

A Study On α -Cuts of Trapezoidal Fuzzy Numbers And α -Cuts of Trapezoidal Fuzzy Number Matrices

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Abstract - α -cuts of trapezoidal fuzzy numbers and α -cuts of trapezoidal fuzzy number matrices are introduced and evaluated.

Keyword - α -cuts of Trapezoidal Fuzzy Numbers (TrFNs), α -cuts of Trapezoidal Fuzzy Number Matrices (TrFNMs)

I. INTRODUCTION

Real world decision making problems are very often uncertain (or) vague in a number of ways. In 1965, Zadeh [5] introduced the concept of fuzzy set theory to meet those problems. The fuzziness can be represented by different ways. One of the most useful representation is the membership function. Depending on the nature of the membership function the fuzzy numbers can be classified in different forms, such as Triangular Fuzzy Numbers (TFNs), Trapezoidal Fuzzy Numbers, Interval Fuzzy Numbers etc. Fuzzy matrices play an important role in scientific development. Fuzzy matrices were introduced by M.G.Thomson [4]. Two new operators and some properties of fuzzy matrices over these new operators are given in [1]. Some new operators on triangular fuzzy numbers and triangular fuzzy number matrices are given in [3].

II. PRELIMINARIES

In this paper, some new elementary operators on α -cuts of Trapezoidal Fuzzy Numbers (TrFNs) and some new operators on α -cuts of Trapezoidal Fuzzy Number Matrices (TrFNMs) are defined. Using these operators, some important properties are proved.

Definition 1.2.1^[2]:

A **Fuzzy set** A in a universe of discourse X is defined as the following set of pairs $A = \{(x, \mu_A(x)) : x \in X\}$

Here $\mu_A : X \rightarrow [0,1]$ is a mapping called the membership value of $x \in X$ in a fuzzy set A.

Definition 1.2.2^[2]:

A **Fuzzy number** is an extension of a regular number in the sense that it does not refer to one single value but rather to a connected set of possible values, where each possible values has its own weight 0 and 1. This weight is called the **membership function**.

Definition 1.2.3^[1]:

A **normal fuzzy number** A with shape function

$$\mu_A = \begin{cases} \left(\frac{x-a}{b-a}\right)^n & \text{when } x \in [a, b], \\ w & \text{when } x \in [b, c], \\ \left(\frac{d-x}{d-c}\right)^n & \text{when } x \in (c, d] \\ 0 & \text{otherwise} \end{cases}$$



where $n > 0$, will be denoted by $A = (a, b, c, d)_n$.

If A be non-normal fuzzy number, it will be denoted by $A = (a, b, c, d; w)_n$.

If $n = 1$, we simply write $A = (a, b, c, d)$, which is known as a **normal Trapezoidal fuzzy number**.

Definition 1.2.4^[5]:

A **Trapezoidal Fuzzy Number (TrFN)** denoted by $\langle m, \alpha, \beta, \gamma \rangle$ has the membership function

$$\mu_A(x) = \begin{cases} 0, & \text{for } x \leq m \\ \frac{x-m}{\alpha-m}, & m \leq x \leq \alpha \\ 1, & \alpha \leq x \leq \beta \\ \frac{\gamma-x}{\gamma-\beta}, & \beta \leq x \leq \gamma \\ 0, & x \geq \gamma \end{cases}$$

$$\text{or, } \mu_A(x) = \max \left(\min \left(\frac{x-m}{\alpha-m}, 1, \frac{\gamma-x}{\gamma-\beta} \right), 0 \right)$$

The point m , with membership grade of 1, is called the **mean value** and α, β are the **left hand spreads** of M respectively.

When $\alpha = \beta$, the trapezoidal fuzzy number coincides with triangular one.

Example 1.2.1:

$\tilde{M} = \langle 3, 6, 7, 9 \rangle$ is a Trapezoidal Fuzzy number.

Definition 1.2.5^[2] :

A **Trapezoidal Fuzzy Number Matrix (TrFNM)** of order $m \times n$

is defined as $M = (M_{ij})_{m \times n}$ where $M_{ij} = \langle m_{ij}, \alpha_{ij}, \beta_{ij}, \gamma_{ij} \rangle$

Example 1.2.2:

$$M = \begin{bmatrix} \langle 2, 3, 4, 5 \rangle & \langle 4, 6, 7, 5 \rangle & \langle 3, 6, 7, 2 \rangle & \langle 2, 5, 8, 3 \rangle \\ \langle 0, 1, 0, 6 \rangle & \langle 1, 2, 3, 4 \rangle & \langle 5, 6, 8, 10 \rangle & \langle 3, 4, 5, 1 \rangle \\ \langle 1, 7, 9, 11 \rangle & \langle 0, 0, 0, 0 \rangle & \langle 2, 5, 3, 4 \rangle & \langle 1, 0, 1, 0 \rangle \\ \langle 1, 1, 3, 5 \rangle & \langle 1, 2, 2, 1 \rangle & \langle 7, 5, 0, 3 \rangle & \langle 2, 2, 0, 1 \rangle \end{bmatrix}$$

$$N = \begin{bmatrix} \langle 0.3, 0.3, 0.6, 0.4 \rangle & \langle 0.8, 0.1, 0.3, 0.4 \rangle \\ \langle 0.4, 0.9, 0.5, 0.1 \rangle & \langle 0.2, 0.1, 0.3, 0.7 \rangle \end{bmatrix}$$

are Trapezoidal fuzzy number matrices.

Definition 1.2.6^[3]:

Let $M = (\tilde{M}_{ij})_{m \times n}$ and $N = (\tilde{N}_{ij})_{m \times n}$ be two Trapezoidal Fuzzy Number Matrices (TrFNM) of the same order.

Then the following **operators** are defined.

- (1) $M \oplus N = (\tilde{M}_{ij} \oplus \tilde{N}_{ij})$
- (2) $M \vee N = (\tilde{M}_{ij} \vee \tilde{N}_{ij})$
- (3) $M \ominus N = (\tilde{M}_{ij} \ominus \tilde{N}_{ij})$
- (4) $M \geq N$ iff $\tilde{M}_{ij} \geq \tilde{N}_{ij} \forall i = 1 \text{ to } m, j = 1 \text{ to } n$.

