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In-Between Forward and Central Difference Approximation for Evenly Spaced Data Using the Shift Operator

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Abstract - *A number of different methods have been developed to construct useful interpolation formulas for evenly and unevenly spaced points. We developed a new interpolation formula obtained through a combination of Newton's Gregory forward interpolation and a modified form of Gauss backward interpolation formula using the shift operator. This was achieved through the reduction in the subscripts of Gauss's backward formula by one unit and replacing s by s-1. Comparison of the newly developed method with their counterparts was also carried out and results show that the new formula is very efficient and possess good accuracy for evaluating functional values between given data.*

Keywords - *Shift operator, forward difference interpolation, Gauss's backward formula, Symbolic method, evenly spaced points.*

[1] INTRODUCTION

The word "interpolation" originated from the Latin verb ''interpolare'', a contraction of "inter," meaning "between," and "polare," meaning "to polish." That is to say, to smooth in between given pieces of information. Interpolation appears to have been used first in a mathematical sense by Wallis in his 1655 book on infinitesimal arithmetic. The forward differences

y_1-y_0, y_2-y_1, y_3-y_2. . .y_n-y_{n-1} when denoted by dy_0, dy_1, dy_2, dy_3 . . . dy_{n-1} respectively.

A number of different methods have been advanced to construct interpolation formulae for evenly and unevenly spaced points. For example, Newton's formula for constructing the interpolation polynomial made use of divided difference formula (Atkinson, K.E. 1989; Conte, S.D. and Boor, C. 1980). The inference drawn from the formula suggests that, there exists many number of interpolation formulas using differences through difference table for evenly space data. The process of finding the value of y corresponding to any value of $x = x_n$ between x_0 and x_n is called interpolation.

The Newton's forward and Gauss's formula belong to the forward and central difference interpolation respectively. The Gauss interpolation formulas consist in the fact that the selection of interpolation nodes ensures the best approximation of the residual term of all possible choices, while the ordering of the nodes by their distances from the interpolation point reduces the numerical errors in the interpolation.

II. REVIEW OF RELATED LITERATURE

A. Background History

The presentation of the two interpolation formulae in the Principia is heavily condensed and contains no proofs. Newton's Method Differential contains a more elaborate treatment, including proofs and several alternative formulae. Three of those formulae for uniformly spaced data were discussed a few years later by Stirling (1719). These are the Gregory–Newton formula and two central-difference formulae, the first of which is now known as the Newton-Stirling formula. It is interesting to note that Brahmagupta's formula is, in fact, the Newton-Stirling formula for the case when the third and higher order differences are zero. A very elegant alternative representation of Newton's general formula that does not require the computation of finite or divided differences was published in 1779 by Waring.

In 1812, Gauss delivered a lecture on interpolation, the substance of which was recorded by his then student, Encke (1830), who first published it not until almost two decades later. Apart from other formulae, he also derived the one which is now known as the Newton-Gauss formula. In the course of the 19th century, two more formulae closely related to Newton-Gauss formula were developed. The first appeared in a paper by Bessel (1824) on computing the motion of the moon and was published by him because, in his own words, he could "not recollect having seen it anywhere." The formula is, however, equivalent to one of Newton's in his Methodus Differentialis, which is the second central-difference formula discussed by Stirling (1719) and has, therefore, been called the Newton–Bessel formula. The second formula, which has frequently been used by statisticians and actuaries, was developed by Everett (1900), (1901) around 1900 and the elegance of this formula lies in the fact that, in contrast with the earlier mentioned formulae, it involves only the even-order differences of the two table entries between which to interpolate.

It is justified to say that "there is no single person who did so much for this field, as for so many others, as Newton", (H. H. Goldstine, 1977). His eagerness becomes clear in a letter he wrote to Oldenburg (1960), where he first describes a method by which certain functions may be expressed in series of powers. The contributions of Newton to the subject are contained in: (1) a letter to Smith in 1675 (I. Newton, 1959); (2) a manuscript entitled MethodusDifferentialis (I. Newton, 1981), published in 1711, although earlier versions were probably written in the middle 1670s; (3) a manuscript entitled Regula Differentiarum, written in 1676, but first discovered and published in the 20th century (D. C. Fraser, 1927); and (4) Lemma V in Book III of his celebrated Principia (I. Newton, 1960), which appeared in 1687. The latter was published first and contains two formulae. The first deals with equal-interval data, which Newton seems to have discovered independently of Gregory. The second formula deals with the more general case of arbitrary-interval data.

The general interpolation formula for equidistant data was first written down in 1670 by Gregory (1939) that can be found in a letter written to Collins. Particular cases of it, but, had been published several decades earlier by Briggs, the man who brought to fruition the work of Napier on logarithms. In the introductory chapters to his major works (H. Briggs, 1624, 1633), he described the precise rules by which he carried out his computations, including interpolations, in constructing the tables contained therein.

By the beginning of the 20th century, the problem of interpolation by finite or divided differences had been studied by astronomers, mathematicians, statisticians, and actuaries. Many of them introduced their own system of notation and terminology, leading to confusion and researchers reformulating existing results. The point was discussed by Joffe (1917), who also made an attempt to standardize yet another system. It is, however, Sheppard's (1899) notation for central and mean differences that has survived in later publications. Most of the now well-known variants of Newton's original formulae had been worked out. This is not to say, however, that there are no more advanced developments to report on quite to the contrary. Already in 1821, Cauchy (1821) studied interpolation by means of a ratio of two polynomials and showed that the solution to this problem is unique, the Waring–Lagrange formula being the special case for the second polynomial equal to one. It was

Cauchy also who, in 1840, found an expression for the error caused by truncating finite-difference interpolation series (A. Cauchy, 1841). The absolute value of this so-called Cauchy remainder term can be minimized by choosing the abscissa as the zeroes of the polynomials introduced later by Tchebychef (1874).

