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Local Bifurcation Analysis of a Stage Structured Epidemiological Model with Michaelis-Menten Predation and Prey Refuge

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Abstract – *In this paper, we determine the conditions for the appearance of local bifurcations like Saddle-node, Transcritical and Pitchfork bifurcation at all the stable equilibrium points of a diseased stage structured prey-predator model with Michealis-Menten type response function and the disease transmission rate being linear. Bifurcation point is found out and by using Sotomayors' theorem the occurrence of the bifurcation is established at the bifurcation point for the system.*

Keywords : *Eco-epidemiological, Local bifurcation, Michaelis Menten, , Prey predator, Stage structure.*

I. INTRODUCTION

The system of differential equations under examination may contain several parameters. It is probable that a modest shift in the parameter may alter the behaviour the solution of the system entirely. In general, in a dynamical system if a parameter is allowed to vary, then the behavior of the differential system may change. The value of the parameter at which these changes occur is known a “bifurcation value” and the parameter that is varied is known as “bifurcation parameter”. At bifurcation points, the local stability properties, periodic orbits or other invariant set changes. The bifurcations are of two types local and global.

Local bifurcation can be investigated through the changes in the local stability features of the equilibria, periodic orbits or other invariant sets as parameter cross through the thresholds. In [2] Chauhan established that in the infectious model with polluted environment, the susceptible population can survive with infection and with pollution the population totally vanishes. Dai et al in 2014 considered the prey-predator model with Michaelis-Menten response function and two delays to study their local property and the occurrence of Hopf bifurcation [3]. Local bifurcation analysis had been carried out with predator harvesting in [4], with stage structure on prey in [6], with epidemiological model in [7] and with contact transmission function in [5]. The qualitative property for the prey predator model considering stage structure on both the species incorporating anti predator behavior and group defense mechanism against predation was studied [8].

The influence of cross diffusion had been analysed in the prey-predator model with refuging prey and Michaelis Menten response function [9]. Stage structure had been introduced for the predator there in and the stability analysis had been done [10]. The goal of this paper is to study the local bifurcation of the predator prey model where the prey has been infected following [11].

This article is structured as follows. In section 2 the model system is presented with the equilibrium points and the conditions for the local stability property [11]. In section 3 we analyse the local bifurcation at the equilibrium points. Section 4 is the conclusion.

II. MATHEMATICAL MODEL [11]

The epidemiological prey-predator model system considered for local bifurcation analysis with positive initial conditions is as follows:

$$\dot{U}_s = RU_s\left[1-\frac{U_s+U_I}{K}\right]-\frac{A_1(1-\lambda)U_sV_2}{U_s(1-\lambda)+K_1V_2}-\beta U_sU_I$$

$$\dot{U}_I = \beta U_sU_I - d_1U_I$$

$$\dot{V}_1 = \frac{e_1(1-n_1)A_1(1-\lambda)U_sV_2}{U_s(1-\lambda)+K_1V_2}-DV_1-d_2V_2$$

$$\dot{V}_2 = DV_1 - d_3V_2 + \frac{e_1n_1A_1(1-\lambda)U_sV_2}{U_s(1-\lambda)+K_1V_2}$$

where

U_s	Density of susceptible prey
U_I	Density of infected prey
V_1	Density of juvenile predator
V_2	Density of adult predator
R	Intrinsic growth rate of prey
K	Carrying capacity of the environment
A_1	Attack rate of the adult predator
λ	Proportion of prey taking refuge $\lambda \in [0,1)$
K_1	Benefit rate of predator cofeeding
β	Disease transmission rate within prey species
d_1	Mortality rate of infected prey
d_2	Mortality rate of juvenile predator
d_3	Mortality rate of adult predator
n_1	Proportion of food shared between the juvenile and adult predator
e_1	Conversion coefficiency of prey biomass
D	Transmission rate from juvenile predator to adult predator

After rescaling the system gets the new form as

$$\begin{aligned}
\dot{u}_s &= u_s\left[1-u_s-(1+b_1)u_I-\frac{b_2v_2}{u_s(1-\lambda)+K_1v_2}\right]=F_1\left(u_s,u_I,v_1,v_2\right)\\
\dot{u}_I &= u_I\left[b_1u_s-b_3\right]=F_2\left(u_s,u_I,v_1,v_2\right)\\
\dot{v}_1 &= \frac{b_4u_sv_2}{u_s(1-\lambda)+K_1v_2}-\left(b_5+b_6\right)v_1=F_3\left(u_s,u_I,v_1,v_2\right)\\
\dot{v}_2 &= \frac{b_7u_sv_2}{u_s(1-\lambda)+K_1v_2}+b_5v_1-b_8v_2=F_4\left(u_s,u_I,v_1,v_2\right)
\end{aligned} \tag{1}$$

where

$\frac{\beta K}{R}=b_1;\quad \frac{A_1(1-\lambda)}{R}=b_2;\quad \frac{d_1}{R}=b_3;\quad \frac{e_1(1-n_1)A_1(1-\lambda)}{R}=b_4;\ \frac{D}{R}=b_5;\quad \frac{d_2}{R}=b_6;\quad \frac{e_1n_1A_1(1-\lambda)}{R}=b_7;$

