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Coefficient Estimates on Laguerre Polynomials Involving λ-Pseudo Starlike Function

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Abstract : *In this investigation, we introduce a new subclass of univalent, Sakaguchi and λ - pseudo starlike functions $LP_\Sigma(\lambda,\mu,b)$ by means of the generalized Laguerre polynomials defined in the open unit disc Ω. Coefficient bounds and Fekete – Szego inequality to the said subclass are obtained.*

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I. Introduction

Let A be the family of functions f that are analytic in the open unit disk $\Omega=\{z\in C:|z|<1\}$ of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \qquad (1)$$

For $g(z)\in A$, given by

$$g(z)=z+\sum_{k=2}^{\infty}g_k z^k$$

Let S mean the subclass of A consisting of univalent functions in Ω. It is well known (refer[2]) that every function of $f\in S$ virtually possesses an inverse of f, defined by $f^{-1}[f(z)]=z,(z\in\Omega)$ and $f[f^{-1}(w)]=w,(|w|<r_0(f);r_0(f)\geq\frac{1}{4})$, where

$$f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+....... \qquad (2)$$

When the function $f\in A$ is bi univalent, both f and f^{-1} are univalent in Ω. Let Σ be the class of bi univalent functions in Ω given by (2). In fact, Srivastava et al.[11] have revived the study of analytic and bi univalent functions in recent years. Many researchers investigated and propounded various subclasses of bi univalent functions and fixed the initial coefficients $|a_2|$ and $|a_3|$ (refer [4, 5, 7, 8, 9, and 10]).

For analytic functions f and g , f is said to be subordinate to g, denoted $f(z)\prec g(z)$, if there is an analytic function w such that $w(0)=0,|w(z)|<1$ and $f(z)=g(w(z))$.

Frasin [3] investigated the inequalities of coefficient for certain classes of Sakaguchi type functions that satisfy geometrical condition as

$$\mathrm{Re}\left\{\frac{(s-t)z(f^{'}(z))}{f(sz)-f(tz)}\right\}>\alpha \quad (3)$$

for complex numbers s, t but s ≠ t and α (0 ≤ α < 1) .

Recently for some $\lambda \geq 1$ in [1] Babalola introduced and investigated the class of λ - pseudo starlike functions of order α , $(0 \leq \alpha < 1)$ denoted by $L_{\lambda}(\alpha)$ as

$$\Re\left(\frac{z(f'(z))^{\lambda}}{f(z)}\right)>\alpha,(z\in\Omega) \quad (4)$$

For real number $\delta > -1$, the polynomial solution $h(\theta)$ of the differential equation (refer [6])

$$\theta h''+(1+\delta-\theta)h'+nh=0. \quad (5)$$

where n is non-negative integers, is called generalized Laguerre polynomial or associated Laguerre polynomial and ir is denoted by $L_n^{\delta}(\theta)$. It has many applications in areas of mathematical physics and quantum mechanics, for example in the integration of Helmholtz's equation in paraboloidal coordinates, in the theory of propagation of electromagnetic oscillations along long lines and so on. These polynomial satisfy certain recurrence relations, namely,

$$L_{n+1}^{\delta}(\theta)=\frac{2n+1+\delta-\theta}{n+1}L_n^{\delta}(\theta)-\frac{n+\delta}{n+1}L_{n-1}^{\delta}(\theta),(n\geq 1) \quad (6)$$

with the initial conditions

$$L_0^{\delta}(\theta)=1, L_1^{\delta}(\theta)=1+\delta-\theta. \quad (7)$$

It can easily derived from (6) that

$$L_2^{\delta}(\theta)=\frac{\theta^2}{2}-(\delta+2)\theta+\frac{(\delta+1)(\delta+2)}{2},$$

$$L_3^{\delta}(\theta)=-\frac{1}{6}\theta^3+\frac{(\delta+3)}{2}\theta^2-\frac{(\delta+2)(\delta+3)}{2}\theta+\frac{(\delta+1)(\delta+2)(\delta+3)}{6}. \quad (8)$$

and so on.

It may be noted that he simply Laguerre polynomials are the special case $\delta = 0$ of generalized Laguerre polynomial i.e. $L_n^{0}(\theta)=L_n(\theta)$.

