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Counting Natural and Integer Solutions to Equations and Inequalities

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***Abstract** - In this paper, formulas for the purpose of counting both natural number and integer solutions to linear equations and inequalities will be gradually derived using combinations with repetitions. General formulas will be derived and each solution will be additionally generalized.*

Keywords - *elementary combinatorics, combinations with repetitions.*

I. INTRODUCTION

When it comes to both secondary and higher mathematical education combinatorics is widely disregarded. One of the reasons might be the fact that the need for applying combinatorial rules and principles arises only within the field of (mathematical) probability. Furthermore, the teaching aspect includes solely the aspects of elementary combinatorics. This implies that these problems can be reduced to three combinatorial rules/principles: the rule of product, the rule of sum and inclusion-exclusion principle. Based on the product rule, three types of arrangements can be defined: permutations, variations and combinations, with or without repetition allowed. To gain complete understanding regarding the application of the derived combinatorial models, it is sufficient to provide answers to the following questions regarding the objects and their arrangements:

1) Are the objects within a certain arrangement distinct, i.e., does the order matter? If the answer is yes, such arrangements are considered to be variations; otherwise, they are observed as combinations.

2) Does the same arrangement allow repetition? If yes, the arrangement is recognized as with repetition allowed; otherwise, there is no repetition allowed. Permutations differ from variations in terms of simultaneous inclusion of all possible elements within an arrangement. see more details in [1]-[3].

Higher-level (advanced) combinatorics is based on partitions, production functions, graphs, recurrence relations, etc. The “higher-level” part implies that it takes more time as well as previously acquired knowledge to understand and master this level of combinatorics. Furthermore, since, in terms of the number of possible arrangements of certain elements, a considerable number of problems is left without an adequate or any solution at all, there is still room for applying new content within the field of combinatorics.

Combinations with repetition play an important role in higher combinatorics in determining the number of partition classes. See for more details in [4]-[7].

This work will demonstrate how the application of procedural steps is more important than general formulas. More complex problems frequently generate formulas which are, in terms of application and memorizing, far more complicated than the application of procedural steps themselves. Hence simple general formulas will be applied where possible; otherwise, only the solution procedures will be demonstrated.

This work will deal with one of the applications of combinations with repetitions allowed. The reason for this lies in the fact that combinations with repetitions are by far the most disregarded counting technique. Further discussion on this subject might easily identify uncritical legacy as the probable reason for disregarding combinations with repetitions. The number of examples involving combinations with repetitions is at least the same as the number of any of the previously mentioned combinatorial models, if not greater. Another reason is of a more practical nature: problems become easier to solve when they are reduced to a certain universal model. One of the models is applied for the purpose of obtaining the solution to the following problem: How many non-negative integer solutions are there to the equation: $x_1 + x_2 + \cdots + x_n = k, n, k \in N.$ (see [8]). If solely the number of positive integer solutions is required, then we must assume that $k \leq n$. Given that this fact is irrelevant, counting the non-negative integer solutions will be assumed.

II. Counting Integer Solutions to Equations

Suppose we have the following problem: In how many ways can k balls be distributed to n boxes. There is an obvious possibility here that some of the boxes are empty. The solution to this problem, as well as to others alike, is reduced to the following question: How many integer solutions are there to the equation:

$$x_1 + x_2 + \cdots + x_k = n,\ \ n, k \in N \tag{1}$$

satisfying the conditions $x_1 \geq 0, x_2 \geq 0, \dots, x_k \geq 0$.

Solution. In general, the simplest way to obtain the solution is to first consider the following cases for $k = 1, 2, 3, \dots$ and determine the number of integer solutions to the equation (1).

The case for k = 1 is trivial, with the only possible solution being obvious.

For k = 2, the equation (1) is given in the form of: $x_1 + x_2 = n, n \in N$.

Solutions to the last equation are the following pairs: (0, n), (1, n - 1), ..., (n, 0). There are n + 1 pairs in total.

For k = 3 the equation takes the form of: $x_1 + x_2 + x_3 = n, n \in N$.

As $0 \leq x_3 \leq n$, the total number of integer solutions are decomposed by assigning values to x_3. The number of solutions to the following equations need to be summed as can be seen from the following table.

Table I

Value of x_3	equation	number of solution
$x_3 = 0$	$x_1 + x_2 = n$	n+1
$x_3 = 1$	$x_1 + x_2 = n - 1$	n
$x_3 = 2$	$x_1 + x_2 = n - 2$	n-1
. . .	. . .	. . .
$x_3 = n$	$x_1 + x_2 = 0$	1

In relation to the previously mentioned, the total number of solutions is given by:

$$1 + 2 + \ \dots + (n + 1) = \frac{(n+1)(n+2)}{2}.$$

Now it is possible to formulate and prove the theorem which determines the total number of solutions to the equation (1).