A generalization of a different nature was published in 1878 by Hermite, who studied and solved the problem of finding a polynomial of which also the first few derivatives assume pre-specified values at given points, where the order of the highest derivative may differ from point to point. Birkhoff (1906) studied the even more general problem: given any set of points, find a polynomial function that satisfies pre-specified criteria concerning its value and/or the value of any of its derivatives for each individual point. Birkhoff interpolation, also known as lacunary interpolation, initially received little attention, until Schoenberg (1966) revived interest in the subject. Hermite and Birkhoff type of interpolation problems—and their multivariate versions, not necessarily on Cartesian grids—have received much attention in the past decades.

III. METHODOLOGY

A. Difference Table

The difference table is the standard format for displaying finite difference its diagonal pattern makes each entry (except for x and f (x)) the difference of its two nearest neighbors to the left.

Table 1: FORWARD DIFFERENCE TABLE FOR NGF

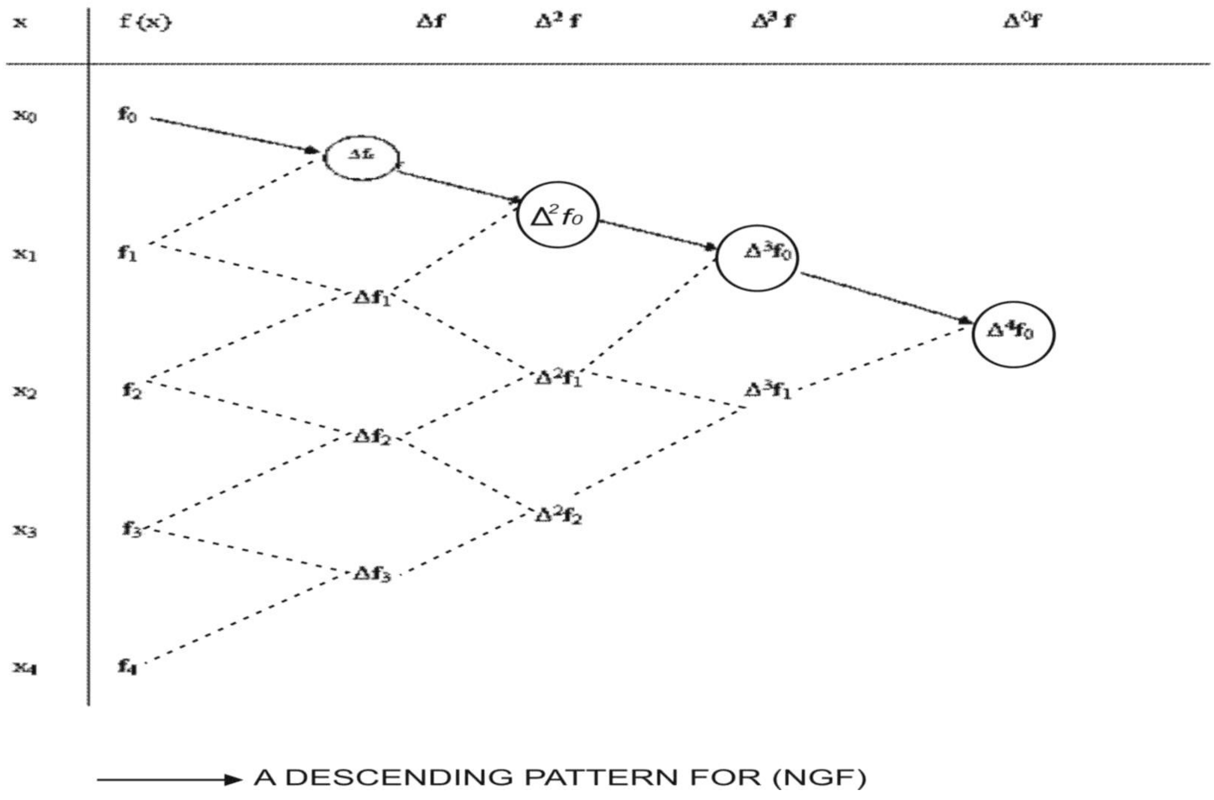
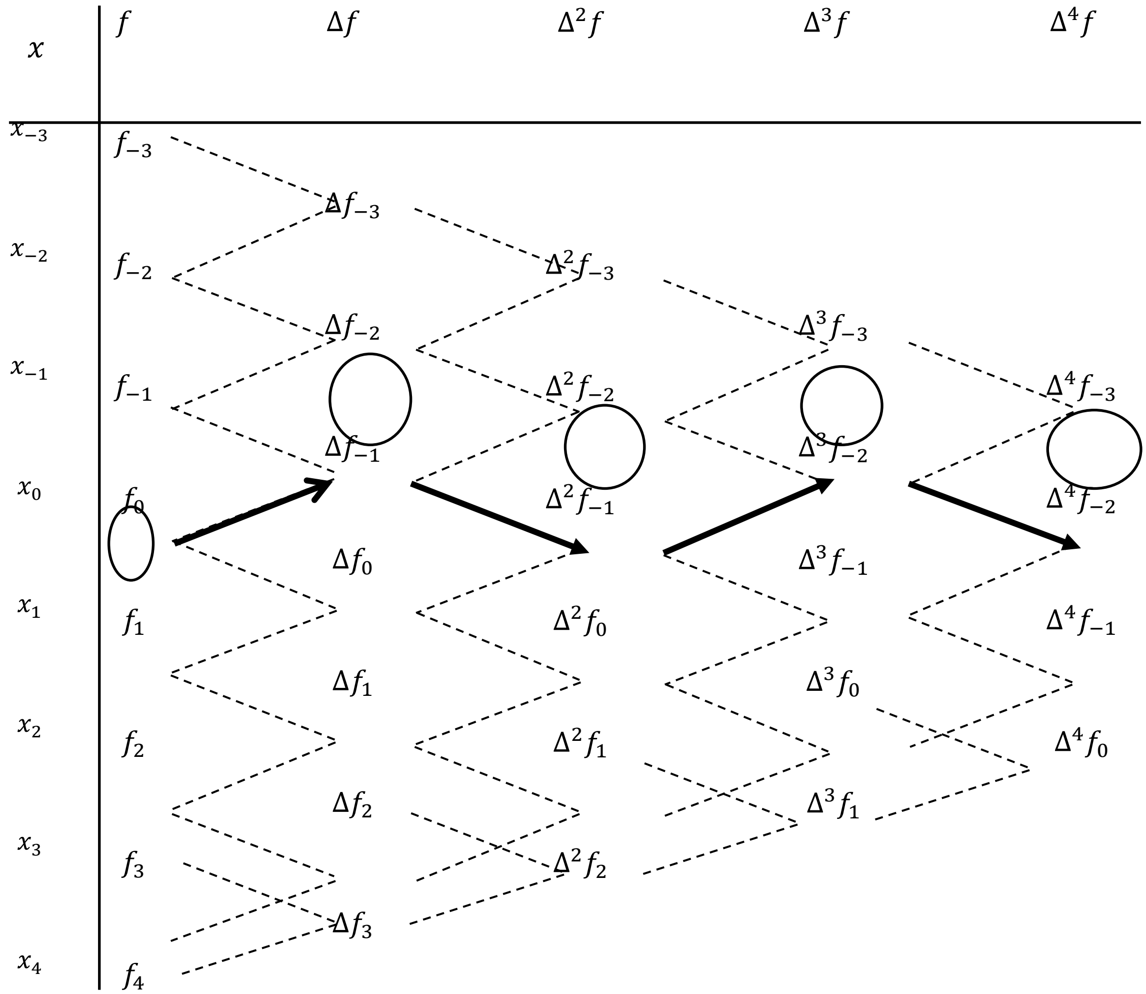


