The Number Line as Geometric Model for Representation of Fractions: The case of the Greek Mathematics Primary Textbooks

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Abstract — *This study presents the research conducted on the Greek Mathematics Primary textbooks about the notion of the sequences of rational numbers on the geometric model of the number line. In other words, which and how many representations have been used in the textbooks of primary school for the above notion. In addition, it is commenting on the frequency and kind of representations, as well their correlation with the difficulties students face, based on surveys. Results show that there is a limited scope of activities on Mathematics textbooks about the sequences of rational numbers on the geometric model of the number line in Greece.*

Keywords — *Fractions, Geometric model, Mathematics textbooks, Number line, Representations*

I. INTRODUCTION

Students, at all levels of education (primary and secondary), both nationally and internationally, face significant difficulties in placing fractions in the geometric model of the number line. These difficulties are summed up in the inability of students to place fractions, which are not similar fractions, on a given number line, to find the fractions at given points on the number line, and to correctly define how many equal parts-intervals the number line is divided into. In addition, other important difficulty presented to students of both levels of education is the understanding of the concept of infinity, that is, the existence of infinite rational numbers between two consecutive fractions. ([1], [4], [5], [25]).

Of particular concern is considered the fact that these difficulties present a timelessness and a rigidity in their change, even though in recent years the curricula and textbooks in Greece have changed. Given this, many researchers have explored the possible reasons why students face these difficulties over time. Thus, the international literature focuses, among other things, on how to teach fractions ([22],[27],[25],[6],[2],[23],[14],[33]), in the use or not of the representations of a mathematical concept in its teaching and the variety of representations in textbooks ([15],[10],[16],[26],[8],[30],[31],[32],[7],[11]) and in the perceptions of the teachers and the candidate teachers ([21],[20],[9],[29],[3],[24],[28],[34]).

We observe, then, those different researchers have each attributed to different factors the difficulties faced by students in fractions. Our research team, believing that these difficulties of students are due to a combination of factors that includes, among others, the findings of the above research, attempts through long-term research to highlight these factors aiming to formulate and propose solutions to reduce these difficulties of students with interventions in the Greek educational system.

In the context of the present research study, because students face difficulty in placing fractions on the number line, our research team has included the study of Greek mathematics textbooks. More specifically, the present work studies the forms of representations and the frequency with which they are presented in the Greek textbooks of elementary school mathematics, for the concept of serialization of fractions in the geometric model of the number-line recording the evolutionary course of these concepts while students moving from one classroom to another. The findings are discussed to determine whether some factors in the structure and content of elementary school textbooks contribute to students' difficulties in explicit numbers.

II. TEXTBOOK RESEARCH-METHODOLOGY

A. Purpose of the Study

The present study aims to research the structure of the Greek textbooks of Mathematics of the elementary school on the expressions. More specifically, we intend to research the forms of representations as well as the frequency presented in the mathematics textbooks for the concept of sequencing fractions in the geometric model of the number line while recording the evolutionary course of these concepts as students move from one class to another. More specifically, the sub-objectives of the

study are as following:
- To identify the representations used in elementary school textbooks in the sense of serialization of fractions in the geometric model of the number-line.
- To examine the representations in the concept of serialization of fractions in the geometric model of the number line in terms of their suitability.
- To make a possible correlation of these representations with the difficulties faced by students on this concept.

B. Research Tools

To achieve the objectives of the research, the Greek books of Mathematics and the six grades of primary school were studied (from the first grade of elementary school to the sixth grade of primary school). The group of 38 books in total studied includes student books, workbooks, and books for the teacher.

C. Data Analysis

In the context of the study, the descriptive analysis of the data collected from the textbooks was followed.