Definition 1.2.7^[3]:

For $\alpha \in [0, 1]$, the **upper α -cut** of the Trapezoidal Fuzzy Number $\tilde{M} = \langle m, \omega, \beta, \gamma \rangle$

is defined as $\tilde{M} = \langle m^{(\alpha)}, \omega^{(\alpha)}, \beta^{(\alpha)}, \gamma^{(\alpha)} \rangle$

and **the lower α -cut** of \tilde{M} is defined as

$$\tilde{M}_{(\alpha)} = \langle m_{(\alpha)}, \omega_{(\alpha)}, \beta_{(\alpha)}, \gamma_{(\alpha)} \rangle$$

Example 1.2.3:

Consider the Trapezoidal fuzzy number \tilde{M} as follows :

$$\tilde{M} = \langle 0.5, 0.1, 0.4, 0.6 \rangle.$$

By taking $\alpha = 0.5$, we get

$$\tilde{M}_{(0.5)} = \langle 0.5, 0, 0, 0.6 \rangle \text{ and}$$

$$\tilde{M}^{(0.5)} = \langle 1, 0, 0, 1 \rangle$$

Definition 1.2.8:

The upper α -cut of a Trapezoidal Fuzzy Number Matrix

$M = (\tilde{M}_{ij})_{m \times n}$ is defined as $M^{(\alpha)} = (\tilde{M}_{ij}^{(\alpha)})_{m \times n}$

and **the lower α -cut of M** is defined as

$$M_{(\alpha)} = (\tilde{M}_{ij(\alpha)})_{m \times n}$$

Theorem 1.1.1:

For any two Trapezoidal fuzzy number matrices M and N ,

$$(i) \quad (M \ominus N)^{(\alpha)} \geq M^{(\alpha)} \ominus N^{(\alpha)}$$

$$(ii) \quad (M \vee N)^{(\alpha)} = M^{(\alpha)} \vee N^{(\alpha)}$$

$$(iii) \quad (M \oplus N)^{(\alpha)} \geq M^{(\alpha)} \oplus N^{(\alpha)}$$

Proof :

Let $M = (\tilde{M}_{ij})_{m \times n}$ where $\tilde{M}_{ij} = \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle$ and

$N = (\tilde{N}_{ij})_{m \times n}$ where $\tilde{N}_{ij} = \langle n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} \rangle$

$$(i) \quad \text{Let } \tilde{E}_{ij} \text{ and } \tilde{D}_{ij} \text{ be the } ij^{\text{th}} \text{ elements of } M^{(\alpha)} \ominus N^{(\alpha)} \text{ and } (M \ominus N)^{(\alpha)}$$

$$\therefore \tilde{E}_{ij} = \tilde{M}_{ij}^{(\alpha)} \ominus \tilde{N}_{ij}^{(\alpha)} \text{ and } \tilde{D}_{ij} = (\tilde{M}_{ij} \ominus \tilde{N}_{ij})^{(\alpha)}$$

Case 1 : $\tilde{M}_{ij} \geq \tilde{N}_{ij} \geq \alpha$

$$\langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle \geq \langle n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} \rangle \geq \alpha$$

$$\therefore m_{ij} \geq n_{ij} \geq \alpha, \omega_{ij} \geq \rho_{ij} \geq \alpha, \beta_{ij} \geq \delta_{ij} \geq \alpha, \gamma_{ij} \geq \sigma_{ij} \geq \alpha$$

$$\text{Therefore, } \tilde{D}_{ij} = (\tilde{M}_{ij} \ominus \tilde{N}_{ij})^{(\alpha)}$$

$$= \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle^{(\alpha)}$$

$$= \langle m_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} \rangle$$

$$= \langle 1, 1, 1, 1 \rangle$$

$$\text{and } \tilde{E}_{ij} = \tilde{M}_{ij}^{(\alpha)} \ominus \tilde{N}_{ij}^{(\alpha)}$$

$$= \langle m_{ij}^{(\alpha)} \ominus n_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)} \ominus \rho_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)} \ominus \delta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} \ominus \sigma_{ij}^{(\alpha)} \rangle$$

$$= \langle 1 \ominus 1, 1 \ominus 1, 1 \ominus 1, 1 \ominus 1 \rangle$$

$$= \langle 0, 0, 0, 0 \rangle$$

i.e., $\tilde{D}_{ij} > \tilde{E}_{ij}$.

Case 2 : $\tilde{M}_{ij} \geq \alpha \geq \tilde{N}_{ij}$

$$\therefore \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle \geq \alpha \geq \langle n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} \rangle$$

$$\therefore m_{ij} \geq \alpha \geq n_{ij}, \omega_{ij} \geq \alpha \geq \rho_{ij}, \beta_{ij} \geq \alpha \geq \delta_{ij}, \gamma_{ij} \geq \alpha \geq \sigma_{ij}$$

$$\text{Then } \tilde{D}_{ij} = (\tilde{M}_{ij} \ominus \tilde{N}_{ij})^{(\alpha)}$$

$$= \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle^{(\alpha)}$$

$$= \langle m_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} \rangle$$

$$= \langle 1, 1, 1, 1 \rangle$$

$$\text{and } \tilde{E}_{ij} = \tilde{M}_{ij}^{(\alpha)} \ominus \tilde{N}_{ij}^{(\alpha)}$$

$$= \langle m_{ij}^{(\alpha)} \ominus n_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)} \ominus \rho_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)} \ominus \delta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} \ominus \sigma_{ij}^{(\alpha)} \rangle$$

$$= \langle 1 \ominus 0, 1 \ominus 0, 1 \ominus 0, 1 \ominus 0 \rangle$$

$$= \langle 1, 1, 1, 1 \rangle$$

$$\therefore \tilde{D}_{ij} = \tilde{E}_{ij}.$$

Case 3 : $\alpha \geq \tilde{M}_{ij} \geq \tilde{N}_{ij}$

$$\alpha \geq \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle \geq \langle n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} \rangle$$

$$\therefore \alpha \geq m_{ij} \geq n_{ij}, \alpha \geq \omega_{ij} \geq \rho_{ij}, \alpha \geq \beta_{ij} \geq \delta_{ij}, \alpha \geq \gamma_{ij} \geq \sigma_{ij}$$

$$\text{Here, } \tilde{D}_{ij} = (\tilde{M}_{ij} \ominus \tilde{N}_{ij})^{(\alpha)}$$

$$= \langle m_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} \rangle$$

$$= \langle 0, 0, 0, 0 \rangle$$

$$\tilde{E}_{ij} = \tilde{M}_{ij}^{(\alpha)} \ominus \tilde{N}_{ij}^{(\alpha)}$$

$$= \langle m_{ij}^{(\alpha)} \ominus n_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)} \ominus \rho_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)} \ominus \delta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} \ominus \sigma_{ij}^{(\alpha)} \rangle$$

$$= \langle 0 \ominus 0, 0 \ominus 0, 0 \ominus 0, 0 \ominus 0 \rangle$$

$$= \langle 0, 0, 0, 0 \rangle$$

$$\text{ie., } \tilde{D}_{ij} = \tilde{E}_{ij}$$

$$\therefore \text{In all the cases, } \tilde{D}_{ij} \geq \tilde{E}_{ij}$$

$$\therefore (\tilde{M}_{ij} \tilde{N}_{ij})^{(\alpha)} \geq \tilde{M}_{ij}^{(\alpha)} \ominus \tilde{N}_{ij}^{(\alpha)}$$

$$\therefore (M \ominus N)^{(\alpha)} \geq M^{(\alpha)} \ominus N^{(\alpha)}.$$

i) Let \tilde{C}_{ij} and \tilde{D}_{ij} be the ij^{th} elements of $M^{(\alpha)} \vee N^{(\alpha)}$ and $(M \vee N)^{(\alpha)}$.