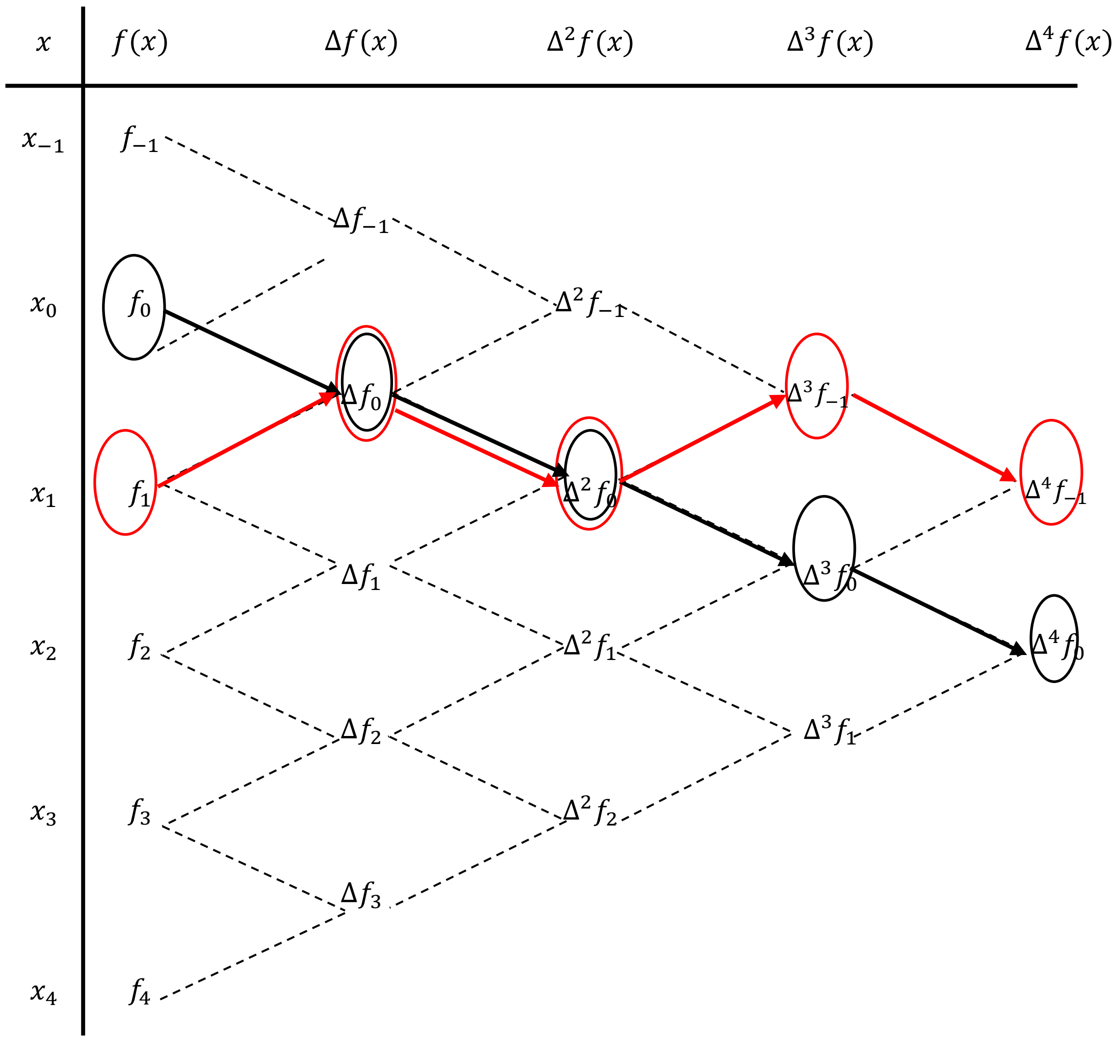
Table 2: CENTRAL DIFFERENCE TABLE FOR GAUSS'S BACKWARD FORMULA (GBF)