III. RESULTS

A. Mathematics Primary books for the 1st, 2nd and 3rd Grade

The number line as a geometric model of representation in general, whether it is natural or explicit numbers, is completely absent from the textbooks of the 1st and 2nd grade of primary schools. It should be noted, of course, that fractions are not taught in these two classes. The placement of fractions in the number line is presented for the first time in the 3rd grade of elementary school, in the chapter 24 "Fractional units and simple fractional numbers", page 63 of the student's book (Figure 1). The main aim of the chapter is "to apply fractions and fractions to straight segments" [19]. More specifically, in the exercise of Figure 1, students are asked based on a quarter of a kilo "to determine on a straight-line segment divided into four parts 1/2, 2/4 or 1/2, 3/4 and 4 / 4 or 1. There is a discussion about the 4/4 that is equal to the unit "[19]. We notice that the directions given in the teacher's book indicate that the straight section is divided into four parts and not into four equal parts. This is important, as research has highlighted the difficulty students face in understanding the need for fractional unit equilibrium [1].
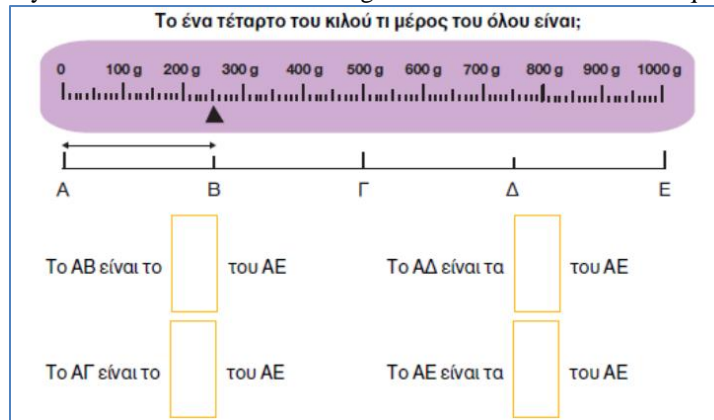

Figure 1. The first number line as a representation of fractions, 3rd Grade, Chapter 34, Student's Book

The next representation of the number line for fractions are presented in the 34th Chapter "Decimal fractions" of the workbook, p. 24 (Figure 2) of the 3rd grade primary school. The pronunciation of the exercise calls the representation straight and in the teacher's book there is no relevant goal for placing decimal fractions on the number line. At this point, we can also notice that the points on the number line are quite densely designed, which makes it difficult for students of this age, who are not familiar enough with the geometric model of the number line.
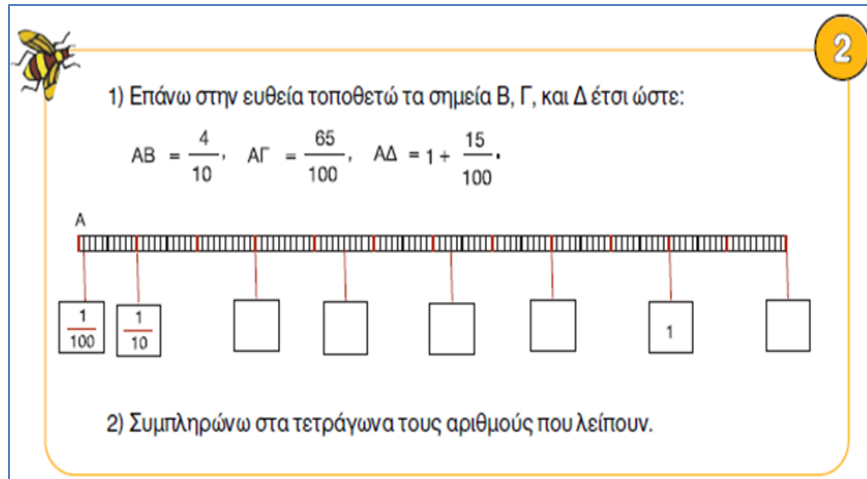
Figure 2. Placement of decimal fractions in the number line, 3rd Grade, Chapter 34, Workbook

Summarizing the findings from the textbooks for 1st, 2nd and 3rd grade primary school, it is observed that the number line as a geometric model of representation of fractions appears for the first time in the 3rd grade primary school, and in total there are only two exercises in the textbooks in which it is used.

