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Hesitant fuzzy interior ideals in $\Gamma-$semigroup

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Abstract

In this paper we introduce the concepts of hesitant fuzzy Γ-subsemigroup, hesitant fuzzy interior ideals, hesitant fuzzy quasi interior-ideals and hesitant fuzzy bi-interior ideals of Γ-semigroup and investigate some of the properties.

Key words: fuzzy set, hesitant fuzzy set, hesitant fuzzy Γ-subsemigroup, hesitant fuzzy ideal, hesitant fuzzy interior ideals, hesitant fuzzy quasi interior-ideal, hesitant fuzzy bi-interior ideal.

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1 Introduction

Handling uncertainties in all fields of life is a very important task. As a number of uncertainties prevail in many real world problems such as Decision making, Computer science, Control Engineering, Management science, Artificial Intelligence, Operations Research, Robotics and many. To handle such uncertainties, some of the mathematical tools such as randomness, rough set and fuzzy set were introduced. Fuzzy sets were proposed by Zadeh[22] as a generalization of basic topics of general topology. Rosenfeld[16] introduced the notion of fuzzy subgroups. Kuroki[10] introduced fuzzy ideals in semigroup. Semigroup play an important role in mathematical analysis, formal language and automata theory, combinatorics. Sen and Saha [17] introduced the notion of Gamma semigroups as a generalization of semigroups. Steinfeld introduced Quasi ideals of a semigroup. Quasi ideals are generalization of left(right) ideals, bi-ideals are generalization of quasi ideals, bi-interior ideals are a generalization of quasi ideal b-ideal and interior ideal. Murali Krishna Rao [11,12,13] introduced fuzzy bi-quasi ideals in $\Gamma-$ semigroups, further studied fuzzy bi-interior ideals of semigroups..

In 2009, Torra and Narukawa[20] and Torra[19] introduced the notion hesitant fuzzy sets as a generalization of fuzzy sets proposed by Zadeh. Hesitant fuzzy set allows the membership of an element of a set to be represented by several possible values which can be a perfect tool for modeling peoples hesitancy. Recently much attention has been gained by many authors and widely used the concepts in clustering analysis and decision making. Jun et.al.[7,8] studied hesitant fuzzy bi-ideals in semigroups. Xia and Xu[21] applied hesitant fuzzy set to decision making. Furthermore, Deepak and John[3] investigated hesitant fuzzy rough sets through hesitant fuzzy relations. Also They studied homomorphisms of hesitant fuzzy subgroups and hesitant fuzzy subrings and ideals. Abbasi et.al.[14]applied the notion of hesitant fuzzy set to po-semigroups and also studied hesitant fuzzy set approach to ideal theory in Γ-semigroups.

2 PRELIMINARIES

In this section, we list some basic definitions and some properties needed in the next sections.

Definition 2.1 *Let M and Γ be nonempty sets, then M is called a Γ-semigroup,if there exists a mapping $MX\Gamma XM \to M$ (images of x, α, y is denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) such that it satisfies*
i) $x\alpha y \in M$
ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$.

Definition 2.2 *Let M be a Γ-semigroup. A nonempty subset A of M is called a sub semigroup if $A\Gamma A \subseteq A$.*

Definition 2.3 *Let M be a Γ-semigroup. A nonempty subset A of M is called left(right) ideal of M if $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$. If it is both, a left and right ideal then it is called an ideal.*

Definition 2.4 *Let M be a Γ-semigroup. A sub semigroup A of M is called a bi-ideal if $A\Gamma M\Gamma A \subseteq A$.*

Definition 2.5 *Let M be a Γ-semigroup. A sub semigroup A of M is called an interior ideal if $M\Gamma A\Gamma M \subseteq A$.*

Definition 2.6 *Let M be a Γ-semigroup. A nonempty subset Q of M is called a quasi-ideal if $Q\Gamma M \cap M\Gamma Q \subseteq Q$.*

Definition 2.7 *Let M be a nonempty set. A mapping $f : M \to [0,1]$ is called a fuzzy subset of a Γ-semigroup M. If f is not a constant function, then f is called a non-empty fuzzy subset.*