Result 1.1(see [6]). Let $\xi(\theta, z)$ be the generating function of the generalized Laguerre polynomial $L_n^{\delta}(\theta)$. Then

$$\xi(\theta,z)=\sum_{n=0}^{\infty}L_n^{\delta}(\theta)z^n=\frac{e^{-\frac{\theta z}{1-z}}}{(1-z)^{\delta+1}},(\theta\in\Re,z\in\Omega) \tag{9}$$

Definition 1.2. A function $f\in\Sigma$ of the form (1) is said to be in the class $LP(\lambda,\mu,b)$ if it satisfies the following subordination conditions:

$$\frac{((1-b)z)(f'(z))^{\lambda}}{\mu z(f'(z)-bf'(bz))+(1-\mu)(f(z)-f(bz))}\prec\xi(\theta,z),(z\in\Omega) \tag{10}$$

and

$$\frac{((1-b)w)(g'(w))^{\lambda}}{\mu w(g'(w)-bg'(bw))+(1-\mu)(g(w)-g(bw))}\prec\xi(\theta,w),(w\in\Omega) \tag{11}$$

where $0\le\mu<1$, $\lambda\ge1,|b|\le1$ but $b\neq1$ and

$$g(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\dots$$

Putting the parameters $\mu=b=0$ and $\lambda=1$ that was studied by T.Panigrahi et al [12].

The main objective of this paper is to obtain the coefficient bounds on the Taylor – Maclaurin coefficients on a_2 and a_3 and the Fekete – Szego inequality for the function of the subclass $LP_{\Sigma}(\lambda,\mu,b)$.

II. Coefficient Bounds

Theorem 2.1. For $0\le\mu<1,\lambda\ge1,|b|\le1$ but $b\neq1$, let the function $f\in\Sigma$ be in the class $LP(\lambda,\mu,b)$. Then

$$|a_2|\le\frac{|1+\delta-\theta|\sqrt{|1+\delta-\theta|}}{\sqrt{\left|\left\{\begin{array}{l}\left[(3\lambda-(1+2\mu)(1+b+b^2)+2\lambda(\lambda-1)+(1+\mu)(1+b)((1+\mu)(1+b)-2\lambda))\right](1+\delta-\theta)^2\\-(2\lambda-(1+\mu)(1+b))^2\left(\frac{\theta^2}{2}-(\delta+2)\theta+\frac{(\delta+1)(\delta+2)}{2}\right)\end{array}\right\}\right|}}\ , \tag{12}$$

and

$$|a_3|\le\frac{(1+\delta-\theta)^2}{(2\lambda-(1+\mu)(1+b))^2}+\frac{1+\delta-\theta}{3\lambda-(1+2\mu)(1+b+b^2)}. \tag{13}$$

Proof. Let $f\in LP_{\Sigma}(\lambda,\mu,b)$ be given by (1) and $g=f^{-1}$. Then there are two analytic functions ϕ,φ such as $\phi(0)=\varphi(0)=0$ and $|\phi(z)|<1,|\varphi(w)<1|$ for all $z,w\in\Omega$, which can be written as

$$\frac{((1-b)z)(f'(z))^{\lambda}}{\mu z(f'(z)-bf'(bz))+(1-\mu)(f(z)-f(bz))}=\xi(\theta,\phi(z)), \tag{14}$$

and

$$\frac{((1-b)w)(g'(w))^{\lambda}}{\mu w(g'(w)-bg'(bw))+(1-\mu)(g(w)-g(bw))}=\xi(\theta,\varphi(w)). \tag{15}$$