$$\therefore \tilde{C}_{ij} = \tilde{M}_{ij}^{(\alpha)} \vee \tilde{N}_{ij}^{(\alpha)} \text{ and } \tilde{D}_{ij} = (\tilde{M}_{ij} \vee \tilde{N}_{ij})^{(\alpha)}$$

Case 1: $\tilde{M}_{ij} \geq \tilde{N}_{ij} \geq \alpha$

$$\text{i.e., } \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle \geq \langle n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} \rangle \geq \alpha$$

$$\therefore m_{ij} \geq n_{ij} \geq \alpha, \omega_{ij} \geq \rho_{ij} \geq \alpha, \beta_{ij} \geq \delta_{ij} \geq \alpha, \gamma_{ij} \geq \sigma_{ij} \geq \alpha$$

$$\begin{aligned}
 \therefore \tilde{D}_{ij} &= (\tilde{M}_{ij} \vee \tilde{N}_{ij})^{(\alpha)} \\
 &= \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle^{(\alpha)} \\
 &= \langle m_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} \rangle \\
 &= \langle 1, 1, 1, 1 \rangle
 \end{aligned}$$

$$\begin{aligned}
 \text{and } \tilde{C}_{ij} &= \tilde{M}_{ij}^{(\alpha)} \vee \tilde{N}_{ij}^{(\alpha)} \\
 &= \langle m_{ij}^{(\alpha)} \vee n_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)} \vee \rho_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)} \vee \delta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} \vee \sigma_{ij}^{(\alpha)} \rangle \\
 &= \langle 1 \vee 1, 1 \vee 1, 1 \vee 1, 1 \vee 1 \rangle \\
 &= \langle 1, 1, 1, 1 \rangle
 \end{aligned}$$

i.e., $\tilde{D}_{ij} = \tilde{C}_{ij}$

Case 2: $\tilde{M}_{ij} \geq \alpha \geq \tilde{N}_{ij}$

$$\begin{aligned}
 \therefore \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle &\geq \alpha \geq \langle n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} \rangle \\
 \therefore \langle m_{ij} \geq \alpha \geq n_{ij}, \omega_{ij} \geq \alpha \geq \rho_{ij}, \beta_{ij} \geq \alpha \geq \delta_{ij}, \gamma_{ij} \geq \alpha \geq \sigma_{ij} \rangle
 \end{aligned}$$

$$\begin{aligned}
 \text{Then } \tilde{D}_{ij} &= (\tilde{M}_{ij} \vee \tilde{N}_{ij})^{(\alpha)} \\
 &= \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle^{(\alpha)} \\
 &= \langle m_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} \rangle \\
 &= \langle 1, 1, 1, 1 \rangle
 \end{aligned}$$

$$\begin{aligned}
 \text{and } \tilde{C}_{ij} &= \tilde{M}_{ij}^{(\alpha)} \vee \tilde{N}_{ij}^{(\alpha)} \\
 &= \langle m_{ij}^{(\alpha)} \vee n_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)} \vee \rho_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)} \vee \delta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} \vee \sigma_{ij}^{(\alpha)} \rangle \\
 &= \langle 1 \vee 0, 1 \vee 0, 1 \vee 0, 1 \vee 0 \rangle \\
 &= \langle 1, 1, 1, 1 \rangle
 \end{aligned}$$

$\therefore \tilde{D}_{ij} = \tilde{C}_{ij}$

Case 3: $\alpha \geq \tilde{M}_{ij} \geq \tilde{N}_{ij}$

$$\begin{aligned}
 \text{i.e., } \alpha &\geq \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle \geq \langle n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} \rangle \\
 \therefore \alpha &\geq m_{ij} \geq n_{ij}, \alpha \geq \omega_{ij} \geq \rho_{ij}, \alpha \geq \beta_{ij} \geq \delta_{ij}, \alpha \geq \gamma_{ij} \geq \sigma_{ij}
 \end{aligned}$$

$$\begin{aligned}
 \text{Then, } \tilde{D}_{ij} &= (\tilde{M}_{ij} \vee \tilde{N}_{ij})^{(\alpha)} \\
 &= \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle^{(\alpha)} \\
 &= \langle m_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} \rangle \\
 &= \langle 0, 0, 0, 0 \rangle
 \end{aligned}$$

$$\begin{aligned}
 \text{and } \tilde{C}_{ij} &= (\tilde{M}_{ij}^{(\alpha)} \vee \tilde{N}_{ij}^{(\alpha)}) \\
 &= \langle m_{ij}^{(\alpha)} \vee n_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)} \vee \rho_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)} \vee \delta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} \vee \sigma_{ij}^{(\alpha)} \rangle \\
 &= \langle 0 \vee 0, 0 \vee 0, 0 \vee 0, 0 \vee 0 \rangle \\
 &= \langle 0, 0, 0, 0 \rangle
 \end{aligned}$$

i.e., $\tilde{D}_{ij} = \tilde{C}_{ij}$.