Definition 2.8 *Let f be a nonempty fuzzy subset of a $\Gamma-$semigroup M and $t \in [0,1]$. Then the set $f_t = \{x \in M | f(x) \geq t$ is called the level subset of M with respect to f.*

Definition 2.9 *A Γ-semigroup M is said to be regular if, for each $a \in M$ there exist $x \in M$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$*

3 Hesitant fuzzy ideals

Definition 3.1 *([19]). Let X be a reference set and let P[0,1] denote the power set of [0,1], then a mapping $h : X \to P[0,1]$ is called a hesitant fuzzy set in X.*
The hesitant fuzzy empty[resp. whole]set, denoted by $h^0[resp.h^1]$, is a hesitant fuzzy set in X defined as, for each $x \in X$, $h^0 = \phi[resp.h^1(x) = [0,1]]$.
In this case, we will denote the set of all hesitant fuzzy sets in X as $HS(X)$.

Definition 3.2 *Let $(X, \cdot)$ be a groupoid and let $h_1, h_2 \in HS(X)$. Then the hesitant fuzzy product of h_1 and h_2, is denoted by $h_1 \circ h_2$, is a hesitant fuzzy set in X defined by : for each $x \in X$,*

$$(h_1 \circ h_2)(x) = \begin{cases} \bigcup_{x=y\alpha z}[h_1(y) \cap h_2(z)] & if \quad y\alpha z = x \\ \phi & otherwise. \end{cases}$$

Definition 3.3 *The hesitant characteristic function of a nonempty subset A of M is defined as $h_A : M \to P([0,1])$, where*

$$h_A(x) = \begin{cases} [0,1], & if x \in A, \\ \phi, & otherwise. \end{cases}$$

Definition 3.4 *The hesitant fuzzy subset of a nonempty subset I of M is defined as $I : M \to P([0,1])/I(x) = [0,1]$.*

Definition 3.5 *Let M be a Γ-semigroup and h be a hesitant fuzzy subset of M. Then h is called a hesitant fuzzy Γ-subsemigroup of M if*

$$h(x\alpha y) \supseteq h(x) \cap h(y) for\ all\ x, y \in M, \alpha \in \Gamma.$$

Definition 3.6 *Let M be a Γ-semigroup. A hesitant fuzzy set h of M is said to be a hesitant fuzzy right(left) ideal of M if h is a hesitant fuzzy Γ-subsemigroup of M and $h(x\alpha y) \supseteq h(x)(h(y))$ for all $x, y \in M$, $\alpha \in \Gamma$.*

Definition 3.7 *Let M be a Γ-semigroup. A hesitant fuzzy set h of M is said to be a hesitant fuzzy bi-ideal of M if*
i) h is a fuzzy Γ-subsemigroup of M.
ii) $h(x\alpha y\beta z) \supseteq h(x) \cap h(z)$ for all $x, y, z \in M$, $\alpha, \beta \in \Gamma$.

Definition 3.8 *Let M be a Γ-semigroup. A hesitant fuzzy Γ-subsemigroup h of M is said to be a hesitant fuzzy interior ideal of M if*
$h(x\alpha y\beta z) \supseteq h(y)$ for all $x, y, z \in M$, $\alpha, \beta \in \Gamma$.

Definition 3.9 *A hesitant fuzzy subset h of M is called a hesitant fuzzy bi-ideal of M if $hoIoh \subseteq h$.*

Definition 3.10 *A hesitant fuzzy subset h of M is called a hesitant fuzzy left(right) bi-quasi ideal of M if $Ioh \cap hoIoh \subseteq h(hoI \cap hoIoh \subseteq h)$.*

Definition 3.11 *Let M be a Γ-semigroup. A hesitant fuzzy subset h of M is said to be a hesitant fuzzy left(right) quasi-interior ideal of M if*
$IohoIoh \subseteq h(hoIohoI \subseteq h)$

Definition 3.12 *Let h be a hesitant fuzzy subset of a Γ-semigroup M and $t \in p([0, 1])$. Then the set $h_t = \{x \in M/h(x) \supseteq t\}$ is called the t-cut of h.*

Remark: Let h be a hesitant fuzzy subset of a Γ-semigroup M and $t_1, t_2 \in p([0, 1])$ such that $t_1 \subseteq t_2$. Then $h_{t_2} \subseteq h_{t_1}$.