It is fairly well – known that if

$$|\phi(z)|=|u_1z+u_2z^2+u_3z^3+...|,(z\in\Omega),$$

and

$$|\varphi(w)|=|v_1w+v_2w^2+v_3w^3+...|,(w\in\Omega).$$

Then

$$|u_k|<1 \text{ and } |v_k<1|, \text{ for } k\in N \tag{16}$$

From the equalities (14) and (15), it is obtained that

$$\frac{((1-b)z)(f'(z))^{\lambda}}{\mu z(f'(z)-bf'(bz))+(1-\mu)(f(z)-f(bz))}=\sum_{n=0}^{\infty}L_n^{\delta}(\theta)|\phi(z)|^n$$

$$=1+L_n^{\delta}(\theta)u_1z+\left[L_n^{\delta}(\theta)u_2+L_n^{\delta}(\theta)u_1^2\right]z^2+.... \tag{17}$$

and

$$\frac{((1-b)w)(g'(w))^{\lambda}}{\mu w(g'(w)-bg'(bw))+(1-\mu)(g(w)-g(bw))}=\sum_{n=0}^{\infty}L_n^{\delta}(\theta)|\varphi(z)|^n$$

$$=1+L_n^{\delta}(\theta)v_1w+\left[L_n^{\delta}(\theta)v_2+L_n^{\delta}(\theta)v_1^2\right]w^2+.... \tag{18}$$

Comparing the coefficients of (14) and (15), it is deduced that

$$(2\lambda-(1+\mu)(1+b))a_2=L_1^{\delta}(\theta)u_1, \tag{19}$$

$$\left.\begin{cases}(3\lambda-(1+2\mu)(1+b+b^2))a_3\\+(2\lambda(\lambda-1)+(1+\mu)(1+b)((1+\mu)(1+b)-2\lambda))a_2^2\end{cases}\right\}=L_1^{\delta}(\theta)u_2+L_2^{\delta}(\theta)u_1^2, \tag{20}$$

$$-(2\lambda-(1+\mu)(1+b))a_2=L_1^{\delta}(\theta)v_1, \tag{21}$$

$$\left\{\begin{array}{l}\left(3\lambda-(1+2\mu)\left(1+b+b^2\right)\right)\left(2a_2^2-a_3\right)\\+\left(2\lambda(\lambda-1)+(1+\mu)(1+b)((1+\mu)(1+b)-2\lambda)\right)a_2^2\end{array}\right\}=L_1^\delta(\theta)v_2+L_2^\delta(\theta)v_1^2. \qquad (22)$$

It follows from (19) and (21) that

$$u_1=-v_1, \qquad (23)$$

and

$$2(2\lambda-(1+\mu)(1+b))^2a_2^2=\left(L_1^\delta(\theta)\right)^2\left(u_1^2+v_1^2\right), \qquad (24)$$

By summing (20) and (21), it is found that

$$\left(\begin{array}{l}2\left(3\lambda-(1+2\mu)\left(1+b+b^2\right)\right)\\+2\left(2\lambda(\lambda-1)+(1+\mu)(1+b)((1+\mu)(1+b)-2\lambda)\right)\end{array}\right)a_2^2=L_1^\delta(\theta)(u_2+v_2)+L_2^\delta(\theta)\left(u_1^2+v_1^2\right), \qquad (25)$$

By substituting the values of $\left(u_1^2+v_1^2\right)$ from (24) in the right side of (25), it is obtained that

$$2\left\{\begin{array}{l}\left[\begin{array}{l}\left(3\lambda-(1+2\mu)\left(1+b+b^2\right)\right)\\+\left(2\lambda(\lambda-1)+(1+\mu)(1+b)((1+\mu)(1+b)-2\lambda)\right)\end{array}\right]\left(L_1^\delta(\theta)\right)^2\\-(2\lambda-(1+\mu)(1+b))^2L_2^\delta(\theta)\end{array}\right\}=\left(L_1^\delta(\theta)\right)^3(u_2+v_2), \qquad (26)$$

By simple computations using (7), (8), (16) and (26), it is determined that

$$|a_2|\le\frac{|1+\delta-\theta|\sqrt{|1+\delta-\theta|}}{\sqrt{\left|\left\{\begin{array}{l}\left[\left(3\lambda-(1+2\mu)\left(1+b+b^2\right)+2\lambda(\lambda-1)+(1+\mu)(1+b)((1+\mu)(1+b)-2\lambda)\right)\right](1+\delta-\theta)^2\\-(2\lambda-(1+\mu)(1+b))^2\left(\frac{\theta^2}{2}-(\delta+2)\theta+\frac{(\delta+1)(\delta+2)}{2}\right)\end{array}\right\}\right|}}.$$