Theorem 1. If the number of required solutions to the equation (1) is designated by B_k^n then:

a) $B_k^n = \sum_{i=1}^{n+1} B_{k-1}^{n-i+1}$

b) $B_k^n = \binom{n+k-1}{k-1}$

c) $B_k^n = \sum_{i=1}^{k} B_{i-1}^{n+i-1}$.

Proof. a) It is possible to prove the given recurrence formula by applying the same steps as for the case k = 3.

Let us assume the following equation: $x_1 + x_2 + \ \dots + x_k = n$. The last variable x_k can only take values satisfying $0 \leq k \leq n$. By distinguishing all the cases for $x_k, k = 0, 1, \dots, n$, we find that the total number of the solutions is the sum $B_{k-1}^n + B_{k-1}^{n-1} + \ \dots \ + B_{k-1}^0$ which in summation notation is: $B_k^n = \sum_{i=1}^{n} B_{k-1}^{n-i}$, as asserted in the theorem. It is possible to calculate the number by summation and get the sum explicitly, which is asserted in part b).

b) The proof will be constructed by applying transfinite induction. It has already been demonstrated that for j = 1, $B_1^n = \binom{n+1-1}{1-1} = 1$. For j = 2, $B_2^n = \binom{n+2-1}{2-1} = n + 1$. Assume that this assertion is true for any $j \leq i \leq m$, i.e., that $B_j^n = \binom{n+j-1}{j-1}$, for all $j \leq m$.

Let us prove that the assertion is true for $j = i$ + 1. For that purpose, start from the equality we proved:

$$B_{m+1}^n = \sum_{i=1}^{n} B_m^{n-i} = B_m^0 + \cdots + B_m^{n-1} + \ B_m^n$$

By using induction hypotheses, the equality is transformed into:

$$\binom{m-1}{m-1} + \ \binom{m}{m-1} + \ \binom{m+1}{m-1} + \cdots + \ \binom{n+m-1}{m-1}. \tag{2}$$

By successive addition we obtain

$$\binom{m-1}{m-1} + \binom{m}{m-1} = \binom{m+1}{m}$$

$$\underbrace{\binom{m-1}{m-1} + \binom{m}{m-1}} + \binom{m+1}{m-1} = \binom{m+1}{m} + \binom{m+1}{m-1} = \binom{m+2}{m}.$$

$$\underbrace{\binom{m-1}{m-1}+\binom{m}{m-1}+\binom{m+1}{m-1}}+\binom{m+2}{m-1}=\binom{m+2}{m}+\binom{m+2}{m-1}=\binom{m+3}{m}.$$

$$\underbrace{\binom{m-1}{m-1}+\binom{m}{m-1}+\cdots+\binom{m+n-2}{m-1}}+\binom{m+n-1}{m-1}=\binom{m+n-1}{m}+\binom{m+n-1}{m-1}=\binom{m+n}{m}.$$

The last formula proves another recurrence formula expressing the recursive relation among values denoted by B_k^n.

c) The recurrence relation represents the immediate consequence of the relation (2) when given in relation to the result obtained in part b).

Example 1. How many solutions are there to the equation $x_1+x_2+\cdots+x_{100}=2021$, where $x_1 \geq 0,\ x_2 \geq 0, \ldots,\ x_{100} \geq 0$.

Solution. Following the derived formula, the answer is $\binom{2021+100-1}{100-1}=\binom{2120}{99}$. The solution is not easily verified. To verify the formula b) for calculating the number of solutions to the equation (1), the reader can consider the example that allows finding the number of solutions by hand. For instance, how many positive integer solutions are there to the equation $x_1+x_2+x_3=2$. The answer is $\binom{2+3-1}{3-1}=6$, which is easily verifiable, since these are the triples: (1,1,0), (1,0,1), (0,1,1), (2,0,0), (0,2,0), (0,0,2).