Theorem 3.1 *Let M be a Γ-semigroup and h be a non empty hesitant fuzzy subset of M. Then h is a hesitant fuzzy sub semigroup of M if and only if $hoh \subseteq h$.*

Proof: *Let $hoh \subseteq h$, then for $x, y \in M, \alpha \in \Gamma$ we have*

$$h(x\alpha y) \supseteq (hoh)(x\alpha y) \supseteq \{h(x) \cap h(y)\}.$$

Therefore h is a hesitant fuzzy subsemigroup of M.
Conversely suppose h is a hesitant fuzzy subsemigroup of M and $a \in M$ then there exists $x, y \in M, \alpha \in \Gamma$ such that $a = x\alpha y$ then

$$\begin{aligned} h(x\alpha y) &\supseteq \{h(x) \cap h(y)\}. \\ \Rightarrow h(x\alpha y) &\supseteq \bigcup_{a=x\alpha y} \{h(x) \cap h(y)\}. \\ &= (hoh)(a) \end{aligned}$$

Hence $(hoh) \subseteq h$. □

Theorem 3.2 *Let h be a non empty hesitant fuzzy subset of a Γ-semigroup M. Then h is a hesitant fuzzy sub semigroup of M if and only if h_t is a sub semigroup for every $t \in P([0,1])$,where $h_t = \{x \in M |\ h(x) \supseteq t\}$.*

Proof: *Let h be a hesitant fuzzy sub semigroup of M.*
Let $t \in P([0,1])$, then for some $a \in M$ such that $h_a = t$.
So $a \in h_t \Rightarrow h_t \neq \phi$
Let $x, y \in h_t$. Then $h(x) \supseteq t, h(y) \supseteq t$. hence $\{h(x) \cap h(y)\} \supseteq t$.
Now for $\alpha \in \Gamma$

$$h(x\alpha y) \supseteq \{h(x) \cap h(y)\} \supseteq t.$$

Hence $x\alpha y \in h_t$ for all $\alpha \in \Gamma$ such that $h_t \Gamma h_t \subseteq h_t$.
Therefore h_t is a sub semigroup of M.
Conversely suppose $h_t's$ are sub semigroups of M for all $t \in P([0,1])$
Let $h(x) = t_1$ and $h(y) = t_2$ and $t_1 \geq t_2$, then $x, y \in h_{t_2}$.
Therefore $x\alpha y \in h_{t_2}$
$\Rightarrow h(x\alpha y) \supseteq t_2 = \{h(x) \cap h(y)\}$.
Therefore h is a hesitant fuzzy sub semigroup of M. □

Theorem 3.3 *Let h be a nonempty hesitant fuzzy Γ-sub semigroup of M. Then the following conditions are equivalent:*
1) h is a hesitant fuzzy left(right) ideal of M.
2) $I \circ h \subseteq h \quad (h \circ I \subseteq h)$ where I is a hesitant fuzzy set and $I(x) = [0,1]$ for all $x \in M$.

Theorem 3.4 *Let M be a* Γ*-semigroup and h be a non empty hesitant fuzzy subset of M. Then the following conditions are equivalent:*
1) h is a hesitant fuzzy bi-ideal of M.
2) $h \circ h \subseteq h$ *and* $h \circ I \circ h \subseteq h$.