By subtracting (22) and (20), it is deduced that

$$2\left(3\lambda-(1+2\mu)\left(1+b+b^2\right)\right)\left(a_3-a_2^2\right)=L_1^\delta(\theta)(u_2-v_2)+L_2^\delta(\theta)\left(u_1^2-v_1^2\right), \qquad (27)$$

Thus, applying (7) and (8), it is concluded that

$$|a_3|\le\frac{(1+\delta-\theta)^2}{(2\lambda-(1+\mu)(1+b))^2}+\frac{1+\delta-\theta}{3\lambda-(1+2\mu)\left(1+b+b^2\right)}. \qquad \square$$

By setting, the parameters $\mu=b=0$ in Theorem 1.1., it is claimed that

Corollary 2.1. Let $f\in LP_\Sigma(\lambda)$, then

$$|a_2| \le \frac{|1+\delta-\theta|\sqrt{|1+\delta-\theta|}}{\sqrt{\left|\left((3\lambda-1)+2\lambda(\lambda-1)+(1-2\lambda)\right)(1+\delta-\theta)^2-(2\lambda-1)^2\left(\frac{\theta^2}{2}-(\delta+2)\theta+\frac{(\delta+1)(\delta+2)}{2}\right)\right|}},$$

and

$$|a_3| \le \frac{(1+\delta-\theta)^2}{(2\lambda-1)^2}+\frac{1+\delta-\theta}{3\lambda-1}.$$

Putting the parameter $\lambda=1$, in the corollary 2.1, that was obtained by T.Panigrahi et al [12].

Remark 2.1. Let $f \in LP(1)$, then

$$|a_2| \le \frac{|1+\delta-\theta|\sqrt{|1+\delta-\theta|}}{\sqrt{\left|(1+\delta-\theta)^2-\left(\frac{\theta^2}{2}-(\delta+2)\theta+\frac{(\delta+1)(\delta+2)}{2}\right)\right|}},$$

and

$$|a_3| \le (1+\delta-\theta)^2+\frac{1+\delta-\theta}{2}.$$

3 Fekete – Szego inequalities for the class $LP_\Sigma(\lambda,\mu,b)$

The Fekete – Szego inequalities for function f in the class $LP_\Sigma(\lambda,\mu,b)$ is given by the following theorem.

Theorem 3.1. Let the function f given by (1) be in the class $LP_\Sigma(\lambda,\mu,b)$. Then for any real number σ, we have

$$|a_3-\sigma a_2^2| \le \begin{cases} \dfrac{|1+\delta-\theta|}{3\lambda-(1+2\mu)(1+b+b^2)}, |\sigma-1| \le \dfrac{|A_1|}{(3\lambda-(1+2\mu)(1+b+b^2))(1+\delta-\theta)^2} \\ \dfrac{|1+\delta-\theta|^2(1-\sigma)}{|A_1|}, |\sigma-1| \ge \dfrac{|A_1|}{(3\lambda-(1+2\mu)(1+b+b^2))(1+\delta-\theta)^2} \end{cases}$$

where

$$A_1 = \left[\left(3\lambda-(1+2\mu)(1+b+b^2)+2\lambda(\lambda-1)+(1+\mu)(1+b)((1+\mu)(1+b)-2\lambda)\right)\right](1+\delta-\theta)^2$$
$$-(2\lambda-(1+\mu)(1+b))^2\left(\frac{\theta^2}{2}-(\delta+2)\theta+\frac{(\delta+1)(\delta+2)}{2}\right)$$

Proof. From (25) and (27), it is observed that

$$a_3-\sigma a_2^2 = a_2^2+\frac{L_1^\delta(\theta)(u_2-v_2)}{2(3\lambda-(1+2\mu)(1+b+b^2))}-\sigma a_2^2=(1-\sigma)a_2^2+\frac{L_1^\delta(\theta)(u_2-v_2)}{2(3\lambda-(1+2\mu)(1+b+b^2))}$$