Key:

→ zig-zag movement pattern for GBF

TABLE 3: COMBINATION OF DESCENDING AND ZIG-ZAG PATTERN OF THE NEW METHOD



Key:

Computational model for NGF

Computational model for GBF

Descending movement pattern for NGF

Zig-Zag movement pattern for GBF

B. Derivation of Methods through the Shift Operator

1. EF (x) = f (x+h): Shifting Operator.
2. Δf (x) = f (x+h) –f (x): Forward Difference Operator.
3. $EF_0 = EF(x_0) = f(x_0 + h) = f(x_1) = f_1$
4. $E^2F(x) = f(x+2h), \ldots, E^n f(x) = f(x+nh)$

   following 1-4 we deduce that:

   $E^s f(x) = f(x+sh)$

5. $\Delta f(x) = EF(x) - f(x) = (E-1) f(x) \Longrightarrow \Delta f(x) = (E-1) f(x)$

   that is, $\Delta = E-1$ or $E = \Delta + 1$

a) Derivation of NGF

The consequence of 1-5 yields:

$$E^s f_0 = (1+\Delta)^s f_0 = [1 + s\Delta + \binom{S}{2}\Delta^2 + \binom{S}{3}\Delta^3 + \ldots + \binom{S}{n}\Delta^n] f_0$$

$$P_n(x_s) = f_0 + s\Delta f_0 + \frac{S(S-1)}{2!}\Delta^2 f_0 + \frac{S(S-1)(S-2)}{3!}\Delta^3 f_0 + \qquad (3.1)$$

Equation (3.1) is the Newton's Gregory Forward formula.

b) Gauss Backward Formula (GBF)

$$P_n(x_s) = f_0 + \binom{S}{1}\Delta f_{-1} + \binom{S+1}{2}\Delta^2 f_{-1} + \binom{S+1}{3}\Delta^3 f_{-2} + \ldots \left(S + [\tfrac{n}{2}]\right)\Delta^n f_{-n}$$

$$= f_0 + s\Delta f_{-1} + \frac{S(S+1)}{2!}\Delta^2 f_{-1} + \frac{S(S2-1)}{3!}\Delta^3 f_{-2} + \frac{S(S2-1)(S+2)}{4!}\Delta^4 f_{-2} + \qquad (3.2)$$

c) Modified GBF (Gauss Backward Formula)

Advancing the subscripts of equation (3.2) (Gauss Backward Formula) by one unit and replacing s by (s-1) yields.

$$P_n(x_s) = f_1 + (s-1)\Delta f_0 + \frac{S(S-1)}{2!}\Delta^2 f_0 + \frac{S(S-1)(S-2)}{3!}\Delta^3 f_{-1} + \frac{S(S-1)(S-2)(S+1)}{4!}\Delta^4 f_{-1+} \qquad (3.3)$$

d) Newly Derived Method

Taking the mean of (3.1) and (3.3) we obtain the new method in the form.

$$P_n(x_s) = \frac{1}{2}(f_0 + f_1) + \frac{(2S-1)}{2}\Delta f_0 + \frac{S(S-1)}{2}\Delta^2 f_0 + \frac{S(S-1)(S-2)}{12}[\Delta^3 f_0 + \Delta^3 f_{-1}] + \frac{S(S-1)(S-2)}{48}[(s-3)\Delta^4 f_0 + (s+1)\Delta^4 f_{-1}] + \qquad (3.4)$$

Equation (3.4) is the newly proposed method.

Other related methods include the following:

e) Stirling's Interpolation method

$$P_n(x_s) = f_0 + \frac{S(\Delta f_{-1} + \Delta f_0)}{2} + \frac{S^2}{2!}\Delta^2 f_{-1} + \frac{S(S^2-1)}{3!}\frac{(\Delta^3 f_{-2} + \Delta^3 f_{-1})}{2} + \frac{S^2(S^2-1)}{4!}\Delta^4 f_{2+} \qquad (3.5)$$

f) Bessel's Formula

$$\mathbf{P_n(x_s) = \frac{(f_0+f_1)}{2} + (s-\frac{1}{2})\,\Delta f_0 + \frac{(S-1)}{2!}(\Delta^2 f_{-1} + \Delta^2 f_0) + \frac{S\left(S-\frac{1}{2}\right)(S-1)}{3!}\Delta^3 f_{-1} + \frac{S(S^2-1)\,(S-2)}{4!}(\Delta^4 f_{-2} + \Delta^4 f_1) + \ldots} \qquad (3.6)$$

3.2.7 Laplace – Everett's Formula

$$P_n(x_s) = [vf_0 + \frac{(v^2-1)\,v}{3!}\Delta^2 f_{-1} + \frac{(v^2-1)\,v\,(v^2-2^2)}{5!}\Delta^4 f_{-2} + \ldots] + [sf_1 + \frac{(S^2-1)\,S}{3!}\Delta^2 f_0 + \frac{S\,(S^2-1)\,(S^2-2^2)}{5!}\Delta^4 f_{-1} + \ldots] \qquad (3.7)$$

where, $s = \frac{(x-x0)}{h}$ and v = 1-s

IV. NUMERICAL EXPERIMENT

Problem 4.1: given the following table of values of e^{-x}, find the value of $e^{-1.9}$

x	1.0	1.25	1.50	1.75	2.00	2.25	2.50
f(x)	0.3679	0.2865	0.2231	0.1738	0.1353	0.1054	0.0821