Proof: *Suppose h is a hesitant fuzzy bi-ideal of M. Let* $a \in M$. *then*

$$\begin{aligned}(hoh)(a) &= \bigcup_{a=x\alpha y} \{h(x) \cap h(y)\}.\\ &\subseteq \bigcup_{a=x\alpha y} h(x\alpha y)\\ &\subseteq \bigcup_{a=x\alpha y} h(a)\\ &= h(a)\end{aligned}$$

Therefore $hoh \subseteq h$.

$$\begin{aligned}(h \circ I \circ h)(a) &= \bigcup_{a=x\alpha y} \{h \circ I(x) \cap h(y)\}\\ &= \bigcup_{a=x\alpha y} \left\{ \bigcup_{x=m\beta n} \{h(m) \cap I(n)\} \cap h(y) \right\}\\ &= \bigcup_{a=x\alpha y} \left\{ \bigcup_{x=m\beta n} \{h(m) \cap [0,1]\} \cap h(y) \right\}\\ &= \bigcup_{a=m\beta n\alpha y} \{h(m) \cap h(y)\}\\ &\subseteq \bigcup_{a=m\beta n\alpha y} h(m\beta n\alpha y)\\ &= h(a).\end{aligned}$$

Hence $(h \circ I \circ h) \subseteq h$.
Conversely suppose $h \circ h \subseteq h$ *and* $h \circ I \circ h \subseteq h$.
Let $hoh \subseteq h$. *Then for* $x, y \in M, \alpha \in \Gamma$,

$$h(x\alpha y) \supseteq (hoh)(x\alpha y) \supseteq \{h(x) \cap h(y)\}.$$

Therefore h is a hesitant fuzzy subsemigroup. Let $x, y, z \in M, \alpha, \beta \in \Gamma$ *and* $a = x\alpha y\beta z$

Since $h \circ I \circ h \subseteq h$ *we have*

$$\begin{aligned} h(x\alpha y\beta z) &= h(a) \supseteq (h \circ I \circ h)(a) \\ &= \bigcup_{a=x\alpha y\beta z} \{(hoI)(x\alpha y) \cap h(z)\} \\ &= \bigcup_{p=x\alpha y} \{h(x) \cap I(y) \cap h(z)\} \\ &\supseteq \bigcup_{a=p\beta z} \{h(p) \cap h(z)\} \\ &\supseteq \bigcup_{a=x\alpha y\beta z} \{h(x) \cap h(z)\} \\ &\supseteq h(x) \cap h(z) \\ \Rightarrow h(x\alpha y\beta z) &\supseteq h(x) \cap h(z). \end{aligned}$$

Hence, h is a hesitant fuzzy bi-ideal of M. □

Theorem 3.5 *Let h be a non empty hesitant fuzzy subset of a regular* Γ-*semigroup M. Then h is a hesitant fuzzy ideal of M if and only if h is a hesitant fuzzy interior ideal of M.*

Proof: *Let M be a regular* Γ*-semigroup and h be a hesitant fuzzy ideal of M. let* $x, y, z \in M\alpha, \beta \in \Gamma$ *then*

$$h(x\alpha y\beta z) \supseteq h(y\beta z) \supseteq h(y)$$

Therefore h is a hesitant fuzzy interior ideal of M.
Conversely suppose h is a hesitant fuzzy interior ideal of M.
Suppose $x, y \in M$ *and* $\gamma \in \Gamma$
since M is regular there exists $\alpha, \beta \in \Gamma$ *and* $y \in M$ *then* $x = x\alpha y\beta x$

$$\begin{aligned} h(x\gamma y) &= h(x\alpha y\beta x\gamma y) \\ &\supseteq h(x). \end{aligned}$$

Similarly we can prove $h(x\gamma y) \supseteq h(y)$.
Therefore h is a hesitant fuzzy ideal of M. □

Definition 3.13 *A Hesitant fuzzy subsemigroup h of a* Γ*-semigroup M is said to be Hesitant fuzzy left(right) quasi-interior ideal of M if* $(IohoIoh \subseteq h(hoIohoI \subseteq h)$.