\therefore In all the cases, $\tilde{D}_{ij} = \tilde{C}_{ij}$

$$\therefore (\tilde{M}_{ij} \vee \tilde{N}_{ij})^{(\alpha)} = \tilde{M}_{ij}^{(\alpha)} \vee \tilde{N}_{ij}^{(\alpha)}$$

$$\therefore (M \vee N)^{(\alpha)} = M^{(\alpha)} \vee N^{(\alpha)}.$$

(iii) Let \tilde{P}_{ij} and \tilde{Q}_{ij} be the ij^{th} elements of $M^{(\alpha)} \oplus N^{(\alpha)}$ and $(M \oplus N)^{(\alpha)}$

$$\therefore \tilde{P}_{ij} = M_{ij}^{(\alpha)} \oplus N_{ij}^{(\alpha)} \text{ and } \tilde{Q}_{ij} = (\tilde{M}_{ij} \oplus \tilde{N}_{ij})^{(\alpha)}$$

Case 1: $\tilde{M}_{ij} \geq \tilde{N}_{ij} \geq \alpha$

$$\langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle \geq \langle n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} \rangle \geq \alpha$$

$$\therefore m_{ij} \geq n_{ij} \geq \alpha, \omega_{ij} \geq \rho_{ij} \geq \alpha, \beta_{ij} \geq \delta_{ij} \geq \alpha, \gamma_{ij} \geq \sigma_{ij} \geq \alpha$$

$$\text{Then } \tilde{Q}_{ij} = (\tilde{M}_{ij} \oplus \tilde{N}_{ij})^{(\alpha)}$$

$$= \langle m_{ij} + n_{ij} - m_{ij} \cdot n_{ij}, \omega_{ij} + \rho_{ij} - \omega_{ij} \cdot \rho_{ij}, \\ \beta_{ij} + \delta_{ij} - \beta_{ij} \cdot \delta_{ij}, \gamma_{ij} + \sigma_{ij} - \gamma_{ij} \cdot \sigma_{ij} \rangle^{(\alpha)}$$

$$= \langle m_{ij} + n_{ij}(1 - m_{ij}), \omega_{ij} + \rho_{ij}(1 - \omega_{ij}), \\ \beta_{ij} + \delta_{ij}(1 - \beta_{ij}), \gamma_{ij} + \sigma_{ij}(1 - \gamma_{ij}) \rangle^{(\alpha)}$$

$$\geq \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle^{(\alpha)}$$

$$> \langle m_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} \rangle$$

$$= \langle 1, 1, 1, 1 \rangle$$

$$\text{and } \tilde{P}_{ij} = \tilde{M}_{ij}^{(\alpha)} \oplus \tilde{N}_{ij}^{(\alpha)}$$

$$= \langle m_{ij}^{(\alpha)} \oplus n_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)} \oplus \rho_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)} \oplus \delta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} \oplus \sigma_{ij}^{(\alpha)} \rangle$$

$$= \langle m_{ij}^{(\alpha)} + n_{ij}^{(\alpha)} - m_{ij}^{(\alpha)} n_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)} + \rho_{ij}^{(\alpha)} - \omega_{ij}^{(\alpha)} \rho_{ij}^{(\alpha)}, \\ \beta_{ij}^{(\alpha)} + \delta_{ij}^{(\alpha)} - \beta_{ij}^{(\alpha)} \delta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} + \sigma_{ij}^{(\alpha)} - \gamma_{ij}^{(\alpha)} \sigma_{ij}^{(\alpha)} \rangle$$

$$= \langle 1+1-1, 1+1-1, 1+1-1, 1+1-1 \rangle$$

$$= \langle 1, 1, 1, 1 \rangle$$

$$\therefore \tilde{Q}_{ij} > \tilde{P}_{ij}$$

Case 2: $\tilde{M}_{ij} \geq \alpha \geq \tilde{N}_{ij}$

$$\therefore \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle \geq \alpha \geq \langle n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} \rangle$$

$$\therefore m_{ij} \geq \alpha \geq n_{ij}, \omega_{ij} \geq \alpha \geq \rho_{ij}, \beta_{ij} \geq \alpha \geq \delta_{ij}, \gamma_{ij} \geq \alpha \geq \sigma_{ij}$$

Then $\tilde{Q}_{ij} =$

$$(\tilde{M}_{ij} \oplus \tilde{N}_{ij})^{(\alpha)}$$

$$= \langle m_{ij} + n_{ij} - m_{ij} \cdot n_{ij}, \omega_{ij} + \rho_{ij} - \omega_{ij} \cdot \rho_{ij}, \\ \beta_{ij} + \delta_{ij} - \beta_{ij} \cdot \delta_{ij}, \gamma_{ij} + \sigma_{ij} - \gamma_{ij} \cdot \sigma_{ij} \rangle^{(\alpha)}$$

$$= \langle m_{ij} + n_{ij}(1 - m_{ij}), \omega_{ij} + \rho_{ij}(1 - \omega_{ij}), \\ \beta_{ij} + \delta_{ij}(1 - \beta_{ij}), \gamma_{ij} + \sigma_{ij}(1 - \gamma_{ij}) \rangle^{(\alpha)}$$

$$\geq \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle^{(\alpha)}$$

$$> \langle m_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} \rangle$$

$$= \langle 1, 1, 1, 1 \rangle$$

$$\begin{aligned} \text{and } \tilde{P}_{ij} &= \tilde{M}_{ij}^{(\alpha)} \oplus \tilde{N}_{ij}^{(\alpha)} \\ &= \langle m_{ij}^{(\alpha)} \oplus n_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)} \oplus \rho_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)} \oplus \delta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} \oplus \sigma_{ij}^{(\alpha)} \rangle \\ &= \langle m_{ij}^{(\alpha)} + n_{ij}^{(\alpha)} - m_{ij}^{(\alpha)} n_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)} + \rho_{ij}^{(\alpha)} - \omega_{ij}^{(\alpha)} \rho_{ij}^{(\alpha)}, \\ &\quad \beta_{ij}^{(\alpha)} + \delta_{ij}^{(\alpha)} - \beta_{ij}^{(\alpha)} \delta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} + \sigma_{ij}^{(\alpha)} - \gamma_{ij}^{(\alpha)} \sigma_{ij}^{(\alpha)} \rangle \\ &= \langle 1+0-0, 1+0-0, 1+0-0, 1+0-0 \rangle \\ &= \langle 1, 1, 1, 1 \rangle \\ \therefore \tilde{Q}_{ij} &> \tilde{P}_{ij} \end{aligned}$$