TABLE 4: DIFFERENCE TABLE FOR e^{-x}

x	f(x)	Δf(x)	Δ^2f(x)	Δ^3f(x)	Δ^4f(x)
1.0	0.3679				
		-0.0814			
1.25	0.2865		0.0180		
		-0.0634		-0.0039	
1.50	0.2231		0.0141		0.0006
		-0.0493		-0.0033	
1.75	0.1738		0.0108		0.0011
		-0.0385		-0.0022	
2.00	0.1353		0.0086		0.0002
		-0.0299		-0.0020	
2.25	0.1054		0.0066		
		-0.0233			
2.50	0.0821				

Gauss Backward formula

$$P_n(x) = f_o + s\Delta f_{-1} + \frac{S(S+1)}{2!}\Delta^2 f_{-1} + \frac{S(S^2-1)}{3!}\Delta^3 f_{-2} + \frac{S(S^2-1)(S+2)}{4!}\Delta^4 f_{-2} + \ldots$$

$$P_4(1.9) = 0.2231 + (1.6)(-0.0634) + \frac{(1.6)(2.6)}{2}(0.0141) + \frac{(1.6)(1.56)(-0.0039}{6} + \frac{(1.6)(1.56)}{24}(3.6)(0.0006) = 0.14959024$$

Newton's Gregory Forward Formula

$$P_n(x_s) = f_0 + S\Delta f_0 + \frac{S(S-1)}{2!}\Delta^2 f_0 + \frac{S(S-1)(S-2)}{3!}\Delta^3 f_0 + \frac{S(S-1)(S-2)(S-3)}{4!}\Delta^4 f_0$$

P_4 (1.9) = 0.2231 + (1.6) (-0.0493) + $\frac{(1.6)(1.6-1)}{2}$ (0.0108) + $\frac{(1.6)(1.6-1)(1.6-2)}{6}$ (-0.0022) + $\frac{(1.6)(1.6-1)(1.6-2)(1.6-3)(0.0002)}{24}$ = 0.14954928

Stirling's Interpolation Formula

$P_n(x_s)=f_0+\frac{S(\Delta f-1+\Delta f-0)}{2}+\frac{S^2}{2!}\Delta^2 f_{-1}+\frac{S(S^2-1)}{3!}\frac{(\Delta^3 f-2+\Delta^3 f-1)}{2}+\frac{S^2(S^2-1)}{4!}\Delta^4 f_{-2}$

P_4 (1.9) = 0.2231+(1.6) $\frac{(-0.0634)+(-0.0493)}{2}$ + $\frac{(1.6)^2}{2}$ (0.0141) + $\frac{(1.6)((1.6)^2-1)}{6}\frac{(-0.0039)+(-0.0033)}{2}$ + $\frac{(1.6)^2((1.6)^2-1)}{24}$ (0.0006) = 0.1495902

Bessel's Interpolation Formula

$P_n(x_s)=\frac{(f0+f1)}{2}+(S-\frac{1}{2})\Delta f_0+\frac{S(S-1)}{2!}\frac{(\Delta^2 f-1+\Delta^2 f0)}{2}+\frac{S(S-\frac{1}{2})(S-1)}{3!}\Delta^3 f_{-1}+\frac{S(S^2-1)(S-2)}{4!}\frac{(\Delta^4 f-2+\Delta^4 f-1)}{2}$

$P_4(1.9)=\frac{(0.2231+0.1738)}{2}+(1.6-\frac{1}{2})$ (-0.0493) + $\frac{(1.6)+(1.6-1)}{2}\frac{(0.0141+0.0108)}{2}$ + $\frac{(1.6)(1.6-\frac{1}{2})(1.6-1)}{6}$(-0.0033) +$\frac{(1.6)((1.6)^2-1)(1.6-2)}{24}\frac{(0.0006+0.0011)}{2}$ = 0.14957984

Laplace Everett's Formula

$P_n(x_s)=[vf_0+\frac{(v^2-1)}{3!}\Delta^2 f_{-1}+\frac{(v^2-1)\,v\,(v^2-2^2)}{5!}\Delta^4 f_{-2}+\ldots]+[s\,f_1+\frac{(S^2-1)}{3!}s\Delta^2 f_0+\frac{S(S^2-1)\,(S^2-2^2)}{5!}\Delta^4 f_{-1}+\ldots]$

$P_4(1.9)=(-0.6)(0.2231)+\frac{((-0.6))^2-1)(-0.6)}{6}(0.0141)+\frac{((-0.6))^2-1)(-0.6)((-0.6))^2-4)}{120}(0.0006)$ +(1.6)(0.1738) + $\frac{((1.6)^2-1)(-1.6)}{6}$ (0.0108) + $\frac{((1.6)^2-1)(1.6)((1.6)^2-4)}{120}$(0.0011) = 0.14957526

New Proposed Formula

$P_n(x_s)=\frac{1}{2}(f_0+f_1)+\frac{(2S-1)}{2}+\Delta f_0+\frac{S(S-1)}{2}\quad\Delta^2 f_0+\frac{S(S-1)(S-2)}{12}(\Delta^3 f_0+\Delta^3 f_{-1})+\frac{S(S-1)(S-2)}{48}[(S-3)\Delta^4 f_0+(S+1)\Delta^4 f_{-1}]$

$P_4(1.9)$ = $\frac{(0.2231+0.1738)}{2}$ + $\frac{(2(1.6)-1)}{2}$ + (-0.0493)+(1.6) $\frac{(1.6-1)}{2}$ (0.0108)+ $\frac{(1.6)(1.6-1)(1.6-2)}{12}$ [(-0.0022) + (-0.0033)] + $\frac{(1.6)(1.6-1)(1.6-2)}{48}$ [(1.6-3) (0.0002)+(1.6+1) (0.0011)] = 0.14955936