B. Mathematics primary books for the 4th, 5th, and 6th Grade

Continuing the analysis of the number-line as a geometric model of representation of the fractions in the other classes, we observe that it is completely absent from the textbooks of the 4th grade. Regarding the 5th grade, the number line as a geometric model for the representation of fractions is shown in chapter 9 "Value of a digit in decimal numbers" in student's book, p. 30 (Figure 3). Here students are asked to place decimal numbers, but the corresponding decimal fractions are also on the number line as aids. The chapter does not aim to familiarize students with placing fractions on the number line.
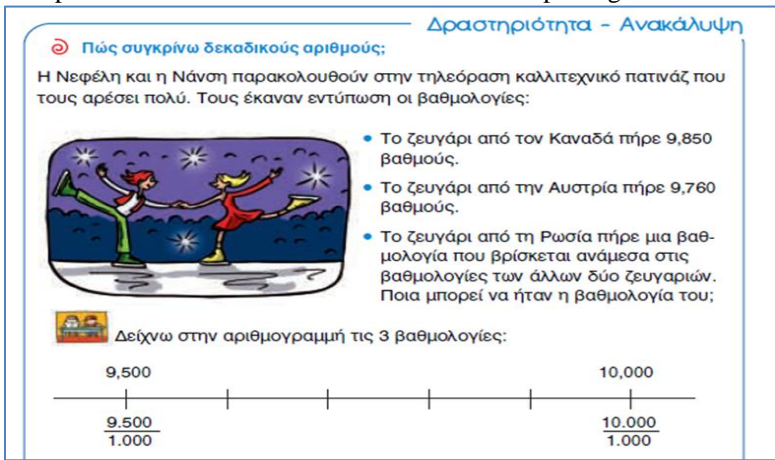

Figure 3. Placement of decimal fractions in the number line, 5th Grade, Chapter 9, Student's Book

The next representation in 5th grade is found in chapter "Fractional Units" in the student's book, p.47 (Figure 4). Students are asked to show on the number line the fractional units that correspond to the geometric shapes. In the same chapter, in the workbook, pp. 10 and 11 there are two pure exercises, which ask students to place fractions on the number line (Figure 5 and 6). In the activity of figure 5, the students are asked to place the fractions with the division strategy, that is, to convert the fractions into decimals with the help of the pocket calculator. In addition to this strategy, the teacher's book presents another fractional serialization strategy of completing the unit (how much more we need to build the unit) and cites as an example: "1/4 and 1/5, 3/4 is less than 4/5, so 1/4> 1/5 " [17]. But what strategy will students apply to conclude that 3/4 is less than 4/5? Is directly comparing fractional units (which is the larger fraction with the smallest denominator) a simpler strategy, for the example that handles fractional units?
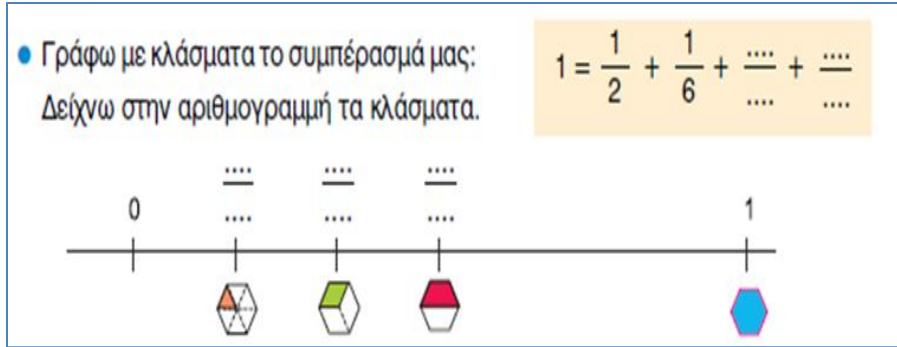
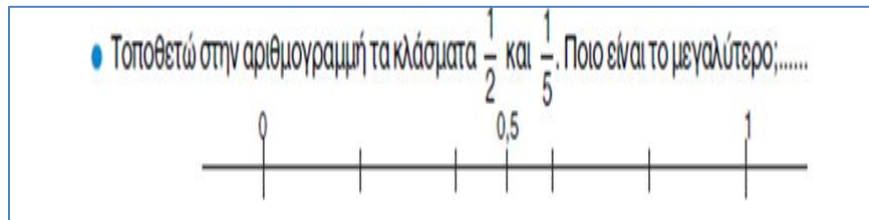**Figure 4. Placement of decimal fractions in the number line, 5th Grade, Chapter 16, Student's Book**

Figure 5. Placement of decimal fractions in the number line, 5th Grade, Chapter 16, Workbook

The representation of Figure 6 is a very good activity of implementing strategies for serialization of fractions using elements of the open problem and the possibility of choosing individual options. That is, students can construct different fractions and order them based on the strategy or strategies that best suit them. Although the above is not explicitly mentioned in the teacher's book, it is nevertheless an activity with many capacities, if used properly by the teacher. We note that in all three representations above the teaching of the number line is not a recorded goal of the chapter.