Definition 3.14 *A Hesitant fuzzy subsemigroup h of a Γ-semigroup M is said to be Hesitant fuzzy quasi-interior ideal of M if it is both, a left quasi-interior ideal and a right quasi-interior ideal of M.*

Theorem 3.6 *Let M be a Γ-semigroup, then every hesitant fuzzy right(left) ideal of M. is a hesitant fuzzy right(left)quasi-interior ideal of M.*

Proof: *Let h be a hesitant fuzzy right ideal of a Γ-semigroup M, $x \in M$, then*

$$\begin{aligned}
(hoI)(x) &= \bigcup_{x=a\alpha b} \{h(a) \cap I(b)\} \\
&= \bigcup_{x=a\alpha b} \{h(a) \cap [0,1]\} \\
&\subseteq \bigcup_{x=a\alpha b} \{h(a)\} \\
&\subseteq \bigcup_{x=a\alpha b} \{h(a\alpha b)\} \\
&\subseteq \bigcup_{x=a\alpha b} \{h(x)\} \\
&\subseteq h(x)
\end{aligned}$$

Thus $(hoI)(x) \subseteq h(x)$. and

$$\begin{aligned}
(hoIohoI)(x) &= \bigcup_{x=a\alpha b\beta c} \{(hoI(a\alpha b) \cap hoI(c)\} \\
&\subseteq \bigcup_{x=a\alpha b\beta c} \{(h(a\alpha b) \cap h(c))\} \\
&\subseteq \bigcup_{x=a\alpha b\beta c} \{(h(a\alpha b) \cap h(c))\} \\
&\subseteq h(x).
\end{aligned}$$

Therefore h is a hesitant fuzzy right quasi-interior ideal of M. □

Theorem 3.7 *Let h be a non empty hesitant fuzzy subset of a Γ-semigroup M. Then h is a hesitant fuzzy left-quasi interior ideal of M if and only if the t-cut h_t of h is a left-quasi interior ideal of M for every $t \in P([0,1])$,where $h_t = \{x \in M \mid h(x) \supseteq t\}$.*

Proof: *Let M be a* Γ*-semigroup and h be a nonempty hesitant fuzzy subset of M, h be a hesitant fuzzy left-quasi interior ideal of M,* $h_t \neq \phi, t \in P([0,1])$ *and* $a, b \in h_t, \alpha \in \Gamma$*. then* $h_a \supseteq t, h_b \supseteq t$*. Thus*

$$h(a\alpha b) \supseteq \{h_a \cap h_b\} \supseteq t.$$

so $a\alpha b \in h_t$
Let $x \in M\Gamma h_t \Gamma M \Gamma h_t$*, then* $x = a\alpha b\beta c\gamma d$ *where* $a, c \in M, b, d \in h_t, \alpha, \beta, \gamma \in \Gamma$.
Then $(h)(x) \supseteq IohoIoh \supseteq t$,
Therefore $x \in h_t$.
Hence h_t *is a left quasi-interior ideal of M.*
Conversely suppose h_t *is a left quasi-interior ideal of M for all* $t \in P([0,1])$
Let $x, y \in M$, $h(x) = t_1$, $h(y) = t_2$ *and* $t_1 \geq t_2$*, then* $x, y \in h_{t_2}$*. and* $x\alpha y \in h_{t_2}$
$\Rightarrow h(x\alpha y) \supseteq t_2 = \{h(x) \cap h(y)\} \supseteq t_2$.
We have $M\Gamma h_k \Gamma M \Gamma h_k \subseteq h_k$*, for all* $k \in P([0,1])$
suppose $t \in P([0,1])$*. Then* $M\Gamma h_t \Gamma M \Gamma h_t \subseteq h_t$.
Therefore $(IohoIoh) \subseteq h$.
Hence h is a hesitant fuzzy left quasi-interior ideal of M.
□

Theorem 3.8 *Let M be a regular* Γ*-semigroup and h be a hesitant fuzzy left quasi-interior ideal of M, then* $IohoIoh = h$

Proof: *Let h be a hesitant fuzzy left quasi-interior ideal of M, then* $IohoIoh \subseteq h$*. And*

$$\begin{aligned}(IohoIoh)(a) &= \bigcup_{a=a\alpha b\beta a} \{(Ioh(a) \cap Ioh(b\beta a)\} \\ &\supseteq \bigcup_{a=a\alpha b\beta a} \{(h(a) \cap h(a)\} \\ &= h(a).\end{aligned}$$