Problem 4.2: From the following table of values, estimate (7.5)

x	1	2	3	4	5	6	7	8
y=f(x)	1	8	27	64	125	216	343	512

$s=\frac{x-x_0}{h}=\frac{7.5-4}{1}=3.5,\quad v=1-s=1-3.5=-2.5$

Gauss Backward Formula

$P_n(x_s)=f_0+s\Delta f_{-1}+\frac{s(s+1)}{2!}\Delta^2 f_{-1}+\frac{s(s-1)}{3!}\Delta^3 f_{-2}+s(s^2-1)\frac{(s+2)}{4!}\Delta^4 f_{-2}$

P_3 (7.5) =64 + (3.5)(37) + $\frac{(3.5)(3.5+1)}{2}$ (24) +$\frac{(3.5)((3.5)^2-1))}{6}$ (6)=421.875

Table 5: Difference Table for Problem 4.2

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_{-3}	1			
		7		
x_{-2}	8		12	
		19		6
x_{-1}	27		18	
		37		6
x_0	64		24	
		61		6
x_1	125		30	
		91		6
x_2	216		36	
		127		6
x_3	343		42	
		169		
x_4	512			

Newton's Gregory Forward Formula

$$P_n(x_s) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0$$

$$P_3(7.5) = 64 + (3.5)(61) + \frac{(3.5)(3.5-1)}{2}(30) + \frac{(3.5)(3.5-1)(3.5-2)}{6} = 421.875$$

Stirling's Interpolation Formula

$$P_n(x_s) = f_0 + s\frac{(\Delta f_{-1}+\Delta f_0)}{2} + \frac{S^2}{2!}\Delta^2 f_{-1} + \frac{S(S^2-1)}{3!}\frac{(\Delta^3 f_{-2}+\Delta^3 f_{-1})}{2} + \frac{S^2(S^2-1)}{4!}\Delta^4 f_{-2}$$

$$P_3(7.5) = 64 + (3.5)\frac{(37+61)}{2} + \frac{(3.5)^2}{2}(24) + \frac{(3.5)(13.5)^2-1)}{6}\frac{(6+6)}{2} = 421.875$$

Bessel's Interpolation Formula

$$P_n(x_s) = \frac{(f0-f1)}{2} + (s-\frac{1}{2})\Delta f_0 + \frac{S(S-1)}{2!}(\Delta^2 f_{-1} + \Delta^2 f_0) + \frac{S\left(S-\frac{1}{2}\right)(S-1)}{3!}\Delta^3 f_{-1}$$

$$P_3(7.5) = \frac{(64+125)}{2} + (3.5-\frac{1}{2})(61) + \frac{(3.5)(3.5-1)}{2}\frac{(24+30)}{2} + (3.5)\frac{(3.5-\frac{1}{2})(3.5-1)}{2}(6) = 421.875$$

Laplace Everett's Formula

$$P_n(x_s) = [vf_0 + \frac{(v^2-1)}{3!}v\Delta^2 f_{-1} + \frac{(v^2-1)\,v\,(v^2-2^2)}{5!}\Delta^4 f_{-2} + \ldots\ldots] + [s\,f_1 + \frac{(S^2-1)}{3!}s\Delta^2 f_0 + \frac{S(S^2-1)\,(S^2-2^2)}{5!}\Delta^4 f_{-1} + \ldots]$$

$v = 1-s = 1-3.5 = -2.5$

$P_3(7.5)=[(-2.5)(64) + \frac{((-0.25)^2-1)(-2.5)}{6}(24) + (-2.5)((-2.5)^2-1)\frac{((-0.25)^2-2^2)(0)}{120}] + [(3.5)(125) + \frac{((3.5)^2-1)(3.5)(30)}{6} + (3.5)\frac{(3.5)^2-1)((3.5)^2-2^2)(0)}{120}] = 421.875$

New Proposed Method

$P_n(x_s) = \frac{1}{2}(f_{0+} f_1) + \frac{(2S-1)}{2}\Delta f_0 + \frac{S(S-1)}{2}\Delta^2 f_0 + \frac{S(S-1)(S-2)}{12}[\Delta^3 f_0 + \Delta^3 f_{-1}]$

$P_3(7.5) = \frac{(64+125)}{2} + \frac{(2(3.5)-1)}{2}(61)\frac{(3.5)(3.5)-1)}{2}(30) + \frac{(3.5)(3.5)-1)(3.5)-2)}{12}(6+6) = 421.875$

Problem 4.3: From the given table of values, estimate the value of $y = \log_{10}^{337.5}$

x	310	320	330	340	350	360
$y=\log_{10}^{x}$	2.4913617	2.5051500	2.5185139	2.5314789	2.544068	2.5563025

Table 6: Difference Table for Problem 4.3

	x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
x_{-2}	310	2.4913617				
			0.0137883			
x_{-1}	320	2.5051500		-0.0004244		
			0.0133639		0.0000255	
x_0	330	2.5185139		-0.0003989		-0.0000025
			0.012965		0.000023	
x_1	340	2.5314789		-0.0003759		-0.0000017
			0.0125891		0.0000213	
x_2	350	2.544068		-0.0003546		
			0.0122345			
x_3	360	2.5563025				

$s = \frac{337.5-330}{10} = 0.75$, $v = 1-s = 1-0.75 = 0.25$

Gauss Backward Formula

$P_n(x_s) = f_0 + s\Delta f_{-1} + \frac{S(S+1)}{2!}\Delta^2 f_{-1} + \frac{S(S^2-1)}{3!}\Delta^3 f_{-2} + \frac{S(S^2-1)S(S+2)}{4!}\Delta^4 f_{-2}$

$P_4(337.5) = 2.5185139 + (0.75)(0.0133639) + \frac{(0.75)(0.75+1)}{2!}(-0.0003989) + \frac{(0.7)((0.75)^2-1)}{6}(0.0000255) + \frac{(0.75)((0.75)^2-1)(0.75+2)}{24}(-0.0000025) = 2.5282737,$