A. *Equations with a distinct lower bound*

If there are certain bounds established for the purpose of obtaining a solution satisfying $x_1 \geq a_1, x_2 \geq a_2, \ldots, x_k \geq a_k$, instead of zero, then by applying substitution $y_1=x_1-a_1, y_2=x_2-a_2, \ldots, y_k=x_k-a_k$, for the equation (1) we are reducing it to the prior case. In that case B_k^n of the equation (1) satisfying the conditions $x_1 \geq a_1, x_2 \geq a_2, \ldots, x_k \geq a_k$ is equal to B_k^n of the equation $y_1+y_2+\ \ldots+y_k=n-(a_1+a_2+\cdots+a_k)$ and equals $\binom{n-\sum_{i=1}^{k}a_i+k-1}{k-1}$. If the upper number on the binomial coefficient is less than or equal to zero, then the value is zero.

B. *Equation with upper bounds*

Let us consider how many solutions to the equation (1) there are in the case of double-bounded components. For instance, how many integer solutions are there to the equation (1) satisfying the condition $0 \leq x_1 \leq a_1, 0 \leq x_2 \leq a_2, \ldots, 0 \leq x_k \leq a_k$. First, consider the following example: How many integer solutions are there to the equation $x_1+x_2+x_3=10$, satisfying the condition $0 \leq x_1 \leq 3, 0 \leq x_2 \leq 4, 0 \leq x_3 \leq 5$. The solution might be obtained by the inclusion-exclusion principle. Total number of solutions to the equation satisfying the condition $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ is denoted by $B(S)$. The total number of solutions to the equation satisfying the condition $0 \leq x_1 \leq 3, x_2 \geq 0, x_3 \geq 0$ is denoted by $B(A)$ and $B(A')$ denotes the total number of solutions to the equation satisfying the condition $x_1 > 3, (x_1 \geq 4), x_2 \geq 0, x_3 \geq 0$. Similarly, $B(B)$ denotes the number of solutions to the equation satisfying the condition $x_1 \geq 0, 0 \leq x_2 \leq 4, x_3 \geq 0$, and $B(B')$ denotes the number of solutions to the equation satisfying the condition $x_1 \geq 0, x_2 > 4, (x_2 \geq 5), x_3 \geq 0$. Further, $B(C)$ denotes number of solutions to the equation satisfying the condition $x_1 \geq 0, x_2 \geq 0, 0 \leq x_3 \leq 5$ and $B(C')$ the number of solutions to the equation satisfying the condition $x_1 \geq 0, x_2 \geq 0, x_3 > 5\ (x_3 \geq 6)$.

Thus, by applying the inclusion-exclusion formula, we obtain:

$$B(ABC)=B(S)-B(A')-B(B')-B(C')+B(A'B')+B(A'C')+B(B'C')-B(A'B'C').$$

Now we find that

$$B(S)=\binom{10+3-1}{3-1}=66, B(A')=\binom{6+3-1}{3-1}=28, B(B')=\binom{5+3-1}{3-1}=21, B(C')=\binom{4+3-1}{3-1}, B(A'B')$$
$$=\binom{1+3-1}{3-1}=3, B(A'C')=\binom{0+3-1}{3-1}=1, B(B'C^1)=0, B(A'B'C')=0.$$

In this way we get: 66 - (28 + 21 + 15) + 3 + 1 + 0 – 0 = 6.

Generally, the given procedure gets more complex only in terms of notation. If the equation (1) satisfies the condition $0 \leq x_1 \leq a_1, 0 \leq x_2 \leq a_2, \ldots, 0 \leq x_k \leq a_k$, then B(S) denotes total number of solutions without an upper bound, and A_1 denotes the number of solutions with the bound given by $0 \leq x_1 \leq a_1, x_2 \geq 0, \ldots, x_k \geq 0$ and $B(A_1)$ denotes the number of solutions to the same equation with the bounds given by $x_1 \geq a_1, x_2 \geq 0, \ldots, x_k \geq 0$, and similar applied to the remaining variables, then the total number is obtained by the formula:

$$B(A_1A_2\ldots A_k)=B(S)-\sum_{i=1}^{k}B(A_i')+\sum_{i\neq j=1}^{k}B(A_i'A_j')+\cdots+(-1)^kB(A_1'A_2'\ldots A_k').$$

Note that $B(S)=\binom{n+k-1}{k-1}$ in all cases.

C. *Counting the solutions to equations with two-sided bounds*

It is possible to go one step further in terms of obtaining the total number of bounded solutions with lower bounds other than zero. For instance: $a_1\le x_1\le b_1, a_2\le x_2\le b_2,\ldots,a_k\le x_k\le b_k$. There are at least two approaches to obtain the solution. The first one assumes breaking down the problem into two parts, each observed in relation to the prior case, and obtaining the total number of solutions as the remainder in the number of solutions to the two inequalities. The second approach includes substitutions: $y_1=x_1-a_1, y_2=x_2-a_2,\ldots,y_k=x_k-a_k$ which reduces the problem directly to section 2.