Therefore $h \subseteq (IohoIoh)$.
Hence $h = IohoIoh$. □

Theorem 3.9 *Let M be a regular* Γ*-semigroup and h be a hesitant fuzzy subset of M. Then h is a hesitant fuzzy left quasi-interior ideal if h is a hesitant fuzzy left quasi-ideal of M.*

Proof: *Let h be a hesitant fuzzy left quasi-interior ideal of M, and $a \in M$. Then $IohoIoh \subseteq h$.*
Suppose $(Ioh)(a) \supset h(a)$ and $(hoI)(a) \supset h(a)$.
Since M is regular, there exist $b \in M$ such that $a = a\alpha b\beta a$. Then

$$\begin{aligned}(hoI)(a) &= \bigcup_{a=a\alpha b\beta a} \{h(a) \cap I(a)\} \\ &= \bigcup_{a=a\alpha b\beta a} \{h(a) \cap [0,1]\} \\ &= \bigcup_{a=a\alpha b\beta a} \{h(a)\} \\ &\supset h(a)\end{aligned}$$

and

$$\begin{aligned}(hoIohoI)(a) &= \bigcup_{a=a\alpha b\beta a} \{(hoI(a) \cap hoI(b\beta a)\} \\ &\supset \bigcup_{a=a\alpha b\beta a} \{(h(a) \cap h(b\beta a)\} \\ &\supset \bigcup_{a=a\alpha b\beta a} \{(h(a) \cap h(a))\} \\ &= h(a).\end{aligned}$$

which is a contradiction.
Therefore h is a a hesitant fuzzy left quasi-ideal of M. □

Definition 3.15 *A Hesitant fuzzy subset h of a Γ-semigroup M is said to be Hesitant fuzzy bi-interior ideal of M if $IohoI \cap hoIoh \subseteq h$.*

Theorem 3.10 *Let M be a Γ-semigroup, then every hesitant fuzzy left ideal of M is a hesitant fuzzy bi-interior ideal of M.*

Proof: *Let h be a hesitant fuzzy left ideal of a Γ-semigroup of M, and $a \in M$.*

Then

$$\begin{aligned}(Ioh)(a) &= \bigcup_{a=x\alpha y}\{I(x)\cap h(y)\}\\ &= \bigcup_{a=x\alpha y}\{[0,1]\cap h(y)\}\\ &= \bigcup_{a=x\alpha y}\{h(y)\}\\ &\subseteq \bigcup_{a=x\alpha y}\{h(x\alpha y)\}\\ &\subseteq \bigcup_{a=x\alpha y}\{h(a)\}\\ &\subseteq h(a)\end{aligned}$$

Hence, $(Ioh)(a) \subseteq h(a)$. *And*

$$\begin{aligned}(hoIoh)(x) &= \bigcup_{x=a\alpha b\beta c}\{(h(a)\cap Ioh(b\beta c)\}\\ &= \bigcup_{x=a\alpha b\beta c}\{(h(a)\cap\{\bigcup_{b\beta c=p\eta q}\{(I(p)\cap h(q))\}\}\\ &= \bigcup_{x=a\alpha b\beta c}\{(h(a)\cap\{\bigcup_{b\beta c=p\eta q}\{([0,1]\cap h(q))\}\}\\ &\subseteq \bigcup_{x=a\alpha b\beta c}\{(h(a)\cap h(c))\}\\ &\subseteq \{h(a\alpha b\beta c)\}\\ &\subseteq h(x).\end{aligned}$$

$$\begin{aligned}((IohoI)\cap(hoIoh))(x) &= \{(IohoI)(x)\cap(hoIoh)(x)\}\\ &\subseteq \{(IohoI)(x)\cap h(x)\}\\ &\subseteq h(x).\end{aligned}$$

Therefore $(IohoI)\cap(hoIoh)\subseteq h$.
Therefore h is a hesitant fuzzy bi-interior ideal of M. □

Theorem 3.11 *Let M be a* Γ*-semigroup, then every hesitant fuzzy right ideal of M is a hesitant fuzzy bi-interior ideal of M.*

Proof: *Let h be a hesitant fuzzy right ideal of a* Γ*-semigroup of M, and* $a \in M$.