Newton's Gregory Forward Formula

$P_n(x_s) = f_0 + s\Delta f_0 + \frac{S(S-1)}{2!}\Delta^2 f_0 + \frac{S(S-1)(S-2)}{3!}\Delta^3 f_0 + \frac{S(S-1)(S-2)(S-3)}{4!}\Delta^4 f_0$

$P_4(337.5) = 2.5185139 + (0.75)(0.012965) + \frac{(0.75)(0.75-1)}{2}(0.0003759) + \frac{(0.75)(0.75-1)(0.75-2)}{6}(0.0000213) + 0 = 2.5282737$

Stirling's Interpolation Formula

$P_n(x_s) = f_0 + s\frac{(\Delta f-1+\Delta f0)}{2} + \frac{S^2}{2!}\Delta^2 f_{-1} + \frac{S(S^2-1)}{3!}\frac{(\Delta^3 f-2+\Delta^3 f-1)}{2} + \frac{S^2(S^2-1)}{4!}\Delta^4 f_{-2}$

$P_4(337.5) = 2.5185139 + (0.75)(0.0133639 + 0.012965) + \frac{(0.75)^2}{2}(-0.0003989) +$

$\frac{(0.75)((0.75)^2-1)}{6}(0.0000255 + 0.000023) + (0.75)^2\frac{((0.75)^2-1)}{24}(-0.0000025) = 2.5282737$

Bessel's Interpolation Formula

$P_n(x_s) = \frac{(f0-f1)}{2} + (s-\frac{1}{2})\Delta f_0 + \frac{S(S-1)}{2!}\frac{(\Delta^2 f_{-1}+\Delta^2 f_0)}{2} + \frac{S(S-\frac{1}{2})(S-1)}{3!}\Delta^3 f_{-1} + \frac{S(S^2-1)(S-2)}{4!}\frac{(\Delta^4 f-2+\Delta^4 f-1)}{2}$

$P_4(337.5) = \frac{(2.5185139+2.5314789)}{2} + (0.75-0.5)(0.012965) + \frac{(0.75)(0.75-1)}{2}$

$\frac{(-0.0003989-0.0003759)}{2} + \frac{(0.75)(0.75-0.5)(0.75-1)}{6}(0.000023) + \frac{(0.75)((0.75)^2-1)(0.75-2)}{24}$

$\frac{(-0.0000025-0.0000017)}{2} = 2.5282737$

Laplace Everett's Formula

$P_n(x_s) = [vf_0 + \frac{(v^2-1)}{3!}v\Delta^2 f_{-1} + \frac{v(v^2-1)(v^2-2^2)}{5!}\Delta^4 f_{-2}] + [S f_1 + \frac{(S^2-1)}{3!}s\Delta^2 f_0 + \frac{S(S^2-1)(S^2-2^2)}{5!}\Delta^4 f_{-1}]$

$P_4(337.5) = [(0.25)(2.5185139) + \frac{((0.25)^2-1)(0.25)}{6}(-0.0003989) + \frac{(0.25)((0.25)^2-1)((0.25)^2-2^2)}{120}-$

$0.0000025] + [(0.75)(2.5314789) + \frac{(0.75)((0.75)^2-1)}{6}(-0.0003759) + \frac{(0.75)((0.75)^2-1)((0.75)^2-2^2)}{120}(-0.0000017)] = 2.5282738$

New Proposed Formula

$P_n(x_s) = \frac{(f_0+f_1)}{2} + \frac{(2S-1)}{2}\Delta f_0 + \frac{S(S-1)}{2}\Delta^2 f_0 + \frac{S(S-1)(S-2)}{12}[\Delta^3 f_0 + \Delta^3 f_{-1}] + \frac{S(S-1)(S-2)}{48}[(s-3)\Delta^4 f_0 + (s+1)\Delta^3 f_{-1}]$

$P_4(337.5) = \frac{(2.5185139+2.5314789)}{2} + \frac{(2(0.75)-1)}{2}(0.012965) + (0.75)\frac{(0.75-1)}{2}(-0.0003759) +$

$\frac{(0.75)(0.75-1)(0.75-2)}{12}(0.0000213 + 0.000023) + \frac{(0.75)(0.75-1)(0.75-2)}{48}[+(0.75-3)(0) + (0.75+1)(-0.0000017)] = 2.5282738$

Problem 4.4: From the following data, find out the value of sin 45^0

x	20^0	30^0	40^0	50^0	60^0	70^0
$\sin x^0$	0.34202	0.5	0.64279	0.76604	0.86603	0.93969

Gauss Backward Formula

$P_n(x_s) = f_0 + s\Delta f_{-1} + \frac{S(S+1)}{2!}\Delta^2 f_{-1} + \frac{S(S^2-1)}{3!}\Delta^3 f_{-2} + \frac{S(S^2-1)(S+2)}{4!}\Delta^4 f_{-2}$

$P_4(45) = 0.64279 + \left(\frac{1}{2}\right)(0.14279) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)}{2}(-0.01954) + \frac{\left(\frac{1}{2}\right)((0.5)^2-1)}{6}(-0.00435) + \frac{\left(\frac{1}{2}\right)\left(\left(\frac{1}{2}\right)^2-1\right)\left(\frac{1}{2}+2\right)}{24}(0.00063) =$

$0.7071309, \frac{((0.25)^2-1)(0.25)}{6} = 0.7071309$

Newton's Gregory Forward Formula

$P_n(x_s) = f_0 + s\Delta f_0 + \frac{S(S-1)}{2!}\Delta^2 f_0 + \frac{S(S-1)(S-2)}{3!}\Delta^3 f_0 + \frac{S(S-1)(S-2)(S-3)}{4!}\Delta^4 f_0$

$P_4(45) = 0.64279 + \left(\frac{1}{2}\right)(0.12325) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2}(-0.02326) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{6}(-0.00307) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)(0)}{24} = 0.7071306$