Case 3: $\alpha \geq \tilde{M}_{ij} \geq \tilde{N}_{ij}$

$$\begin{aligned} \text{i.e., } \alpha &\geq \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle \geq \langle n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} \rangle \\ \therefore \alpha &\geq m_{ij} \geq n_{ij}, \alpha \geq \omega_{ij} \geq \rho_{ij}, \alpha \geq \beta_{ij} \geq \delta_{ij}, \alpha \geq \gamma_{ij} \geq \sigma_{ij} \end{aligned}$$

$$\begin{aligned} \text{Then } \tilde{Q}_{ij} &= (\tilde{M}_{ij} \oplus \tilde{N}_{ij})^{(\alpha)} \\ &= \langle m_{ij} + n_{ij} - m_{ij} \cdot n_{ij}, \omega_{ij} + \rho_{ij} - \omega_{ij} \cdot \rho_{ij}, \\ &\quad \beta_{ij} + \delta_{ij} - \beta_{ij} \cdot \delta_{ij}, \gamma_{ij} + \sigma_{ij} - \gamma_{ij} \cdot \sigma_{ij} \rangle^{(\alpha)} \end{aligned}$$

$$\begin{aligned} &\geq \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle^{(\alpha)} \\ &> \langle m_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} \rangle \\ &= \langle 0, 0, 0, 0 \rangle \end{aligned}$$

$$\begin{aligned} \text{and } \tilde{P}_{ij} &= \tilde{M}_{ij}^{(\alpha)} \oplus \tilde{N}_{ij}^{(\alpha)} \\ &= \langle m_{ij}^{(\alpha)} \oplus n_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)} \oplus \rho_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)} \oplus \delta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} \oplus \sigma_{ij}^{(\alpha)} \rangle \\ &= \langle m_{ij}^{(\alpha)} + n_{ij}^{(\alpha)} - m_{ij}^{(\alpha)} n_{ij}^{(\alpha)}, \omega_{ij}^{(\alpha)} + \rho_{ij}^{(\alpha)} - \omega_{ij}^{(\alpha)} \rho_{ij}^{(\alpha)}, \\ &\quad \beta_{ij}^{(\alpha)} + \delta_{ij}^{(\alpha)} - \beta_{ij}^{(\alpha)} \delta_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)} + \sigma_{ij}^{(\alpha)} - \gamma_{ij}^{(\alpha)} \sigma_{ij}^{(\alpha)} \rangle \\ &= \langle 0+0-0, 0+0-0, 0+0-0, 0+0-0 \rangle \\ &= \langle 0, 0, 0, 0 \rangle \end{aligned}$$

$$\therefore \tilde{Q}_{ij} = \tilde{P}_{ij}.$$

\therefore In all the cases, $\tilde{Q}_{ij} \geq \tilde{P}_{ij}$

$$\therefore (\tilde{M}_{ij} \oplus \tilde{N}_{ij})^{(\alpha)} \geq \tilde{M}_{ij}^{(\alpha)} \oplus \tilde{N}_{ij}^{(\alpha)}$$

Thus, $(\mathbf{M} \oplus \mathbf{N})^{(\alpha)} \geq \mathbf{M}^{(\alpha)} \oplus \mathbf{N}^{(\alpha)}$.

Theorem 1.1.2:

For any two Trapezoidal Fuzzy Number matrices M and N,

- (i) $(M \vee N)_{(\alpha)} = M_{(\alpha)} \vee N_{(\alpha)}$
- (ii) $(M \ominus N)_{(\alpha)} = M_{(\alpha)} \ominus N_{(\alpha)}$
- (iii) $(M \oplus N)_{(\alpha)} \geq M_{(\alpha)} \oplus N_{(\alpha)}$

Proof:

Let $M = (\tilde{M}_{ij})_{(m \times n)}$ where $\tilde{M}_{ij} = \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle$ and

$N = (\tilde{N}_{ij})_{(m \times n)}$ where $\tilde{N}_{ij} = \langle n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} \rangle$

(i) Let \tilde{C}_{ij} and \tilde{D}_{ij} be the ij^{th} elements of $M_{(\alpha)} \vee N_{(\alpha)}$ and $(M \vee N)_{(\alpha)}$

$$\therefore \tilde{C}_{ij} = \tilde{M}_{ij(\alpha)} \vee \tilde{N}_{ij(\alpha)} \text{ and } \tilde{D}_{ij} = (\tilde{M}_{ij} \vee \tilde{N}_{ij})_{(\alpha)}$$

Case 1: $\tilde{M}_{ij} \geq \tilde{N}_{ij} \geq \alpha$

$$\langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle \geq \langle n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} \rangle \geq \alpha$$

$$\therefore m_{ij} \geq n_{ij} \geq \alpha, \omega_{ij} \geq \rho_{ij} \geq \alpha, \beta_{ij} \geq \delta_{ij} \geq \alpha, \gamma_{ij} \geq \sigma_{ij} \geq \alpha$$

$$\text{Here } \tilde{D}_{ij} = (\tilde{M}_{ij} \vee \tilde{N}_{ij})_{(\alpha)}$$

$$= \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle_{(\alpha)}$$

$$= \langle m_{ij(\alpha)}, \omega_{ij(\alpha)}, \beta_{ij(\alpha)}, \gamma_{ij(\alpha)} \rangle$$

$$= \langle x, x, x, x \rangle$$

$$\text{and } \tilde{C}_{ij} = \tilde{M}_{ij(\alpha)} \vee \tilde{N}_{ij(\alpha)}$$

$$= \langle m_{ij(\alpha)} \vee n_{ij(\alpha)}, \omega_{ij(\alpha)} \vee \rho_{ij(\alpha)}, \beta_{ij(\alpha)} \vee \delta_{ij(\alpha)}, \gamma_{ij(\alpha)} \vee \sigma_{ij(\alpha)} \rangle$$

$$= \langle x \vee x, x \vee x, x \vee x, x \vee x \rangle$$

$$= \langle x, x, x, x \rangle$$

$$\text{i.e., } \tilde{D}_{ij} = \tilde{C}_{ij}$$

Case2: $\tilde{M}_{ij} \geq \alpha \geq \tilde{N}_{ij}$

$$\therefore \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle \geq \alpha <$$

$$n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} >$$

$$\therefore m_{ij} \geq \alpha \geq n_{ij}, \omega_{ij} \geq \alpha \geq \rho_{ij}, \beta_{ij} \geq \alpha \geq \delta_{ij}, \gamma_{ij} \geq \alpha \geq \sigma_{ij}$$

$$\text{Then } \tilde{D}_{ij} = (\tilde{M}_{ij} \vee N_{ij})_{(\alpha)}$$

$$= \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle_{(\alpha)}$$

$$= \langle m_{ij(\alpha)}, \omega_{ij(\alpha)}, \beta_{ij(\alpha)}, \gamma_{ij(\alpha)} \rangle$$

$$= \langle x, x, x, x \rangle$$

$$\text{And } \tilde{C}_{ij} = \tilde{M}_{ij(\alpha)} \vee \tilde{N}_{ij(\alpha)}$$

$$= \langle m_{ij(\alpha)} \vee n_{ij(\alpha)}, \omega_{ij(\alpha)} \vee \rho_{ij(\alpha)}, \beta_{ij(\alpha)} \vee \delta_{(\alpha)}, \gamma_{ij(\alpha)} \vee \sigma_{ij(\alpha)} \rangle$$