$\frac{d_3}{R}=b_8$ with $u_s(0)\ge 0,\ u_I(0)\ge 0,\ v_1(0)\ge 0,\ v_2(0)\ge 0$

The equilibrium points are

1. $E_0(0,0,0,0)$ 2. $E_1(1,0,0,0)$ 3. $E_2(\hat{u}_s,\hat{u}_I,0,0)$ 4. $E_3(\hat{\hat{u}}_s,0,\hat{\hat{v}}_1,\hat{\hat{v}}_2)$ 5. $E_4(\tilde{u}_s,\tilde{u}_I,\tilde{v}_1,\tilde{v}_2)$. The conditions for local stability at these equilibrium points are given in [11].

III. BIFURCATION

In this section, the effect of varying the parameter values on the dynamical behavior of the system around each equilibrium point is studied. The existence of non-hyperbolic point of the system (1) is the necessary but not sufficient condition for bifurcation to occur. We analyse the local bifurcation using Sotomayor's theorem.

The jacobian matrix of the system (1) at any of the equilibrium point (u_s,u_I,v_1,v_2) is given by

$$J=\begin{bmatrix} 1-2u_s-(1+b_1)u_I-\dfrac{b_1K_1v_2^2}{\left[u_s(1-\bar{\lambda})+K_1v_2\right]^2} & -(1+b_1)u_s & 0 & -\dfrac{b_2u_s^2(1-\bar{\lambda})}{\left[u_s(1-\bar{\lambda})+K_1v_2\right]^2} \\ b_1u_I & b_1u_s-b_3 & 0 & 0 \\ \dfrac{b_4K_1v_2^2}{\left[u_s(1-\bar{\lambda})+K_1v_2\right]^2} & 0 & -(b_5+b_6) & \dfrac{b_4u_s^2(1-\bar{\lambda})}{\left[u_s(1-\bar{\lambda})+K_1v_2\right]^2} \\ \dfrac{b_7K_1v_2^2}{\left[u_s(1-\bar{\lambda})+K_1v_2\right]^2} & 0 & b_5 & \dfrac{b_7u_s^2(1-\bar{\lambda})}{\left[u_s(1-\bar{\lambda})+K_1v_2\right]^2}-b_8 \end{bmatrix} \qquad (2)$$

For any non-zero vector $X=\left(X_1,X_2,X_3,X_4\right)^T$,

$D^2F_\alpha(Y,\alpha)(X,X)=\begin{bmatrix}\Omega_1\\ \Omega_2\\ \Omega_3\\ \Omega_4\end{bmatrix}$, where $Y=(u_s,u_I,v_1,v_2)$ and α is any bifurcation parameter.

where $$\Omega_1=-2\left[1-\frac{K_1v_2^2(1-\bar{\lambda})}{\left[u_s(1-\bar{\lambda})+K_1v_2\right]^3}\right]X_1^2-2(1+b_1)X_1X_2-\frac{2K_1(b_1+b_2)(1-\bar{\lambda})u_sv_2}{\left[u_s(1-\bar{\lambda})+K_1v_2\right]^3}X_1X_4+\frac{2b_2K_1(1-\bar{\lambda})u_s^2}{\left[u_s(1-\bar{\lambda})+K_1v_2\right]^3}X_4^2$$

$$\Omega_2=2b_1X_1X_2$$

$$\Omega_3=-\frac{2b_4K_1(1-\bar{\lambda})v_2^2}{\left[u_s(1-\bar{\lambda})+K_1v_2\right]^3}X_1^2+\frac{4b_4K_1(1-\bar{\lambda})u_sv_2}{\left[u_s(1-\bar{\lambda})+K_1v_2\right]^3}X_1X_4-\frac{2b_4K_1(1-\bar{\lambda})u_s^2}{\left[u_s(1-\bar{\lambda})+K_1v_2\right]^3}X_4^2$$

$$\Omega_4=-\frac{2b_7K_1(1-\bar{\lambda})v_2^2}{\left[u_s(1-\bar{\lambda})+K_1v_2\right]^3}X_1^2+\frac{4b_7K_1(1-\bar{\lambda})u_sv_2}{\left[u_s(1-\bar{\lambda})+K_1v_2\right]^3}X_1X_4-\frac{2b_7K_1(1-\bar{\lambda})u_s^2}{\left[u_s(1-\bar{\lambda})+K_1v_2\right]^3}X_4^2$$

