p + \sup_{t>0} t^{-\alpha} w_k(f,t)_p \qquad (2.18)$$

for every $\alpha > 0$ with $k = [\alpha]+1$.

For the special case when $0 < \alpha < 1, B_\infty^\alpha(L_p)$ space reduces to $H_{(\alpha,p)}$ space due to Das et al. [1] and the corresponding norm is given by

$$\| f \|_{B_\infty^\alpha(L_p)} = \| f \|_{(\alpha,p)} = \| f \|_p + \sup_{t>0} t^{-\alpha} w(f,t)_p, 0 < \alpha < 1. \qquad (2.19)$$

For $\alpha = 1$, the norm is given by

$$\| f \|_{B_\infty^1(L_p)} = \| f \|_p + \sup_{t>0} t^{-\alpha} w_2(f,t)_p. \qquad (2.20)$$

Note that $\| f \|_{B_\infty^1(L_p)}$ is not same as $\| f \|_{(1,p)}$ and the space $B_\infty^1(L_p)$ includes the space $H(1,p),\ p \geq 1$. If we further specialize by taking $p = \infty, B_\infty^\alpha, 0 < \alpha < 1$, coincides with H_α space due to Prossodorf [3] and the norm is given by

$$\| f \|_{B_\infty^\alpha(L_\infty)} = \| f \|_\alpha = \| f \|_c + \sup_{t>0} t^{\alpha} w(f,t),\ 0 < \alpha < 1. \qquad (2.21)$$

For $\alpha = 1,\ p = \infty$, the norm is given by

$$\| f \|_{B_\infty^1(L_\infty)} = \| f \|_c + \sup_{t>0} t^{-1} w_2(f,t),\ \alpha = 1 \qquad (2.22)$$

which is different from $\| f \|_1$ and $B_\infty^1(L_\infty)$ includes the H_1 space.

III. Main Result:

We prove the following theorem.

Theorem : Let $0 < \alpha < 2$ and $0 \leq \beta < \alpha$. If $f \in B_q^\alpha(L_p), p \geq 1$ and $1 < q \leq \infty$ and let $\tilde{t}_n(x)$ be the Riesz transformation of the conjugate series, then

Case 1:(For $1 < q < \infty$)

$$\| \tilde{T}_n(\cdot) \|_{B_q^\beta(L_p)} = O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}\right) + O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\left(\frac{B_{k+1}}{k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}}$$

Case 2:(For $q = \infty$)

$$\| \tilde{T}_n(\cdot) \|_{B_\infty^\beta(L_p)} = O\left(\frac{1}{n^{\alpha-\beta}}\right) + O\left(\frac{1}{B_n}\right)\sum_{k=1}^{n}\frac{B_{k+1}}{k^{\alpha-\beta}}$$

IV. Additional Notations and Lemmas:

We need the following additional notations for the proof of the theorem.

$$\psi(x,t,u)=\begin{cases}\psi_{x+t}(u)-\psi_x(u) & \\ \psi_{x+u}(t)-\psi_x(t), & 0<\alpha<1\\ \psi_{x+t}(u)+\psi_{x-t}(u)-2\psi_x(u) & \\ \psi_{x+u}(t)+\psi_{x-u}(t)-2\psi_x(t), & 1\le\alpha<2\end{cases}\tag{4.1}$$

For $k=[\alpha]+1$, we have for $p\ge 1$

$$w_k(f,t)_p=\begin{cases}w_1(f,t)_p, & 0<\alpha<1\\ w_2(f,t)_p, & 1\le\alpha<2\end{cases}\tag{4.2}$$

Let

$$\tilde{T}_n(x,t)=\begin{cases}\tilde{T}_n(x+t)-\tilde{T}_n(x) & 0<\alpha<1\\ \tilde{T}_n(x+t)+\tilde{T}_n(x-t)-2\tilde{T}_n(x) & 1\le\alpha<2\end{cases}\tag{4.3}$$

Using (4.3) and definition of $w_k(f,t)_p$, we have

$$w_k(\tilde{T}_n,t)_p=\|\tilde{T}_n(\cdot,t)\|_p\tag{4.4}$$

Using (1.12) and (4.1) respectively for the expressions $\tilde{T}_n(x)$ and $\psi(x,t,u)$, we have

$$\tilde{T}_n(x,t)=\frac{-1}{\pi}\int_0^{\frac{\pi}{n}}\psi(x,t,u)\tilde{K}_n(u)du+\frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi}\psi(x,t,u)H_n(u)du\tag{4.5}$$

We need the following Lemmas to prove the theorem.

Lemma 1 *Let* $1\le p\le\infty$ *and* $0<\alpha<2$. *If* $f\in L_p[0,2\pi]$, *then for* $0<t,u\le\pi$

(i) $\|\psi(\cdot,t,u)\|_p\le 2w_k(f,t)_p$

(ii) $\|\psi(\cdot,t,u)\|_p\le 2w_k(f,u)_p$

(iii) $\|\psi_{\cdot}(u)\|_p\le 2w_k(f,u)_p$,

where $k=[\alpha]+1$.

Proof: We first consider the case $0<\alpha<1$.