$$= \langle x \vee x, x \vee x, x \vee x, x \vee x \rangle$$

$$= \langle x, x, x, x \rangle$$

$$\therefore \tilde{D}_{ij} = \tilde{C}_{ij}$$

Case 3: $\alpha \geq \tilde{M}_{ij} \geq \tilde{N}_{ij}$

$$\text{i.e., } \alpha \geq \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle \geq \langle n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} \rangle$$

$$\therefore \alpha \geq m_{ij} \geq n_{ij}, \alpha \geq \omega_{ij} \geq \rho_{ij}, \alpha \geq \beta_{ij} \geq \delta_{ij}, \alpha \geq \gamma_{ij} \geq \sigma_{ij}$$

$$\text{Then } \tilde{D}_{ij} = \tilde{M}_{ij} \vee \tilde{N}_{ij(\alpha)}$$

$$= \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle_{(\alpha)}$$

$$= \langle m_{ij(\alpha)}, \omega_{ij(\alpha)}, \beta_{ij(\alpha)}, \gamma_{ij(\alpha)} \rangle \\ = \langle 0, 0, 0, 0 \rangle$$

$$\text{and } \tilde{C}_{ij} = (\tilde{M}_{ij(\alpha)} \vee \tilde{N}_{ij(\alpha)}) \\ = \langle m_{ij(\alpha)} \vee n_{ij(\alpha)}, \omega_{ij(\alpha)} \vee \rho_{ij(\alpha)}, \beta_{ij(\alpha)} \vee \delta_{ij(\alpha)}, \gamma_{ij(\alpha)} \vee \sigma_{ij(\alpha)} \rangle \\ = \langle 0 \vee 0, 0 \vee 0, 0 \vee 0, 0 \vee 0 \rangle \\ = \langle 0, 0, 0, 0 \rangle$$

i.e., $\tilde{D}_{ij} = \tilde{C}_{ij}$

\therefore In all the cases, $\tilde{D}_{ij} = \tilde{C}_{ij}$

$$\therefore \tilde{M}_{ij} \vee \tilde{N}_{ij(\alpha)} = \tilde{M}_{ij(\alpha)} \vee \tilde{N}_{ij(\alpha)}$$

$$\therefore (M \vee N)_{(\alpha)} = M_{(\alpha)} \vee N_{(\alpha)}$$

(ii) Let \tilde{E}_{ij} and \tilde{D}_{ij} be the ij^{th} elements of $M_{(\alpha)} \Theta N_{(\alpha)}$ and $(M \Theta N)_{(\alpha)}$.

$$\therefore \tilde{E}_{ij} = M_{ij(\alpha)} \Theta N_{ij(\alpha)} \text{ and } \tilde{D}_{ij} = (\tilde{M}_{ij} \Theta \tilde{N}_{ij})_{(\alpha)}$$

Case 1: $\tilde{M}_{ij} \geq \tilde{N}_{ij} \geq \alpha$

$$\langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle \geq \langle n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} \rangle \geq \alpha \\ m_{ij} \geq n_{ij} \geq \alpha, \omega_{ij} \geq \rho_{ij} \geq \alpha, \beta_{ij} \geq \delta_{ij} \geq \alpha, \gamma_{ij} \geq \sigma_{ij} \geq \alpha$$

Therefore, $\tilde{D}_{ij} = (\tilde{M}_{ij} \Theta \tilde{N}_{ij})_{(\alpha)}$

$$= \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle_{(\alpha)} \\ = \langle m_{ij(\alpha)}, \omega_{ij(\alpha)}, \beta_{ij(\alpha)}, \gamma_{ij(\alpha)} \rangle \\ = \langle x, x, x, x \rangle$$

and $\tilde{E}_{ij} = \tilde{M}_{ij(\alpha)} \Theta \tilde{N}_{ij(\alpha)}$

$$= \langle m_{ij(\alpha)} \Theta n_{ij(\alpha)}, \omega_{ij(\alpha)} \Theta \rho_{ij(\alpha)}, \beta_{ij(\alpha)} \Theta \delta_{ij(\alpha)}, \gamma_{ij(\alpha)} \Theta \sigma_{ij(\alpha)} \rangle \quad = \langle$$

$x \Theta x, x \Theta x, x \Theta x, x \Theta x \rangle$

$$= \langle 0, 0, 0, 0 \rangle$$

i.e., $\tilde{D}_{ij} = \tilde{E}_{ij}$

Case 2: $\tilde{M}_{ij} \geq \alpha \geq \tilde{N}_{ij}$

$$\langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle \geq \alpha \geq \langle n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} \rangle \\ m_{ij} \geq \alpha \geq n_{ij}, \beta_{ij} \geq \alpha \geq \rho_{ij}, \beta_{ij} \geq \alpha \geq \delta_{ij}, \gamma_{ij} \geq \alpha \geq \sigma_{ij}$$

Then $\tilde{D}_{ij} = (\tilde{M}_{ij} \Theta \tilde{N}_{ij})_{(\alpha)}$

$$= \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle_{(\alpha)} \\ = \langle m_{ij(\alpha)}, \omega_{ij(\alpha)}, \beta_{ij(\alpha)}, \gamma_{ij(\alpha)} \rangle \\ = \langle x, x, x, x \rangle$$

and $\tilde{E}_{ij} = \tilde{M}_{ij(\alpha)} \Theta \tilde{N}_{ij(\alpha)}$

$$= \langle m_{ij(\alpha)} \Theta n_{ij(\alpha)}, \omega_{ij(\alpha)} \Theta \rho_{ij(\alpha)}, \beta_{ij(\alpha)} \Theta \delta_{ij(\alpha)}, \gamma_{ij(\alpha)} \Theta \sigma_{ij(\alpha)} \rangle$$

$$\begin{aligned}
 &= \langle x \oplus 0, x \oplus 0, x \oplus 0, x \oplus 0 \rangle \\
 &= \langle x, x, x, x \rangle \\
 \therefore \widetilde{D}_{ij} &= \widetilde{E}_{ij}.
 \end{aligned}$$