D. *Counting solutions to the equations with coefficients other than one*

If the coefficients of the initial equation have value other than one, for instance, the integer solutions to the following equation:

$$\alpha_1x_1+\alpha_2x_2+\ \ldots+\alpha_kx_k=n,\qquad n,k\in N.$$

are to be counted. This is the case with the distribution of n candies among $\alpha_1,\alpha_2,\ldots,\alpha_k$ children successively living in k houses and n candies are to be distributed in a way such that each child inside a certain house gets an equal number of candies (which does not have to apply to distinct houses). By substitution: $y_1=\alpha_1x_1, y_2=\alpha_2x_2,\ldots,y_k=\alpha_kx_k$, the problem is reduced to its initial form. In order for any k-tuple to be the solution to the equation, it needs to have the form of $(\alpha_1t_1,\alpha_2t_2,\ldots,a_kt_k)$ and $\alpha_1t_1+\alpha_2t_2+\ \ldots+\ \alpha_kt_k=n$. As this equation is reduced to the equation (1), we need to finalize obtaining the total number of solutions by applying the inclusion-exclusion principle. The process is similar to that of section 2. All we need to do is properly form each component. $B(A_1)$ denotes the number of solutions to the equation $\alpha_1x_1+\ x_2+\ \ldots+\ x_k=n,\ \ \alpha_1\in N, x_1\ge 0, x_2\ge 0,\ldots,x_k\ge 0$. Although at first glance it is similar to prior cases, we are facing great complexity when it comes to determining the number denoted by $B(A_1)$ in the general sense. Determining the total number and reducing it to a single formula is not an easy task to do. Let us assume the most basic case of the equation: $2x_1+x_2+\ \ldots+\ x_3=n$. At first, we consider the first variable coefficient to be irrelevant. Let us determine the total number of the solutions with coefficient 1, and then the number of those with the even number as the first component. In order to obtain the total number of *k*- tuples, two cases have to be distinguished: *n* is either even or odd. If *n* is even, then the number equals the sum: $0+2+4+\cdots+n=\frac{(n+2)^2}{4}$. If n is odd, then the number is obtained from the sum $1+3+\ \ldots+n=\frac{(n+1)(n+3)}{4}$. Similarly, when the coefficient equals 3, three cases need to be distinguished and three formulas derived. In the case of *n* = 3*m* **-** 1, the solution is obtained by summing: $0+3+\ \ldots+3m=\frac{3m(m+1)}{2}=\frac{(n+1)(n+4)}{6}$. In the case of *n* = 3*m*, the solution is obtained by summing: $1+4+\ \ldots+(3m+1)=\frac{(n+2)(n+3)}{6}$. In the case of *n* = 3*m* + 1, the solution is obtained by summing: $2+5+\ \ldots+(3m+2)=\frac{(n+2)(n+3)}{6}$. We can now calculate the general case and verify the formula using the prior derivation. Let us assume that *n* = *pm* + *q*. Then the following needs to be summed:

$$(q+1)+\underbrace{p+(q+1)}+\underbrace{2p+(q+1)}+\cdots+\underbrace{mp+(q+1)}=\frac{(n+p-q)(n+q+2)}{2}.$$

Example 2. How many solutions are there to the equation $6x_1+x_2+x_3=28$ satisfying the conditions $x_1\ge 0, x_2\ge 0, x_3\ge 0$. By applying the derived formula, we obtain that there are $\frac{(28+2)(28+6)}{12}=70$ solutions.

Alternative approach. It is possible to take a recursive approach as well. Let us consider one example. How many negative integer solutions are there to the inequality $3x_1+2x_2+4x_3=10$? If we combine the first two variables into one and introduce the substitution $y_2=4x_3$ for the third, we get the equation: $y_1+y_2=10$. The equation has 11 solutions, but only the three with y_2 is divisible by 4. Now we are solving the following equations: $3x_1+2x_2=0$, $3x_1+2x_2=4$ and $3x_1+2x_2=8$. There is a single solution to the first, and when it comes to the second and the third, we repeat the prior procedure. Then we substitute the second equation with $y_1+y_2=4$ with 5 possible solutions, of which only three of them include y_2 divisible by 2. Furthermore, the equation is decomposed to three equations, with only the first one having a single solution, and for the other two the number of solutions is determined trivially. The last step assumes the summation of the obtained numbers.