Clearly $k=1$ and we can express by virtue of (4.1)

$$\psi(x,t,u)=\begin{cases}\psi_{x+t}(u)-\psi_x(u)\\ \psi_{x+u}-\psi_x(t)\end{cases}$$

as follows:

$$\psi(x,t,u)=\begin{cases}\{f(x+t+u)-f(x+t-u)\}-\{f(x+u)-f(x-u)\}\\ \{f(x+t+u)-f(x+u-t)\}-\{f(x+t)-f(x-t)\}\end{cases}$$

$$=\begin{cases}\{f(x+t+u)-f(x+u)\}-\{f(x-u+t)-f(x-u)\}\\ \{f(x+t+u)-f(x+t)\}-\{f(x-t+u)-f(x-t)\}\end{cases}\tag{4.6}$$

Applying Minkowski's inequality to (??), we get for $p\ge 1$

$$\| \psi(\cdot,t,u) \|_p \leq \| f(\cdot+t+u) - f(\cdot+u) \|_p + \| f(\cdot+t-u) - f(\cdot-u) \|_p$$
$$\leq 2w_1(f,t)_p,\ 0<\alpha<1$$

Similarly applying Minkowski's inequality to (??), we get

for $p \geq 1$

$$\| \psi(\cdot,t,u) \|_p \leq 2w_1(f,u)_p.$$

When $1 \leq \alpha < 2$, clearly $k=2$ and we can write

$$\psi(x,t,u) = \begin{cases} \{f(x+t+u) - f(x+t-u)\} + \{f(x-t+u) \\ + f(x-t-u)\} - 2\{f(x+u) - f(x-u)\} \\ \{f(x+t+u) - f(x-t+u)\} + \{f(x+t-u) \\ + f(x-t-u)\} - 2\{f(x+t) - f(x-t)\} \end{cases}$$

$$= \begin{cases} \{f(x+t+u) + f(x+u-t) - 2f(x+u)\} \\ -\{f(x-u+t) + f(x-t-u) - 2f(x-u)\} \\ \{f(x+t+u) + f(x+t-u) - 2f(x+t)\} \\ -\{f(x-t+u) + f(x-t-u) - 2f(x-t)\} \end{cases} \tag{4.7}$$

Applying Minkowski's inequality to (4.7), we obtain for $p \geq 1$

$$\| \psi(\cdot,t,u) \|_p \leq \| f(\cdot+t+u) + f(\cdot+u-t) - 2f(\cdot+u) \|_p$$
$$+ \| f(\cdot-u+t) + f(\cdot-t-u) - 2f(\cdot-u) \|_p$$
$$\leq 2w_2(f,t)_p$$

Similarly, applying Minkowski's inequality to (4.7), we obtain for $p \geq 1$

$$\| \psi(\cdot,t,u) \|_p \leq 2w_2(f,u)_p$$

and this completes the proof of part (i) and (ii).

The proof of (iii) follows from

$$\psi_{\cdot}(u) = \{f(\cdot+u) - f(\cdot)\} - \{f(\cdot-u) - f(\cdot)\}.$$

Lemma 2 *Let* $0<\alpha<2$, $0 \leq \beta < \alpha$. *If* $f \in B_q^{\alpha}(L_p)$,

$p \geq 1, 1<q<\infty$, then

(i) $$\int_0^{\frac{\pi}{n}} |\tilde{K}_n(u)| \left(\int_0^u \frac{\| \psi(\cdot,t,u) \|_p^q}{t^{\beta q}} \frac{dt}{t} \right)^{\frac{1}{q}} du = O(1) \left\{ \int_0^{\frac{\pi}{n}} \left(u^{\alpha-\beta} |\tilde{K}_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

(ii) $$\int_0^{\frac{\pi}{n}} |\tilde{K}_n(u)| \left(\int_u^{\pi} \frac{\| \psi(\cdot,t,u) \|_p^q}{t^{\beta q}} \frac{dt}{t} \right)^{\frac{1}{q}} du = O(1) \left\{ \int_0^{\frac{\pi}{n}} \left(u^{\alpha-\beta+\frac{1}{q}} |\tilde{K}_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

(iii) $$\int_0^{\frac{\pi}{n}} |H_n(u)| \left(\int_0^u \frac{\|\psi(\cdot,t,u)\|_p^q}{t^{\beta q}} \frac{dt}{t} \right)^{\frac{1}{q}} du = O(1) \left\{ \int_{\frac{\pi}{n}}^{\pi} \left(u^{\alpha-\beta+\frac{1}{q}} |H_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

(iv) $$\int_{\frac{\pi}{n}}^{\pi} |H_n(u)| \left(\int_u^{\pi} \frac{\|\psi(\cdot,t,u)\|_p^q}{t^{\beta q}} \frac{dt}{t} \right)^{\frac{1}{q}} du = O(1) \left\{ \int_{\frac{\pi}{n}}^{\pi} \left(u^{\alpha-\beta+\frac{1}{q}} |H_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

Proof: Applying Lemma 1(i), we get

$$\int_0^{\frac{\pi}{n}} |\tilde{K}_n(u)| \left(\int_0^u \frac{\|\psi(\cdot,t,u)\|_p^q}{t^{\beta q+1}} dt \right)^{\frac{1}{q}} du$$

$$= O(1) \int_0^{\frac{\pi}{n}} |\tilde{K}_n(u)| \left(\int_0^u \left(\frac{w_k(f,t)_p}{t^{\alpha}} \right)^q t^{(\alpha-\beta)q} \frac{dt}{t} \right)^{\frac{1}{q}} du$$

$$= O(1) \int_0^{\frac{\pi}{n}} |\tilde{K}_n(u)| u^{(\alpha-\beta)} \left(\int_0^u \frac{w_k(f,t)_p}{t^{\alpha}} \frac{dt}{t} \right)^{\frac{1}{q}} du$$

$$= O(1) \int_0^{\frac{\pi}{n}} |\tilde{K}_n(u)| u^{(\alpha-\beta)} du$$

the inner integral being finite as $f \in B_q^{\alpha}(L_p)$. Applying Holders inequality

$$= O(1) \left\{ \int_0^{\frac{\pi}{n}} \left(|\tilde{K}_n(u)| u^{(\alpha-\beta)} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \left(\int_0^{\pi} 1^q du \right)^{\frac{1}{q}}$$

$$= O(1) \left\{ \int_0^{\frac{\pi}{n}} \left(|\tilde{K}_n(u)| u^{(\alpha-\beta)} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