Figure 6. Placement of decimal fractions in the number line, 5th Grade, Chapter 16, Workbook

Continuing with the findings of the books for the 5th grade, in chapter 18 "Conversion of a fraction to a decimal" of the workbook, p. 14 the last numerical line is presented as a geometric model of representation of fractions (Figure 7). The strategy here in the teacher's book is to convert the fraction to a decimal number. Nor is this chapter recorded in the teacher's book with the aim to acquaint the students with the placement of fractions in the number line.
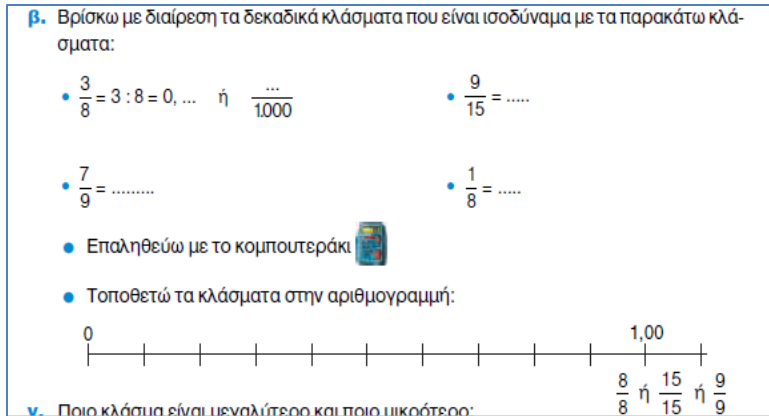
Figure 7. Placement of decimal fractions in the number line, 5th Grade, Chapter 18, Workbook

Regarding the textbooks of Mathematics for the 6th grade of primary school, the number line as a geometric model of representation of fractions is presented in chapter 3 titled "Conversion of decimals into fractions and vice versa" of the student's book, p. 13 (Figure 8). The activity aims to understand the need to convert fractions to decimal numbers and not to manage fractions on the number line.

Figure 8. Placement of decimal fractions in the number line, 6th Grade, Chapter 3, Student's Book

In 20th chapter titled "The fraction as an exact quotient of division" a chapter appears for the first time in primary school textbooks with the aim of students noting the position of the fraction on the number line (from its decimal value). More specifically, through the activities in Figure 9, 10 and 11, students become familiar with the strategy of converting a fraction to a decimal to distinguish where in the number-line corresponds the number expressed by a fraction.

Figure 9. Placement of decimal fractions in the number line, 6th Grade, Chapter 20, Student's Book
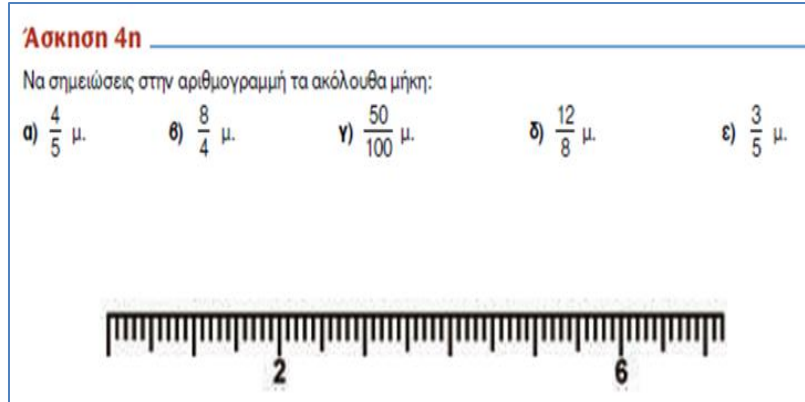
Figure 10. Placement of decimal fractions in the number line, 6th Grade, Chapter 20, Workbook
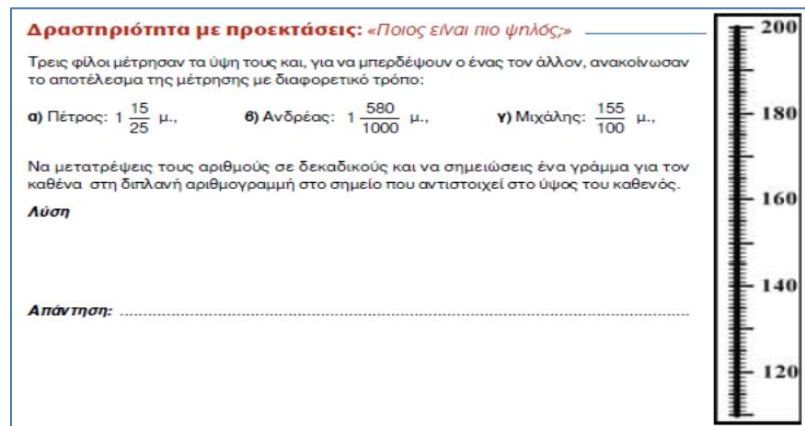