Case 3: $\alpha \geq \widetilde{\mathbf{M}}_{ij} \geq \widetilde{\mathbf{N}}_{ij}$

$$\begin{aligned}
 \alpha &\geq \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle \geq \langle n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} \rangle \\
 \alpha &\geq m_{ij} \geq n_{ij}, \alpha \geq \omega_{ij} \geq \rho_{ij}, \alpha \geq \beta_{ij} \geq \delta_{ij}, \alpha \geq \gamma_{ij} \geq \sigma_{ij}
 \end{aligned}$$

$$\begin{aligned}
 \text{Here, } \widetilde{D}_{ij} &= (\widetilde{\mathbf{M}}_{ij} \oplus \widetilde{\mathbf{N}}_{ij})_{(\alpha)} \\
 &= \langle m_{ij(\alpha)}, \omega_{ij(\alpha)}, \beta_{ij(\alpha)}, \gamma_{ij(\alpha)} \rangle \\
 &= \langle 0, 0, 0, 0 \rangle
 \end{aligned}$$

$$\begin{aligned}
 \text{and } \widetilde{E}_{ij} &= \widetilde{\mathbf{M}}_{ij(\alpha)} \oplus \widetilde{\mathbf{N}}_{ij(\alpha)} \\
 &= \langle m_{ij(\alpha)} \oplus n_{ij(\alpha)}, \omega_{ij(\alpha)} \oplus \rho_{ij(\alpha)}, \beta_{ij(\alpha)} \oplus \delta_{ij(\alpha)}, \gamma_{ij(\alpha)} \oplus \sigma_{ij(\alpha)} \rangle \\
 &= \langle 0 \oplus 0, 0 \oplus 0, 0 \oplus 0, 0 \oplus 0 \rangle \\
 &= \langle 0, 0, 0, 0 \rangle
 \end{aligned}$$

i.e., $\widetilde{D}_{ij} = \widetilde{E}_{ij}$

In all the cases, $\widetilde{D}_{ij} \geq \widetilde{E}_{ij}$

$$\therefore \widetilde{\mathbf{M}}_{ij(\alpha)} \oplus \widetilde{\mathbf{N}}_{ij(\alpha)} \geq \widetilde{\mathbf{M}}_{ij(\alpha)} \oplus \widetilde{\mathbf{N}}_{ij(\alpha)}$$

$$\therefore (\mathbf{M} \oplus \mathbf{N})_{(\alpha)} \geq \mathbf{M}_{(\alpha)} \oplus \mathbf{N}_{(\alpha)}.$$

(iii) Let \tilde{P}_{ij} and \tilde{Q}_{ij} be the ij^{th} elements of $M_{(\alpha)} \oplus N_{(\alpha)}$ and $(M \oplus N)_{(\alpha)}$

$$\begin{aligned}
 \therefore \tilde{P}_{ij} &= \widetilde{\mathbf{M}}_{ij(\alpha)} \oplus \widetilde{\mathbf{N}}_{ij(\alpha)} \\
 \text{and } \tilde{Q}_{ij} &= (\widetilde{\mathbf{M}}_{ij} \oplus \widetilde{\mathbf{N}}_{ij})_{(\alpha)}
 \end{aligned}$$

Case 1: $\widetilde{\mathbf{M}}_{ij} \geq \widetilde{\mathbf{N}}_{ij} \geq \alpha$

$$\begin{aligned}
 \therefore \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle &\geq \langle n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} \rangle \geq \alpha \\
 \therefore m_{ij} &\geq n_{ij} \geq \alpha, \omega_{ij} \geq \rho_{ij} \geq \alpha, \beta_{ij} \geq \delta_{ij} \geq \alpha, \gamma_{ij} \geq \sigma_{ij} \geq \alpha
 \end{aligned}$$

$$\begin{aligned}
 \text{Then } \tilde{Q}_{ij} &= (\widetilde{\mathbf{M}}_{ij} \oplus \widetilde{\mathbf{N}}_{ij})_{(\alpha)} \\
 &= \langle m_{ij} + n_{ij} - m_{ij} \cdot n_{ij}, \omega_{ij} + \rho_{ij} - \omega_{ij} \cdot \rho_{ij}, \\
 &\quad \beta_{ij} + \delta_{ij} - \beta_{ij} \cdot \delta_{ij}, \gamma_{ij} + \sigma_{ij} - \gamma_{ij} \cdot \sigma_{ij} \rangle_{(\alpha)} \\
 &= \langle m_{ij} + n_{ij}(1 - m_{ij}), \omega_{ij} + \rho_{ij}(1 - \omega_{ij}), \\
 &\quad \beta_{ij} + \delta_{ij}(1 - \beta_{ij}), \gamma_{ij} + \sigma_{ij}(1 - \gamma_{ij}) \rangle_{(\alpha)} \\
 &\geq \langle m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} \rangle_{(\alpha)} \\
 &> \langle m_{ij(\alpha)}, \omega_{ij(\alpha)}, \beta_{ij(\alpha)}, \gamma_{ij(\alpha)} \rangle \\
 &= \langle x, x, x, x \rangle
 \end{aligned}$$

$$\begin{aligned}
 \text{and } \tilde{P}_{ij} &= \widetilde{\mathbf{M}}_{ij(\alpha)} \oplus \widetilde{\mathbf{N}}_{ij(\alpha)} \\
 &= \langle m_{ij(\alpha)} \oplus n_{ij(\alpha)}, \omega_{ij(\alpha)} \oplus \rho_{ij(\alpha)}, \beta_{ij(\alpha)} \oplus \delta_{ij(\alpha)}, \gamma_{ij(\alpha)} \oplus \sigma_{ij(\alpha)} \rangle
 \end{aligned}$$

$$\beta_{ij(\alpha)} + \delta_{ij(\alpha)} - \gamma_{ij(\alpha)} \cdot$$

$$\begin{aligned} &= < m_{ij(\alpha)} + n_{ij(\alpha)} - m_{ij(\alpha)} \cdot n_{ij(\alpha)}, \omega_{ij(\alpha)} + \rho_{ij(\alpha)} - \omega_{ij(\alpha)} \cdot \rho_{ij(\alpha)}, \\ &\delta_{ij(\alpha)}, \gamma_{ij(\alpha)} + \sigma_{ij(\alpha)} - \gamma_{ij(\alpha)} \cdot \sigma_{ij(\alpha)} > \\ &= < x + x - x, x + x - x, x + x - x, x + x - x > \\ &= < x, x, x, x > \\ &\therefore \tilde{Q}_{ij} > \tilde{P}_{ij} \end{aligned}$$