Applying Lemma 1(ii), we get

$$\int_0^{\frac{\pi}{n}} |\tilde{K}_n(u)| \left(\int_u^{\pi} \frac{\|\psi(\cdot,t,u)\|_p^q}{t^{\beta q+1}} dt \right)^{\frac{1}{q}} du$$

$$= O(1) \int_0^{\frac{\pi}{n}} |\tilde{K}_n(u)| \left\{ \int_u^{\pi} \left(\frac{w_k(f,u)_p}{t^{\beta}} \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}} du$$

$$= O(1)\int_0^{\frac{\pi}{n}} |\tilde{K}_n(u)|\, w_k(f,u)_p\, du\left(\int_u^{\pi} \frac{dt}{t^{\beta q+1}}\right)^{\frac{1}{q}}$$

$$= O(1)\int_0^{\frac{\pi}{n}} |\tilde{K}_n(u)|\, w_k(f,u)_p\, u^{-\beta} du$$

$$= O(1)\int_0^{\frac{\pi}{n}} \left(\frac{w_k(f,u)_p}{u^{\alpha+\frac{1}{q}}}\right) u^{\alpha-\beta+\frac{1}{q}} |\tilde{K}_n(u)|\, du$$

Applying Holder's inequality

$$= O(1)\left\{\int_0^{\frac{\pi}{n}} \left(\frac{w_k(f,u)_p}{u^{\alpha}}\right)^q \frac{du}{u}\right\}^{\frac{1}{q}} \left\{\int_0^{\frac{\pi}{n}} \left(u^{\alpha-\beta+\frac{1}{q}} |\tilde{K}_n(u)|\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= O(1)\left\{\int_0^{\frac{\pi}{n}} \left(u^{\alpha-\beta+\frac{1}{q}} |\tilde{K}_n(u)|\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

As the first integral on the above is finite by hypothesis. Third part and 4th part of the Lemma follows from above replacing $\tilde{K}_n(u)$ by $H_n(u)$.

Lemma 3 *Let* $0 < \alpha < 2$. *Suppose that* $0 \le \beta < \alpha$. *If* $f \in B_q^{\alpha}(L_p)$, $p \ge 1$ *and* $q = \infty$, *then*

$$\sup_{0<t,u\le\pi} t^{-\beta} \| \psi(\cdot,t,u) \|_p = O(u^{\alpha-\beta})$$

Proof: For $0 < t \le u \le \pi$, applying Lemma 1(i), we have

$$\sup_{\substack{t,\\ 0<t\le u\le\pi}} t^{-\beta} \| \phi(\cdot,t,u) \|_p = \sup_{\substack{t,\\ 0<t\le u\le\pi}} t^{\alpha-\beta}(t^{-\alpha} \| \phi(\cdot,t,u) \|_p)$$

$$\le 4u^{\alpha-\beta} \sup_t (t^{-\alpha} w_k(f,t)_p)$$

$$= O(u^{\alpha-\beta}), \quad \textit{by the hypothesis}.$$

Next for $0 < u \le t \le \pi$, applying Lemma 1(ii), we get

$$\sup_{\substack{t,\\ 0<u\le t\le\pi}} t^{-\beta} \| \phi(\cdot,t,u) \|_p \le 4w_k(f,u)_p \sup_{\substack{t,\\ 0<u\le t\le\pi}} t^{-\beta}$$

$$\le 4u^{\alpha-\beta} \sup_u (u^{-\alpha} w_k(f,u)_p)$$

$$= O(u^{\alpha-\beta}), \quad \textit{by the hypothesis}$$

and this completes the proof.

Lemma 4 *Let* $b_n \ge 0$ *and non-increasing and kernel* $\tilde{K}_n(u)$ *and* $H_n(u)$ *of the conjugate Fourier series be defined as in (1.7) and (1.8). Then for* $0 < u \le \pi$ *and* $m = \left[\frac{\pi}{u}\right]$

$$\tilde{K}_n(u) = O\left(\frac{1}{u}\right).$$

$$H_n(u) = O\left(\frac{B_m}{u^2 B_n}\right)$$

Proof. From (1.7), we have

$$\tilde{K}_n(u) = \frac{1}{B_n}\sum_{k=1}^{n} b_k \tilde{D}_k(u)$$

$$= O\left(\frac{1}{u}\right)\frac{1}{B_n}\sum_{k=1}^{n} b_k \left(since\ \tilde{D}_k(u) = O\left(\frac{1}{u}\right)\right)$$

$$= O\left(\frac{1}{u}\right)\left(since\ \frac{1}{B_n}\sum_{k=1}^{n} b_k = 1\right)$$