III. Counting the Integer Number of Solutions to Inequalities

We might have presumed that some of the boxes were empty due to the fact that the number of boxes could be greater than the number of balls. However, there is the possibility that all the balls do not need to be distributed in boxes. In that case, the problem would be modeled by determining the number of all integer solutions to the inequalities:

$$x_1 + x_2 + \dots + x_k \le n, \;\; n, k \in N, \qquad (3)$$

satisfying the condition $x_1 \ge 0, x_2 \ge 0, \dots, x_k \ge 0$.

Theorem 2. If N_k^n denotes the required number of the solutions for (3), then $N_k^n = \binom{n+k+1}{k+1}$.

Proof. The total number of solutions to the equation is obtained by summation of all solutions to the equations:

$$x_1 + x_2 + \dots + x_k = 0,$$
$$x_1 + x_2 + \dots + x_k = 1,$$
$$x_1 + x_2 + \dots + x_k = 2,$$
$$\dots$$
$$x_1 + x_2 + \dots + x_k = n,$$

Following the result obtained through Theorem 1, the sum $\binom{k}{0} + \binom{k+1}{1} + \dots + \binom{n+k-1}{k-1}$ is to be obtained. Following the procedure applied when proving Theorem 1, we find that the sum equals to $\binom{n+k}{k}$.

Example 3. How many integer solutions are there to the following inequality: $x_1 + x_2 + x_3 \le 2$, $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$.

Solution. The number of solutions is the sum of solutions to the equations: $x_1 + x_2 + x_3 = 0$, $x_1 + x_2 + x_3 = 1$, $x_1 + x_2 + x_3 = 2$. These are successively 1, 3 and 6, with the sum of 10, which is the required answer. Following the formula we get $\binom{5}{3} = 10$.

A. *Counting integer solutions to inequalities*

It is possible to apply all prior procedures to inequalities as well. The remainder will arise only in the case of formulas used for obtaining the total number of solutions. If we compare the results for the case of the basic equation (1) and inequality (3) we can observe that the number of solutions to the equation (1) is given by $\binom{n+k-1}{k-1}$, while the number of solutions to the inequality (3) is given by $\binom{n+k}{k}$.

B. *Counting the solutions to equations with negative coefficients*

Although the essay title includes the word equation, the problem can substantially be reduced to inequalities. Assuming that a coefficient in the equation (1) has negative value, then, under the assumption that we count solely integer solutions, the answer is always that there is an infinite number of solutions. To make the number of solutions finite, for a negative coefficient variable, a two-sided bound is to be established. Assume we need to count integer solutions to the equation: $x_1 + x_2 + \dots + x_{j-1} - x_j + x_{j+1} + \dots + x_k = n$, $n, k \in N$ satisfying the condition: $x_1 \ge 0, x_2 \ge 0, \dots, a_1 \le x_j \le b_1, \dots, x_k \ge 0$. As the coefficient for x_j is negative, then the total number of integer solutions is at the same time the number of solutions to the inequality: $n - a_1 \le x_1 + x_2 + \dots + x_{j-1} + x_{j+1} + \dots + x_k \le n + b_1$.

Example 3. How many integer solutions are there to the following inequality: $x_1 - x_2 + x_3 = 2$, satisfying the condition $x_1 \ge 0, -2 \le x_2 \le 2, x_3 \ge 0$? As previously mentioned, the number is equal to the number of solutions to the inequality: $0 \le x_1 + x_3 \le 4$. The number of solutions is 5, and with each quintuple, one value from -2 to 2 is assigned to x_2. In that way we get 25 solution triples to the initial equation.

REFERENCES

[1] Fred Roberts, Barry Tesman, Fred S. Roberts., Applied Combinatorics (2nd Edition), (2021).
[2] Grimaldi R.P., Discrete and Combinatorial Mathematics, 5th edition ,(2005).
[3] M Hall, Combinatorial theory. Blaisdell, (1967).
[4] Aleksa Srdanov, The universal formulas for the number of partitions. , Proc. Indian Acad. Sci. (Math. Sci.), 128(40) (2018).
[5] Aleksa Srdanov, Fractal form of the partition functions p (n). AIMS Mathematics, 5(3) (2020) 2539-2568.
[6] Aleksa Srdanov, Invariants in partition classes. AIMS Mathematics, 5(6) (2020) 6233-6243.
[7] Aleksa Srdanov, One combinatorial problem, many solutions, Collection of papers, Tehnical college, Pozarevac 1 (2011).
[8] Rade Dacic, Elementary combinatorics, Mathematical institute, Belgrade, (1977).