Case 2: $\tilde{M}_{ij} \geq \alpha \geq \tilde{N}_{ij}$

$$\begin{aligned} &\therefore < m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} > \geq \alpha \geq < n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} > \\ &\therefore m_{ij} \geq \alpha \geq n_{ij}, \omega_{ij} \geq \alpha \geq \rho_{ij}, \beta_{ij} \geq \alpha \geq \delta_{ij}, \gamma_{ij} \geq \alpha \geq \sigma_{ij} \end{aligned}$$

$$\begin{aligned} \text{Then } \tilde{P}_{ij} &= (\tilde{M}_{ij} \oplus \tilde{N}_{ij})_{(\alpha)} \\ &= < m_{ij} + n_{ij} - m_{ij} \cdot n_{ij}, \omega_{ij} + \rho_{ij} - \omega_{ij} \cdot \rho_{ij}, \\ &\beta_{ij} + \delta_{ij} - \beta_{ij} \cdot \delta_{ij}, \gamma_{ij} + \sigma_{ij} - \gamma_{ij} \cdot \sigma_{ij} >_{(\alpha)} \\ &= < m_{ij} + n_{ij}(1 - m_{ij}), \omega_{ij} + \rho_{ij}(1 - \omega_{ij}), \\ &\beta_{ij} + \delta_{ij}(1 - \beta_{ij}), \gamma_{ij} + \sigma_{ij}(1 - \gamma_{ij}) >_{(\alpha)} \\ &\geq < m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} >_{(\alpha)} \\ &> < m_{ij(\alpha)}, \omega_{ij(\alpha)}, \beta_{ij(\alpha)}, \gamma_{ij(\alpha)} > \\ &= < x, x, x, x > \end{aligned}$$

$$\begin{aligned} \text{and } \tilde{P}_{ij} &= \tilde{M}_{ij(\alpha)} \oplus \tilde{N}_{ij(\alpha)} \\ &= < m_{ij(\alpha)} \oplus n_{ij(\alpha)}, \omega_{ij(\alpha)} \oplus \rho_{ij(\alpha)}, \beta_{ij(\alpha)} \oplus \delta_{ij(\alpha)}, \gamma_{ij(\alpha)} \oplus \sigma_{ij(\alpha)} > \\ &= < m_{ij(\alpha)} + n_{ij(\alpha)} - m_{ij(\alpha)} \cdot n_{ij(\alpha)}, \omega_{ij(\alpha)} + \rho_{ij(\alpha)} - \omega_{ij(\alpha)} \cdot \rho_{ij(\alpha)}, \\ &\beta_{ij(\alpha)} + \delta_{ij(\alpha)} - \beta_{ij(\alpha)} \cdot \delta_{ij(\alpha)}, \gamma_{ij(\alpha)} + \sigma_{ij(\alpha)} - \gamma_{ij(\alpha)} \cdot \sigma_{ij(\alpha)} > \\ &= < x + 0 - 0, x + 0 - 0, x + 0 - 0, x + 0 - 0 > \\ &= < x, x, x, x > \end{aligned}$$

$$\therefore \tilde{Q}_{ij} > \tilde{P}_{ij}$$

Case 3: $\alpha \geq \tilde{M}_{ij} \geq \tilde{N}_{ij}$

$$\begin{aligned} \text{i.e., } \alpha &\geq < m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} > \geq < n_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} > \\ \alpha &\geq m_{ij} \geq n_{ij}, \alpha \geq \omega_{ij} \geq \rho_{ij}, \alpha \geq \beta_{ij} \geq \delta_{ij}, \alpha \geq \gamma_{ij} \geq \sigma_{ij} > \end{aligned}$$

$$\begin{aligned} \text{Then } \tilde{Q}_{ij} &= (\tilde{M}_{ij} \oplus \tilde{N}_{ij})_{(\alpha)} \\ &= < m_{ij} + n_{ij} - m_{ij} \cdot n_{ij}, \omega_{ij} + \rho_{ij} - \omega_{ij} \cdot \rho_{ij}, \\ &\beta_{ij} + \delta_{ij} - \beta_{ij} \cdot \delta_{ij}, \gamma_{ij} + \sigma_{ij} - \gamma_{ij} \cdot \sigma_{ij} >_{(\alpha)} \\ &= < m_{ij} + n_{ij}(1 - m_{ij}), \omega_{ij} + \rho_{ij}(1 - \omega_{ij}), \\ &\beta_{ij} + \delta_{ij}(1 - \beta_{ij}), \gamma_{ij} + \sigma_{ij}(1 - \gamma_{ij}) >_{(\alpha)} \\ &= < m_{ij}, \omega_{ij}, \beta_{ij}, \gamma_{ij} >_{(\alpha)} \\ &= < 0, 0, 0, 0 > \end{aligned}$$

$$\text{and } \tilde{P}_{ij} = \tilde{M}_{ij(\alpha)} \oplus \tilde{N}_{ij(\alpha)}$$

$$\begin{aligned}
 &= \langle m_{ij(\alpha)} \oplus n_{ij(\alpha)}, \omega_{ij(\alpha)} \oplus \rho_{ij(\alpha)}, \beta_{ij(\alpha)} \oplus \delta_{ij(\alpha)}, \gamma_{ij(\alpha)} \oplus \sigma_{ij(\alpha)} \rangle \\
 &= \langle m_{ij(\alpha)} + n_{ij(\alpha)} - m_{ij(\alpha)} \cdot n_{ij(\alpha)}, \omega_{ij(\alpha)} + \rho_{ij(\alpha)} - \omega_{ij(\alpha)} \cdot \rho_{ij(\alpha)}, \\
 &\quad \beta_{ij(\alpha)} + \delta_{ij(\alpha)} - \beta_{ij(\alpha)} \cdot \delta_{ij(\alpha)}, \gamma_{ij(\alpha)} + \sigma_{ij(\alpha)} - \gamma_{ij(\alpha)} \cdot \sigma_{ij(\alpha)} \rangle \\
 &= \langle 0+0-0, 0+0-0, 0+0-0, 0+0-0 \rangle \\
 &= \langle 0, 0, 0, 0 \rangle \\
 \therefore \tilde{Q}_{ij} &= \tilde{P}_{ij}. \\
 \therefore \text{In all the cases, } \tilde{Q}_{ij} &\geq \tilde{P}_{ij} \\
 \therefore (\tilde{\mathbf{M}}_{ij} \oplus \tilde{\mathbf{N}}_{ij})_{(\alpha)} &\geq \tilde{\mathbf{M}}_{ij(\alpha)} \oplus \tilde{\mathbf{N}}_{ij(\alpha)}
 \end{aligned}$$

Thus, $(\mathbf{M} \oplus \mathbf{N})_{(\alpha)} \geq \mathbf{M}_{(\alpha)} \oplus \mathbf{N}_{(\alpha)}$.

III. CONCLUSION

Trapezoidal fuzzy number has been applied in many fields such as risk analysis, decision-making, and evaluation. Trapezoidal fuzzy number is applied in collaborative filtering recommendation system, in which trapezoidal fuzzy number is used to express the users' comprehensive evaluation on items. Case demonstration and simulation on Movies Lens show that trapezoidal fuzzy number can express users' comprehensive evaluation on items and the collaborative filtering recommendation accuracy can be higher.

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