Now, from (1.8), we have Again

$$H_n(u) = \frac{1}{B_n}\sum_{k=1}^{n} b_k \frac{\cos(k+\frac{1}{2})u}{\sin\frac{u}{2}}$$

$$= \frac{1}{B_n \sin\frac{u}{2}}\left(\sum_{k=1}^{m} + \sum_{k=m+1}^{n}\right) b_k \cos(k+\frac{1}{2})u$$

$$\Rightarrow |H_n(u)| \le \left|\frac{1}{B_n \sin\frac{u}{2}}\right| \left[\left|\sum_{k=1}^{m}\right| + \left|\sum_{k=m+1}^{n}\right|\right] b_k \cos(k+\frac{1}{2})u$$

$$= O\left(\frac{1}{uB_n}\right)(A+B) \tag{4.8}$$

where

$$A = \left|\sum_{k=1}^{m} b_k \cos(k+\frac{1}{2})u\right| \le \sum_{k=1}^{m} b_k \left|\cos(k+\frac{1}{2})u\right|$$

$$\le \sum_{k=1}^{m} b_k = B_m.$$

$$B = \left|\sum_{k=m+1}^{n} b_k \cos(k+\frac{1}{2})u\right| \le b_m \max_{m,m' \le n} \sum_{k=m'}^{m} \cos(k+\frac{1}{2})u$$

(as b_k is *monotonically decreasing*)

$$= O\left(b_m \cdot \frac{1}{u}\right)$$

Now

$$|H_n(u)| \le O\left(\frac{1}{uB_n}\right)(A+B)$$
$$= O\left(\frac{1}{uB_n}\right)\left[B_m + \frac{b_m}{u}\right]$$
$$= O\left(\frac{B_m}{uB_n}\right) + O\left(\frac{b_m}{u^2 B_n}\right)$$
$$= O\left(\frac{B_m}{u^2 B_n}\right)(\because nb_n \le B_n, m \le n)$$

V. Proof of Theorem

Case 1: For $(1 < q < \infty)$

We first consider the case $1 < q < \infty$.

We have for $p \ge 1$ and $0 \le \beta < \alpha < 2$, by use of Besov norm defined in (2.17) for $B_q^\alpha(L_p)$ is

$$\| \tilde{f} \|_{B_q^\alpha(L_p)} = \| \tilde{f} \|_p + \| w_k(\tilde{f}, \cdot) \|_{\alpha,q} \tag{5.1}$$

$$\| \tilde{T}_n(\cdot) \|_{B_q^\beta(L_p)} = \| \tilde{T}_n(\cdot) \|_p + \| w_k(\tilde{T}_n, \cdot) \|_{\beta,q} \tag{5.2}$$

Applying Lemma 1(iii) in equation (5.2), we have

$$\| \tilde{T}_n(\cdot) \|_p \le \frac{1}{\pi}\int_0^{\frac{\pi}{n}} \| \psi_\cdot(u) \|_p | \tilde{K}_n(u) | du + \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi} \| \psi_\cdot(u) \|_p | H_n(u) | du$$
$$\le \frac{2}{\pi}\left[\int_0^{\frac{\pi}{n}} | \tilde{K}_n(u) | w_k(f,u)_p du + \int_{\frac{\pi}{n}}^{\pi} | H_n(u) | w_k(f,u)_p du\right]$$

Applying H$o$lder's inequality, we have

$$\| \tilde{T}_n(\cdot) \|_p \le \frac{2}{\pi}\left[\left\{\int_0^{\frac{\pi}{n}}\left(| \tilde{K}_n(u) | u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}\left\{\int_0^{\frac{\pi}{n}}\left(\frac{w_k(f,u)_p}{u^{\alpha+\frac{1}{q}}}\right)^q du\right\}^{\frac{1}{q}}\right.$$
$$\left. + \left\{\int_{\frac{\pi}{n}}^{\pi}\left(| H_n(u) | u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}\left\{\int_{\frac{\pi}{n}}^{\pi}\left(\frac{w_k(f,u)_p}{u^{\alpha+\frac{1}{q}}}\right)^q du\right\}^{\frac{1}{q}}\right]$$

$$= O(1)\left[\left\{\int_0^{\frac{\pi}{n}}\left(| \tilde{K}_n(u) | u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}\right. \tag{5.3}$$

$$+\left\{\int_{\frac{\pi}{n}}^{\pi}\left(|H_n(u)|u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\Bigg]$$

$$=O(1)[I+J],\quad (say) \tag{5.4}$$

By using Lemma 4 in I of (5.3), we have

$$I=\left\{\int_0^{\frac{\pi}{n}}(|\tilde{K}_n(u)|u^{\alpha+\frac{1}{q}})^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}$$

$$=O(1)\left\{\int_0^{\frac{\pi}{n}}\left(u^{\alpha+\frac{1}{q}-1}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}$$

$$=O(1)\left\{\int_0^{\frac{\pi}{n}}\left(u^{\frac{q\alpha}{q-1}-1}\right)du\right\}^{1-\frac{1}{q}}$$

$$=O\left(\frac{1}{n^{\alpha}}\right) \tag{5.5}$$

Applying Lemma 4 in J of (5.3), we have

$$J=\left\{\int_{\frac{\pi}{n}}^{\pi}\left(|H_n(u)|u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}$$

$$=O\left(\frac{1}{B_n}\right)\left\{\int_{\frac{\pi}{n}}^{\pi}\left(B_m u^{\alpha+\frac{1}{q}-2}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}$$

$$=O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n-1}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}B_m^{\frac{q}{q-1}}u^{(\alpha+\frac{1}{q}-2)\cdot\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}$$

$$=O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}B_m^{\frac{q}{q-1}}u^{\frac{q}{q-1}(\alpha-1)-1}du\right\}^{1-\frac{1}{q}}$$

$$=O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}B_{k+1}^{\frac{q}{q-1}}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}u^{\frac{q}{q-1}(\alpha-1)-1}du\right\}^{1-\frac{1}{q}}$$

$$=O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}B_{k+1}^{\frac{q}{q-1}}\frac{1}{k^{\frac{q}{q-1}(\alpha-1)+1}}\right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\left(\frac{B_{k+1}}{k^{\alpha-\frac{1}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \tag{5.6}$$