Then

$$\begin{aligned}(hoI)(a) &= \bigcup_{a=x\alpha y} \{h(x) \cap I(y)\} \\ &= \bigcup_{a=x\alpha y} \{h(x) \cap [0,1]\} \\ &= \bigcup_{a=x\alpha y} \{h(x)\} \\ &\subseteq \bigcup_{a=x\alpha y} \{h(x\alpha y)\} \\ &\subseteq \bigcup_{a=x\alpha y} \{h(a)\} \\ &\subseteq h(a)\end{aligned}$$

Therefore $(hoI)(a) \subseteq h(a)$. *And*

$$\begin{aligned}(hoIoh)(x) &= \bigcup_{x=a\alpha b\beta c} \{(hoI(a\alpha b) \cap h(c)\} \\ &= \bigcup_{x=a\alpha b\beta c} \{h(a\alpha b) \cap h(c)\} \\ &\subseteq \bigcup_{x=a\alpha b\beta c} \{(h(a) \cap h(c))\} \\ &\subseteq \{h(a\alpha b\beta c)\} \\ &\subseteq h(x).\end{aligned}$$

$$\begin{aligned}((IohoI) \cap (hoIoh))(x) &= \{(IohoI)(x) \cap (hoIoh)(x)\} \\ &\subseteq \{(IohoI)(x) \cap h(x)\} \\ &\subseteq h(x).\end{aligned}$$

Therefore $(IohoI) \cap (hoIoh) \subseteq h$.
Therefore h is a hesitant fuzzy bi-interior ideal of M. □

Theorem 3.12 *Let h be a non empty hesitant fuzzy subset of a* Γ*-semigroup M. Then h is a hesitant fuzzy bi-interior ideal of M if and only if the t-cut* h_t *of h is a bi-interior ideal of M for every* $t \in P([0,1])$*,where* $h_t = \{x \in M \mid h(x) \supseteq t\}$.

Proof: *Let M be a* Γ*-semigroup and h be a nonempty hesitant fuzzy subset of M.*

Suppose h be a hesitant fuzzy bi-interior ideal of M, $h_t \neq \phi, t \in P([0,1])$.
and $n, p \in h_t, \alpha \in \Gamma$.
Let $x \in M\Gamma h_t \Gamma M \cap h_t \Gamma M \Gamma h_t$ *then* $x = m\alpha n\kappa u = p\beta q\gamma r$ *where* $m, q, u \in M, n, p, r \in h_t, \alpha, \beta, \gamma \in \Gamma$. *Then* $(IohoI)(x) \supseteq t$ *and* $(hoIoh)(x) \supseteq t$
$\Rightarrow h(x) \supseteq (IohoI \cap hoIoh)(x) \supseteq t$.
Therefore $x \in h_t$.
Hence h_t *is a bi-interior ideal of M.*
Conversely suppose h_t *is a bi-interior ideal of M for all* $t \in P([0,1])$
Let $x, y \in M$, $h(x) = t_1$, $h(y) = t_2$ *and* $t_1 \geq t_2$, *then* $x, y \in h_{t_2}$. *and* $x\alpha y \in h_{t_2}$.
$\Rightarrow h(x\alpha y) \supseteq t_2 = \{h(x) \cap h(y)\} \supseteq t_2$.
We have $M\Gamma h_a \Gamma M \cap h_a \Gamma M \Gamma h_a \subseteq h_a$, *suppose* $t \in P([0,1])$. *Then* $M\Gamma h_t \Gamma M \cap h_t \Gamma M \Gamma h_t \subseteq h_t$.
Therefore $(IohoI \cap hoIoh)(x) \supseteq h$.
Hence h is a hesitant fuzzy bi-interior ideal of M. □

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