$$=\frac{(1-\sigma)\left(L_1^\delta(\theta)\right)^3(u_2+v_2)}{\left\{\begin{array}{l}\left[\left(\begin{array}{l}3\lambda-(1+2\mu)(1+b+b^2)\\+2\lambda(\lambda-1)\\+(1+\mu)(1+b)((1+\mu)(1+b)-2\lambda)\end{array}\right)\right]\left(L_1^\delta(\theta)\right)^2\\-(2\lambda-(1+\mu)(1+b))^2L_2^\delta(\theta)\end{array}\right\}}+\frac{L_1^\delta(\theta)(u_2-v_2)}{2(3\lambda-(1+2\mu)(1+b+b^2))}$$

$$=L_1^\delta(\theta)\left[\left(\eta(\sigma,\theta)+\frac{1}{2(3\lambda-(1+2\mu)(1+b+b^2))}\right)u_2+\left(\eta(\sigma,\theta)-\frac{1}{2(3\lambda-(1+2\mu)(1+b+b^2))}\right)v_2\right]$$

where

$$\eta(\sigma,\theta)=\frac{(1-\sigma)\left(L_1^\delta(\theta)\right)^2}{\left[\left(\begin{array}{l}3\lambda-(1+2\mu)(1+b+b^2)\\+2\lambda(\lambda-1)+(1+\mu)(1+b)((1+\mu)(1+b)-2\lambda)\end{array}\right)\right]\left(L_1^\delta(\theta)\right)^2-(2\lambda-(1+\mu)(1+b))^2L_2^\delta(\theta)}$$

In view of (7) and (8), it is concluded that

$$\left|a_3-\sigma a_2^2\right|\le\begin{cases}\dfrac{|1+\delta-\theta|}{3\lambda-(1+2\mu)(1+b+b^2)}, & 0\le|\eta(\sigma,\theta)|\le\dfrac{1}{2(3\lambda-(1+2\mu)(1+b+b^2))}\\ |\eta(\sigma,\theta)|, & |\eta(\sigma,\theta)|\ge\dfrac{1}{2(3\lambda-(1+2\mu)(1+b+b^2))}\end{cases}.$$

This completes the proof of the Theorem 3.1. □

By taking the parameters $\mu=b=0$ in the above Theorem 3.1, it is found that

Corollary 3.1. For $\eta\in\Re$, let the function $f\in LP_\Sigma(\lambda,\mu,b)$. Then

$$\left|a_3-\sigma a_2^2\right| \le \begin{cases} \dfrac{|1+\delta-\theta|}{3\lambda-1}, |\sigma-1| \le \dfrac{|A_2|}{(3\lambda-1)(1+\delta-\theta)^2} \\ \dfrac{|1+\delta-\theta|^2(1-\sigma)}{|A_2|}, |\sigma-1| \ge \dfrac{|A_2|}{(3\lambda-1)(1+\delta-\theta)^2} \end{cases}$$

where $A_2=\left(2\lambda^2-\lambda\right)(1+\delta-\theta)^2-(2\lambda-1)^2\left(\dfrac{\theta^2}{2}-(\delta+2)\theta+\dfrac{(\delta+1)(\delta+2)}{2}\right)$

By putting $\lambda=1$ in Corollary 3.1, which was investigated by T.Panigrahi et al [12].

Remark 3.1. Let the function $f\in\Sigma$ given by (1) be in the class $LP(1)$. Then for $\eta\in\Re$, it is obtained that

$$\left|a_3-\sigma a_2^2\right| \le \begin{cases} \dfrac{|1+\delta-\theta|}{2}, \\ |\sigma-1| \le \dfrac{1}{2}\left|1-\dfrac{\left(\dfrac{\theta^2}{2}-(\delta+2)\theta+\dfrac{(\delta+1)(\delta+2)}{2}\right)}{(1+\delta-\theta)^2}\right| \\ \dfrac{|1+\delta-\theta|^2(1-\sigma)}{\left|(1+\delta-\theta)^2-\left(\dfrac{\theta^2}{2}-(\delta+2)\theta+\dfrac{(\delta+1)(\delta+2)}{2}\right)\right|}, \\ |\sigma-1| \ge \dfrac{1}{2}\left|1-\dfrac{\left(\dfrac{\theta^2}{2}-(\delta+2)\theta+\dfrac{(\delta+1)(\delta+2)}{2}\right)}{(1+\delta-\theta)^2}\right| \end{cases}$$

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