Using (5.5)and (5.6) and we have from (5.3),

$$\| \tilde{T}_n(\cdot) \|_p = O\left(\frac{1}{n^{\alpha}}\right) + O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\left(\frac{B_{k+1}}{k^{\alpha-\frac{1}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \tag{5.7}$$

By using Besov space, we have

$$\| w_k(\tilde{T}_n,\cdot) \|_{\beta,q} = \left\{\int_0^{\pi}\left(t^{-\beta} w_k(\tilde{T}_n,t)_p\right)^q \frac{dt}{t}\right\}^{\frac{1}{q}}$$

$$\leq \left[\int_0^{\pi}\left(\frac{\| \tilde{T}_n(\cdot,t) \|_p}{t^{\beta}}\right)^q \frac{dt}{t}\right]^{\frac{1}{q}} = \left[\int_0^{\pi}\left\{\int_0^{\pi} | \tilde{T}_n(x,t) |^p \, dx\right\}^{\frac{q}{p}} \frac{dt}{t^{\beta q+1}}\right]^{\frac{1}{q}}$$

$$= \left[\int_0^{\pi}\left\{\int_0^{\pi}\left|\frac{-1}{\pi}\int_0^{\frac{\pi}{n}}\psi(x,t,u)\tilde{K}_n(u)du + \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi}\psi(x,t,u)H_n(u)du\right|^p dx\right\}^{\frac{q}{p}} \frac{dt}{t^{\beta q+1}}\right]^{\frac{1}{q}}$$

$$\leq \frac{1}{\pi}\left[\int_0^{\pi}\frac{dt}{t^{\beta q+1}}\left\{\int_0^{\pi}\left|\int_0^{\frac{\pi}{n}}\psi(x,t,u)\tilde{K}_n(u)du + \int_{\frac{\pi}{n}}^{\pi}\psi(x,t,u)H_n(u)du\right|^p dx\right\}^{\frac{q}{p}}\right]^{\frac{1}{q}}$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}\frac{dt}{t^{\beta q+1}}\left\|\int_0^{\frac{\pi}{n}}\psi(\cdot,t,u)\tilde{K}_n(u)du + \int_{\frac{\pi}{n}}^{\pi}\psi(\cdot,t,u)H_n(u)du\right\|_p^q\right]^{\frac{1}{q}}$$

$$\| w_k(\tilde{T}_n,\cdot) \|_{\beta,q} \leq \frac{1}{\pi}\left[\int_0^{\pi}\left(\frac{\| \int_0^{\frac{\pi}{n}}\psi(\cdot,t,u)\tilde{K}_n(u)du \|_p + \| \int_{\frac{\pi}{n}}^{\pi}\psi(\cdot,t,u)H_n(u)du \|_p}{t^{\beta+\frac{1}{q}}}\right)^q dt\right]^{\frac{1}{q}}$$

by Minkowski’s inequality.

Again applying Minkowski’s inequality, we get

$$\| w_k(\tilde{T}_n,\cdot)\|_{\beta,q} \le \frac{1}{\pi}\left[\int_0^{\pi}\left(\frac{\| \int_0^{\frac{\pi}{n}}\psi(\cdot,t,u)\tilde{K}_n(u)du\|_p}{t^{\beta+\frac{1}{q}}}\right)^q dt\right]^{\frac{1}{q}}$$

$$+\frac{1}{\pi}\left[\int_0^{\pi}\left(\frac{\| \int_{\frac{\pi}{n}}^{\pi}\psi(\cdot,t,u)H_n(u)du\|_p}{t^{\beta+\frac{1}{q}}}\right)^q dt\right]^{\frac{1}{q}}$$

$$= O(1)[I'+J'], \quad (say) \tag{5.8}$$

$$I' = \left[\int_0^{\pi}\left(\frac{\| \int_0^{\frac{\pi}{n}}\psi(\cdot,t,u)\tilde{K}_n(u)du\|_p}{t^{\beta+\frac{1}{q}}}\right)^q dt\right]^{\frac{1}{q}}$$

$$= \left\{\int_0^{\pi}\left(\int_0^{\pi}\left|\int_0^{\frac{\pi}{n}}\psi(x,t,u)\tilde{K}_n(u)du\right|^p dx\right)^{\frac{q}{p}}\frac{dt}{t^{\beta q+1}}\right\}^{\frac{1}{q}} \tag{5.9}$$

By generalized Minkowski's inequality, we get

$$I' = \left[\int_0^{\pi}\left\{\int_0^{\frac{\pi}{n}}\left(\int_0^{\pi}\left|\psi(x,t,u)\tilde{K}_n(u)\right|^p dx\right)^{\frac{1}{p}} du\right\}^q \frac{dt}{t^{\beta q+1}}\right]^{\frac{1}{q}}$$

$$= \left[\int_0^{\pi}\left\{\int_0^{\frac{\pi}{n}}\frac{\| \psi(x,t,u)\|_p |\tilde{K}_n(u)|du}{t^{\beta+\frac{1}{q}}}\right\}^q dt\right]^{\frac{1}{q}}$$