Stirling's Interpolation Formula

$P_n(x_s) = f_0 + S\frac{(\Delta f-1+\Delta f0)}{2} + \frac{S^2}{2!}\Delta^2 f_{-1} + \frac{S(S^2-1)}{3!}\frac{(\Delta^3 f-2+\Delta^3 f-1)}{2} + \frac{S^2(S^2-1)}{4!}\Delta^4 f_{-2}$

$P_4(45) = 0.64279 + \left(\frac{1}{2}\right)\frac{(0.14279+0.12325)}{2} + \frac{\left(\frac{1}{2}\right)^2}{2}(-0.01954) + \frac{\left(\frac{1}{2}\right)\left(\left(\frac{1}{2}\right)^2-1\right)}{6}\frac{(-0.00435-0.00372)}{2} + \frac{\left(\frac{1}{2}\right)^2\left(\left(\frac{1}{2}\right)^2-1\right)}{24}(0.00063) = 0.7071048$

Bessel's Interpolation Formula

$P_n(x_s) = \frac{(f0+f1)}{2} + (S-\frac{1}{2})\Delta f_0 + \frac{S(S-1)}{2!}\frac{(\Delta^2 f-1+\Delta^2 f-2)}{2} + \frac{S\left(S-\frac{1}{2}\right)(S-1)}{3!}\Delta^3 f_{-1}\frac{S(S^2-1)S(S-2)}{4!}\frac{(\Delta^4 f-2+\Delta^4 f-1)}{2}$

$P_4(45) = \frac{(0.64279+0.76604)}{2}\left(\frac{1}{2}-\frac{1}{2}\right)(0.12352) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2}\frac{(-0.01954-0.01519)}{2} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{6}(-0.00372) + \frac{\left(\frac{1}{2}\right)\left(\left(\frac{1}{2}\right)^2-1\right)\left(\frac{1}{2}-2\right)}{24}\frac{(0.0063+0.00065)}{2} = 0.7066006$

Laplace Everett's Formula

$P_n(x_s) = [vf_0 + \frac{(v^2-1)}{3!}v\Delta^2 f_{-1} + \frac{v(v^2-1)\ (v^2-2^2)}{5!}\Delta^4 f_{-2}] + [S f_1 + \frac{(S^2-1)}{3!}s\,\Delta^2 f_0 + \frac{S(S^2-1)\ (S^2-2^2)}{5!}\Delta^4 f_{-1} +] + \ldots$

$P_4(45) = [(0.5)(0.64279) + \frac{(0.5)^2-1)\,(0.5)}{6}(-0.01954) + \frac{(0.5)\,(0.5)^2-1)\ (0.5)^2-4)}{120}(0.00063)] + [(0.5)(0.76604) + \frac{(0.5)^2-1)\,(0.5)}{6}(-0.02326) + \frac{(0.5)\,(0.5)^2-1)\ (0.5)^2-4)}{120}(0.00065)] = 0.707105$

New Proposed Method or Formula

$P_n(x_s) = \frac{1}{2}(f_{0+} f_1) + \frac{(2S-1)}{2}\Delta f_0 + \frac{S(S-1)}{2}\Delta^2 f_0 + \frac{S(S-1)(S-2)}{12}[\Delta^3 f_0 + \Delta^3 f_{-1}]\ \frac{S(S-1)(S-2)}{48}[(s-3)\Delta^4 f_0 + (s+1)\Delta^4 f_{-1}]$

$P_4(45) = \frac{(0.64279+0.76604)}{2} + \frac{\left(2\left(\frac{1}{2}\right)-1\right)}{2}(0.12325) + \frac{\left(\frac{1}{2}\right)\left(\left(\frac{1}{2}\right)-1\right)}{2}(-0.02326) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{12}(-0.00307 - 0.00372) + \left(\frac{1}{2}\right)\frac{\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{48}\left[\left(\frac{1}{2}-3\right)(0) + \left(\frac{1}{2}+1\right)(0.00065)\right] = 0.7071179$

V. ANALYSIS OF RESULTS

Table 7: Comparison of methods

Problem	Exact solution	Gauss Backward	NGF Newton GF	Stirling's	Bessel	Laplace Everett's	New Method
4.1.1	0.149568619	0.14959024	0.14954930	01495902	0.1495798	0.1495753	0.14955936
4.1.2	421.875	421.875	421.875	421.875	421.875	421.875	421.875
4.1.3	2.5282738	2.5282757	2.5282737	2.5282737	2.5282737	2.5282738	2.5282738
4.1.4	0.7071068	0.7071309	0.7071306	0.7071048	0.7066006	0.707105	0.7071179

Table 8: Absolute Errors in Computation of methods

Problem	Gauss Backward	NGF Newton GF	Stirling's	Bessel	Laplace Everett	New Method
4.1.1	2.1621×10^{-5}	1.933×10^{6}	2.159×10^{-6}	1.123×10^{-6}	6.65×10^{-7}	9.2590×10^{-6}
4.1.2	0	0	0	0	0	0
4.1.4	1×10^{8}	1×10^{8}	1×10^{8}	1×10^{8}	0	0
4.1.5	2.41×10^{-6}	2.38×10^{-6}	2.0×10^{7}	5.062×10^{5}	1.8×10^{7}	1.11×10^{-6}

VI. CONCLUSION

We proposed a new interpolation formula (3.4) which was based on central and forward difference interpolation. Derivation of the new method was obtained from a combination of modified Gauss's Backward Formula and Newton's Gregory forward formula in which we advanced the subscripts in Gauss's backward Formula by one unit and replacing s by s-1. Comparison of existing interpolation formulas with the new method was established and results show that the new method was very efficient and convergences to the analytical solution faster than existing method. The newly derived method possessed good accuracy.

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