Figure 11. Placement of decimal fractions in the number line, 6th Grade, Chapter 20, Workbook

In chapter 22 titled "Comparison - Arrangement of fractions", of the student book, p.51, the textbook continues the effort to acquaint students with strategies for serialization of fractions in the number line through a specific goal set by the teacher's book: "The goal of the chapter is the student to place fractions on the number line" [18]. Achieving the goal is accomplished through the activity in Figure 12, where students arrange fractions using the homonymous fraction strategy and learn that they can find which fraction is interspersed between two others by comparing their numerators.
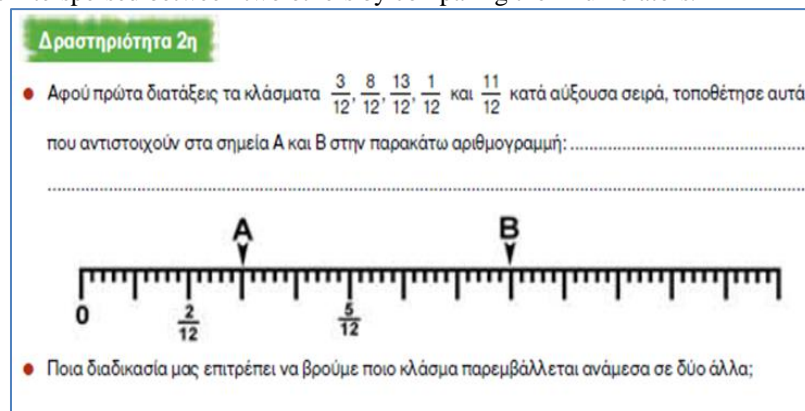

Figure 12. Placement of decimal fractions in the number line, 6th Grade, Chapter 22, Student's Book

The above strategy is also used in activity 5 of Figure 13 of chapter 40 "Estimating the percentage" of the workbook, p.14.
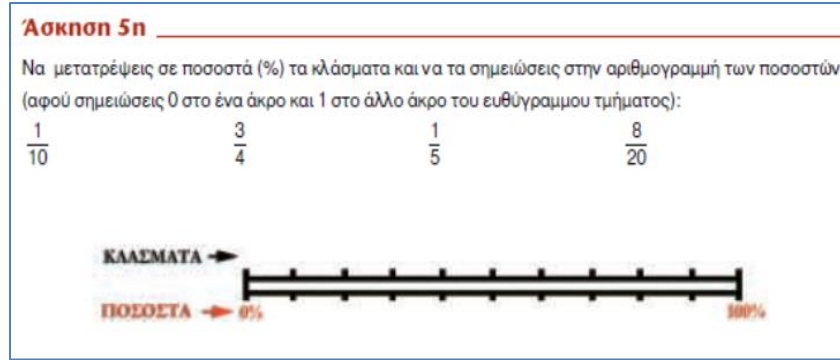
Figure 13. Placement of decimal fractions in the number line, 6th Grade, Chapter 40, Student's Book

With the representation of figure 14 of the repetitive chapter "Numbers and operations" of student's book, p. 58 ends the section with fractions which is the only section that introduces the students to strategies of serialization of fractions in the number line. And in this case, it is noted that these activities are throughout book seven and are limited to four chapters (chapter 3, 20, 22 and 40), perhaps not enough for the respective difficulties faced by students in placing fractions on the number line.
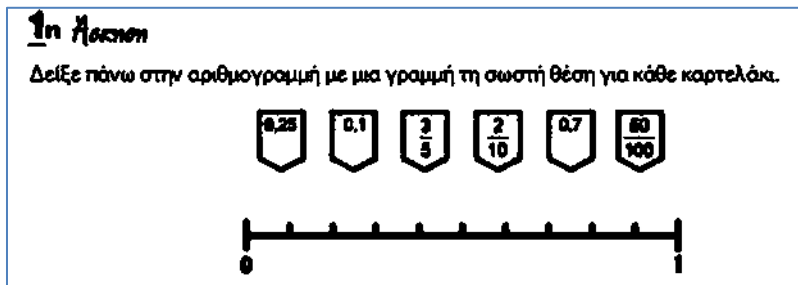