Again applying generalized Minkowski's inequality, we get

$$I' \le \int_0^{\frac{\pi}{n}}\left(\int_0^{\pi}\frac{\| \psi(x,t,u)\|_p^q |\tilde{K}_n(u)|^q}{t^{\beta q+1}}dt\right)^{\frac{1}{q}} du$$

$$= \int_0^{\frac{\pi}{n}}|\tilde{K}_n(u)|du\left(\int_0^{\pi}\frac{\| \psi(x,t,u)\|_p^q}{t^{\beta q+1}}dt\right)^{\frac{1}{q}}$$

$$\leq \int_0^{\frac{\pi}{n}} |\tilde{K}_n(u)| \left(\int_0^u \frac{\|\psi(x,t,u)\|_p^q}{t^{\beta q+1}} dt \right)^{\frac{1}{q}} du + \int_0^{\frac{\pi}{n}} |\tilde{K}_n(u)| \left(\int_u^\pi \frac{\|\psi(x,t,u)\|_p^q}{t^{\beta q+1}} dt \right)^{\frac{1}{q}} du$$

by the inequality $(x+y)^r \leq x^r + y^r, 0 < r < 1.$

$$I' = I_1' + I_2', \quad (say) \tag{5.10}$$

Applying Lemma 2(i)

$$I_1' = O(1)\left\{ \int_0^{\frac{\pi}{n}} \left(|\tilde{K}_n(u)| u^{\alpha-\beta} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

Applying Lemma 4, we get

$$I_1' = O(1)\left\{ \int_0^{\frac{\pi}{n}} \left(u^{\alpha-\beta-1} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

$$= O(1)\left\{ \int_0^{\frac{\pi}{n}} u^{\frac{q}{q-1}(\alpha-\beta-1)} du \right\}^{1-\frac{1}{q}}$$

$$= O(1)\left\{ \int_0^{\frac{\pi}{n}} u^{\frac{q}{q-1}(\alpha-\beta-\frac{1}{q})-1} du \right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}} \right) \tag{5.11}$$

$$I_2' = \int_0^{\frac{\pi}{n}} |\tilde{K}_n(u)| \left(\int_u^\pi \frac{\|\psi(x,t,u)\|_p^q}{t^{\beta q+1}} dt \right)^{\frac{1}{q}} du$$

Applying Lemma 2(ii)

$$I_2' = O(1)\left\{ \int_0^{\frac{\pi}{n}} \left(|\tilde{K}_n(u)| u^{\alpha-\beta+\frac{1}{q}} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

Applying Lemma 4, we get

$$I_2' = O(1)\left\{ \int_0^{\frac{\pi}{n}} \left(u^{\alpha-\beta+\frac{1}{q}-1} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

$$= O(1)\left\{ \int_0^{\frac{\pi}{n}} u^{\frac{q}{q-1}(\alpha-\beta-1+\frac{1}{q})} du \right\}^{1-\frac{1}{q}}$$

$$= O(1)\left\{\int_0^{\frac{\pi}{n}} u^{\frac{q}{q-1}(\alpha-\beta)-1} du\right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{n^{\alpha-\beta}}\right) \quad (5.12)$$

From (5.10), (5.11) and (5.12), we get

$$I' = O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}\right) + O\left(\frac{1}{n^{\alpha-\beta}}\right) = O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}\right) \quad (5.13)$$

$$J' = \left[\int_0^{\pi}\left(\frac{\| \int_{\frac{\pi}{n}}^{\pi} \psi(\cdot,t,u) H_n(u) du \|_p}{t^{\beta+\frac{1}{q}}}\right)^q dt\right]^{\frac{1}{q}}$$

$$= \left\{\int_0^{\pi}\left(\int_0^{\pi}\left|\int_{\frac{\pi}{n}}^{\pi} \psi(x,t,u) H_n(u) du\right|^p dx\right)^{\frac{q}{p}} \frac{dt}{t^{\beta q+1}}\right\}^{\frac{1}{q}}$$

Proceeding as above as in I'.

$$J' \le \int_{\frac{\pi}{n}}^{\pi} |H_n(u)| \left(\int_u^{\pi} \frac{\| \psi(x,t,u) \|_p^q}{t^{\beta q+1}} dt\right)^{\frac{1}{q}} du \le \int_{\frac{\pi}{n}}^{\pi} |H_n(u)| \left(\int_0^{u} \frac{\| \psi(x,t,u) \|_p^q}{t^{\beta q+1}} dt\right)^{\frac{1}{q}} du$$

$$+ \int_{\frac{\pi}{n}}^{\pi} |H_n(u)| \left(\int_u^{\pi} \frac{\| \psi(x,t,u) \|_p^q}{t^{\beta q+1}} dt\right)^{\frac{1}{q}} du$$

$$= J_1' + J_2', \quad (say) \quad (5.14)$$

Now

$$J_1' = \int_{\frac{\pi}{n}}^{\pi} |H_n(u)| \left(\int_0^{u} \frac{\| \psi(x,t,u) \|_p^q}{t^{\beta q+1}} dt\right)^{\frac{1}{q}} du$$

Applying Lemma 2(iii)

$$J_1' = O(1)\left\{\int_{\frac{\pi}{n}}^{\pi} \left(| H_n(u) | u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

Applying Lemma 4, we get

$$J_1' = O\left(\frac{1}{B_n}\right)\left\{\int_{\frac{\pi}{n}}^{\pi} \left(B_m u^{\alpha-\beta-2}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n-1}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\left(B_m u^{\alpha-\beta-2}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\left(B_m u^{\alpha-\beta-2}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}} \qquad (5.15)$$

Let $g(u)=\left(B_m u^{\alpha-\beta-2}\right)^{\frac{q}{q-1}}$ and $G(u)$ is a primitive of $g(u)$, then

$$\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\left(B_m u^{\alpha-\beta-2}\right)^{\frac{q}{q-1}}du = \int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} g(u)du$$

$$= G\left(\frac{\pi}{k}\right) - G\left(\frac{\pi}{k+1}\right)$$

$$= \left(\frac{\pi}{k}-\frac{\pi}{k+1}\right)g(c) \ for some \frac{\pi}{k+1} < c < \frac{\pi}{k}$$

$$= O(1)\frac{1}{k^2}\left(\frac{B_{k+1}}{k^{\alpha-\beta-2}}\right)^{\frac{q}{q-1}} = O(1)\left(\frac{B_{k+1}}{k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}$$

$$J_1' = O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\left(\frac{B_{k+1}}{k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \qquad (5.16)$$

Now

$$J_2' = \int_{\frac{\pi}{n}}^{\pi} |H_n(u)| \left(\int_u^{\pi}\frac{\|\psi(x,t,u)\|_p^q}{t^{\beta q+1}}dt\right)^{\frac{1}{q}}du$$

Applying Lemma 2(iv), we get

$$J_2' = O(1)\left\{\int_{\frac{\pi}{n}}^{\pi}\left(|H_n(u)|\, u^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}$$

Applying Lemma 4, we get

$$J_2' = O\left(\frac{1}{B_n}\right)\left\{\int_{\frac{\pi}{n}}^{\pi}\left(B_m u^{\alpha-\beta-2+\frac{1}{q}}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n-1}\int_{\frac{\pi}{n}}^{\pi}\left(B_m u^{\alpha-\beta-2+\frac{1}{q}}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\left(B_m u^{\alpha-\beta-2+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

Proceeding as in J_1', we have

$$J_2' = O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\left(\frac{B_{k+1}}{k^{\alpha-\beta-\frac{1}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \tag{5.17}$$

$$J' = O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\left(\frac{B_{k+1}}{k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \tag{5.18}$$

From (5.10), (5.13) and (5.18), we get

$$\| w_k(\tilde{T}_n,\cdot)\|_{\beta,q} = O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}\right) + O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\left(\frac{B_{k+1}}{k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \tag{5.19}$$

From (5.2),(5.15) and (5.19), for $1 < q < \infty$, $o \le \beta < \alpha < 2$, $f \in B_q^{\alpha}(L_p)$, $p \ge 1$, we have

$$\|\tilde{T}_n(\cdot)\|_{B_q^{\beta}(L_p)} = O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}\right) + O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\left(\frac{B_{k+1}}{k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \tag{5.20}$$

This completes the proof of Case 1.

Case 2 ($q = \infty$)

Now, we consider the case $q = \infty$.

$$\|\tilde{T}_n(\cdot)\|_{B_\infty^{\beta}(L_p)} = \|\tilde{T}_n(\cdot)\|_p + \| w_k(\tilde{T}_n,\cdot)\|_{\beta,\infty} \tag{5.21}$$

$$\| w_k(\tilde{T}_n,\cdot)\|_{\beta,\infty} = \sup_{t>0}\frac{\|\tilde{T}_n(\cdot,t)\|_p}{t^{\beta}}$$

$$= \sup_{t>0}\frac{t^{-\beta}}{\pi}\left\{\int_0^{\pi}\left|-\int_0^{\frac{\pi}{n}}\psi(x,t,u)\tilde{K}_n(u)du + \int_{\frac{\pi}{n}}^{\pi}\psi(x,t,u)H_n(u)du\right|^p dx\right\}^{\frac{1}{p}}$$

Applying Minkowski's inequality, we have

$$\| w_k(\tilde{T}_n,\cdot)\|_{\beta,\infty} \le \sup_{t>0} \frac{t^{-\beta}}{\pi}\left\{\left(\int_0^{\pi}\left|\int_0^{\frac{\pi}{n}}\psi(x,t,u)\tilde{K}_n(u)du\right|^p dx\right)^{\frac{1}{p}} + \left(\int_0^{\pi}\left|\int_{\frac{\pi}{n}}^{\pi}\psi(x,t,u)H_n(u)du\right|^p dx\right)^{\frac{1}{p}}\right\}$$

Applying Generalized Minkowski's inequality, we have

$$\| w_k(\tilde{T}_n,\cdot)\|_{\beta,\infty} = \sup_{t>0} \frac{t^{-\beta}}{\pi}\left\{\int_0^{\frac{\pi}{n}}\left(\int_0^{\pi}|\psi(x,t,u)|^p\ |\tilde{K}_n(u)|\,dx\right)^{\frac{1}{p}} du + \int_{\frac{\pi}{n}}^{\pi}\left(\int_0^{\pi}|\psi(x,t,u)|^p\ |H(u)|^p\ dx\right)^{\frac{1}{p}} du\right\}$$

$$= \sup_{t>0} \frac{t^{-\beta}}{\pi}\left\{\int_0^{\frac{\pi}{n}}\|\psi(x,t,u)\|_p |\tilde{K}_n(u)|\,du + \int_{\frac{\pi}{n}}^{\pi}\|\psi(x,t,u)\|_p |H(u)|\,du\right\}$$

$$\le \frac{1}{\pi}\left\{\int_0^{\frac{\pi}{n}}|\tilde{K}_n(u)|\left(\sup_{t>0}\frac{\|\psi(x,t,u)\|_p}{t^{\beta}}\right)du + \int_{\frac{\pi}{n}}^{\pi}|H_n(u)|\left(\sup_{t>0}\frac{\|\psi(x,t,u)\|_p}{t^{\beta}}\right)du\right\}$$