Figure 14. Placement of decimal fractions in the number line, 6th Grade, Summary, Student's Book

IV. DISCUSSION

Summarizing the findings of the textbooks of Greek Mathematics, we notice that the number line as a geometric model of the representation of fractions does not exist in the 1st and 2nd grade of primary school, as fractions are not taught in these grades. It appears for the first time in the 3rd grade of primary school and in total there are two representations in the books of this class, one in chapter 24 and one in chapter 34. Regarding the 1st, 2nd, and 6th grade of primary school, we observe that the number line as a geometric model of the representation of fractions is completely absent from the books of the 4th grade, while in the 5th grade there are five representations in three chapters (chapter 9, 16 and 18) but which aim to familiarize students with decimal numbers. However, strategies for serialization of fractions in the numerical line are presented such as comparison concerning half, unit complement, common numerators. However, there are not enough and obvious activities to support these strategies.

Thus, the systematic and conscious learning of the number line for the placement of fractions takes place for the first time in the 6th grade of the primary school in a more organized context, with specific set goals and implementation of specific strategies to achieve them. More specifically, in the sixth-grade, two strategies for serialization of fractions in the number line are used, for which there are specific activities (a total of seven in four chapters) for their implementation and are presented with specific objectives in both the BM and the teacher's book. These strategies are the conversion of a fraction to a decimal number and the homonymous fractions. Although in the sixth-grade book there are the first organized sections for teaching the placement of fractions on the number line, however, both the length of the chapters and the strategies used (only two strategies) can be considered sufficient for such an important, but also a difficult concept.

In addition, for some of the difficulties of the students mentioned in the literature, regarding the placement of fractions in the number line, such as to correctly define how many equal parts-intervals the number line is and the difficulty of understanding the concept of infinity, that is, of the existence of infinite explicit numbers between two successive fractions, there do not seem to be any related activities in the textbooks. Thus, these difficulties of the students remain throughout primary school.

The significant difficulties that students face in placing fractions in the arithmetic line can be said to be due to the lack of fields of representation that textbooks have, as research has shown that the more often the student comes in contact with a form of representation so the more familiar it becomes to him and the better he learns it ([13],[16]). An example is the research carried out by Hodgen et al. [13] who used for their research in 2008 the essays of an earlier study of 1977 to compare the results regarding the understanding of fractions in their decimal form in about 3,000 students aged 11-14 years. The results showed that the students' performance in placing the decimal numbers on the number line improved, as in 1977 the success rates were 50%, and in 2008 it was 83%. The explanation for this difference in success rates was the fact that number line representation has been widely used in primary school curricula in recent years, which was not the case 30 years ago.

It would be useful, therefore, to introduce more dynamically the concept of number line regarding fractions in Greek primary school textbooks and to make its wider use, as this concept is related to the construction of fractional numbers that pave the way for the development of a sense of coherence and continuity of numbers [12].

V. DISCUSSION

In the present study, all the chapters of the Greek textbooks of the primary school mathematics were presented, which include the number line as a geometric model for the representation of fractions. Also presented how this concept evolves within the textbooks of all six grades of the primary school, the representations with which the concept is introduced, as well as their frequency.

According to the findings of the study, the representation of the number line for the serialization of the fractions appears for the first time in the 3rd grade of the primary school through two activities, which are the only ones, while in the 4th grade there is no relevant representation at all. In the 5th grade, the activities that use the number line as a geometric model for the representation of the fractions increase from two in the 3rd grade to five. The conscious introduction of the students in this sense takes place in the 6th grade with the presence of seven representations that use the number line for the serialization of fractions.

In conclusion, in the Greek textbooks of Primary Mathematics, there are relatively few activities related to the number line as a geometric model of representation of fractions in proportion to the difficulty presented by this concept to students of both levels of education. This incomplete presence of representations may be related to some extent to the difficulties that students face in placing fractions on the number line, as research suggests.

Summarizing and as already mentioned in the introduction, the representations in the books are not the only factor responsible for these difficulties, as well as the way of teaching and the knowledge of the teachers are an important reason for the students' difficulties in Mathematics. The harmonious combination of these three elements, books, teaching, teacher, should be the pillar that will support the effort to reduce students' difficulties in fractions.

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