Using Lemma 3 and Lemma 4, we have

$$\| w_k(\tilde{T}_n,\cdot)\|_{\beta,\infty} \le O(1)\int_0^{\frac{\pi}{n}}|\tilde{K}_n(u)|\,u^{\alpha-\beta}du + O(1)\int_{\frac{\pi}{n}}^{\pi}|H_n(u)|\,u^{\alpha-\beta}du$$

$$= O(1)[I'' + J''], \quad (say) \tag{5.22}$$

$$I'' = O(1)\int_0^{\frac{\pi}{n}}|\tilde{K}_n(u)|\,u^{\alpha-\beta}du$$

$$= O(1)\int_0^{\frac{\pi}{n}}u^{\alpha-\beta-1}du$$

$$= O\left(\frac{1}{n^{\alpha-\beta}}\right) \tag{5.23}$$

$$J'' = O(1)\int_{\frac{\pi}{n}}^{\pi}|H_n(u)|\,u^{\alpha-\beta}du$$

$$= O\left(\frac{1}{B_n}\right)\int_{\frac{\pi}{n}}^{\pi}B_m u^{\alpha-\beta-2}du$$

$$= O\left(\frac{1}{B_n}\right)\sum_{k=1}^{n-1}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}B_m u^{\alpha-\beta-2}du$$

$$= O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\frac{B_{k+1}}{k^{\alpha-\beta}}\right\} \tag{5.24}$$

From (5.22), (5.23) and (5.24), we have

$$\| w_k(\tilde{T}_n,\cdot)\|_{\beta,\infty} = O\left(\frac{1}{n^{\alpha-\beta}}\right)+O\left(\frac{1}{B_n}\right)\sum_{k=1}^{n}\left(\frac{B_{k+1}}{k^{\alpha-\beta}}\right) \tag{5.25}$$

Now,

$$\|\tilde{T}_n(\cdot)\|_p \le \frac{1}{\pi}\int_0^{\frac{\pi}{n}}\|\psi_\cdot(u)\|_p|\tilde{K}_n(u)|\,du+\frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi}\|\psi_\cdot(u)\|_p|H_n(u)|\,du \tag{5.26}$$

Applying Lemma 1(iii), we have

$$\|\tilde{T}_n(\cdot)\|_p \le \frac{2}{\pi}\int_0^{\frac{\pi}{n}}w_k(f,u)_p\,|\tilde{K}_n(u)|\,du+\frac{2}{\pi}\int_{\frac{\pi}{n}}^{\pi}w_k(f,u)_p\,|H_n(u)|\,du$$

$$= O(1)\int_0^{\frac{\pi}{n}}u^{\alpha}\,|\tilde{K}_n(u)|\,du+O(1)\int_{\frac{\pi}{n}}^{\pi}u^{\alpha}\,|H_n(u)|\,du$$

$$= I''' + J''', \quad (say) \tag{5.27}$$

$$I''' = O(1)\int_0^{\frac{\pi}{n}}u^{\alpha}\,|\tilde{K}_n(u)|\,du$$

$$= O(1)\int_0^{\frac{\pi}{n}}u^{\alpha-1}du = O\left(\frac{1}{n^{\alpha}}\right) \tag{5.28}$$

$$J''' = O(1)\int_{\frac{\pi}{n}}^{\pi}u^{\alpha}\,|H_n(u)|\,du$$

$$= O\left(\frac{1}{B_n}\right)\int_{\frac{\pi}{n}}^{\pi}B_m u^{\alpha-2}du$$

$$= O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n-1}\left(\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}B_m u^{\alpha-2}du\right)\right\}$$

$$= O\left(\frac{1}{B_n}\right)\left\{\sum_{k=1}^{n}\left(\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}B_m u^{\alpha-2}du\right)\right\}$$

$$= O\left(\frac{1}{B_n}\right)\sum_{k=1}^{n}\frac{B_{k+1}}{k^{\alpha}} \tag{5.29}$$

From (5.27), (5.28) and (5.29), we have

$$\|\tilde{T}_n(\cdot)\|_p = O\left(\frac{1}{n^{\alpha}}\right)+O\left(\frac{1}{B_n}\right)\sum_{k=1}^{n}\left(\frac{B_{k+1}}{k^{\alpha}}\right) \tag{5.30}$$

From (5.25) and (5.30), for $q=\infty$, $0\le\beta<\alpha<2$, $f\in B_q^{\alpha}(L_p)$, $p\ge 1$, we have

$$\|\tilde{T}_n(\cdot)\|_{B_\infty^{\beta}(L_p)} = O\left(\frac{1}{n^{\alpha-\beta}}\right)+O\left(\frac{1}{B_n}\right)\sum_{k=1}^{n}\frac{B_{k+1}}{k^{\alpha-\beta}}$$

This completes the proof of Case 2.
Combining the Case 1 and Case 2, we obtain the proof of the theorem.

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