A. Local bifurcation analysis near $E_1(1,0,0,0)$

Theorem 1: If the parameter value b_1 passes through the value $b_1^*=b_3$, then the system at the axial equilibrium point $E_1(1,0,0,0)$ possesses:

- No saddle-node bifurcation
- Transcritical bifurcation
- No Pitch-fork bifurcation

Proof: At $b_1=b_1^*$, the variational matrix (2) has a zero eigen value $\lambda_{1u_I}=0$ and the variational matrix $J(E_1)$ with $b_1=b_1^*$ becomes

$$\hat{J}(E_1)=J_{E_1}(b_1=b_1^*)=\begin{bmatrix} -1 & -(1+b_1^*) & 0 & -\dfrac{b_2}{(1-\lambda)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(b_5+b_6) & \dfrac{b_4}{(1-\lambda)} \\ 0 & 0 & b_5 & \dfrac{b_7}{(1-\lambda)}-b_8 \end{bmatrix}$$

Let $X^{[1]}=\left(X_1^{[1]},X_2^{[1]},X_3^{[1]},X_4^{[1]}\right)^T$ be the eigen vector corresponding to the eigen value $\lambda_{1u_I}=0$. Thus $\left(\hat{J}_{E_1}-\lambda_{1u_I}I\right)X^{[1]}=0$ gives $X^{[1]}=\left(-(1+b_1)X_2^{[1]},X_2^{[1]},0,0\right)^T$ where $X_2^{[1]}\neq 0$.

Let $W^{[1]} = \left(W_1^{[1]}, W_2^{[1]}, W_3^{[1]}, W_4^{[1]}\right)^T$ be the eigen vector associated with the eigen value $\lambda_{1u_I} = 0$ of the matrix $\left[\hat{J}_{E_1}\right]^T$. Then by solving the matrix $\left(\left[\hat{J}_{E_1}\right]^T - \lambda_{1u_I} I\right) W^{[1]} = 0$, we get $W^{[1]} = \left(0, W_2^{[1]}, 0, 0\right)^T$ where $W_2^{[1]} \neq 0$.

Now,

Hence $\frac{\partial F}{\partial b_1}(E_1, b_1^*) = (0,0,0,0)^T$ and so $\left[W^{[1]}\right]^T \frac{\partial F}{\partial b_1}(E_1, b_1^*) = 0$. $\frac{\partial F}{\partial b_1} = F_{b_1}(Y, b_1) = \left(\frac{\partial F_1}{\partial b_1}, \frac{\partial F_2}{\partial b_1}, \frac{\partial F_3}{\partial b_1}, \frac{\partial F_4}{\partial b_1}\right) = (-u_s u_I, u_s u_I, 0, 0)^T$.

Therefore, according to Sotomayer's theorem, the Saddle node bifurcation cannot occur. But the first condition for the Transcritical bifurcation is satisfied.

Now, $\left[W^{[1]}\right]^T DF(E_1, b_1^*) X^{[1]} = \left[0 \quad W_2^{[1]} \quad 0 \quad 0\right] \begin{bmatrix} -X_2^{[1]} \\ X_2^{[1]} \\ 0 \\ 0 \end{bmatrix} = W_2^{[1]} X_2^{[1]} \neq 0$.

The second condition for the Transcritical bifurcation is satisfied. Now,

$$D^2 F_{b_1}(E_1, b_1^*)(X^{[1]}, X^{[1]}) = \begin{bmatrix} -2[X_1^{[1]}]^2 - 2(1+b_3) X_1^{[1]} X_2^{[1]} + \frac{2b_2 K_1}{(1-\lambda)^2}[X_4^{[1]}]^2 \\ 2b_3 X_1^{[1]} X_2^{[1]} \\ -\frac{2b_4 K_1}{(1-\lambda)^2}[X_4^{[1]}]^2 \\ -\frac{2b_7 K_1}{(1-\lambda)^2}[X_4^{[1]}]^2 \end{bmatrix}$$

$$[W^{[1]}]^T D^2 F_{b_1}(E_1, b_1^*)(X^{[1]}, X^{[1]}) = 2b_3 X_1^{[1]} X_2^{[1]} W_2^{[1]} \neq 0$$

Thus, according to Sotomayor Theorem, the system experiences Transcritical bifurcation at the point $E_1(1,0,0,0)$ when the parameter b_1 crosses the value b_1^*.

Also, the system does not experience Pitchfork bifurcation at E_1 when $b_1 = b_1^*$.

B. Local bifurcation Analysis near $E_2(\hat{u}_s, \hat{u}_I, 0, 0)$

Theorem 2: If the parameter value b_4 passes through the value $b_4^* = b_4$, then the system at the predator free equilibrium point $E_2(\hat{u}_s, \hat{u}_I, 0, 0)$ possesses:

- No saddle-node bifurcation
- Transcritical bifurcation
- No Pitch-fork bifurcation

where $b_4^* = \frac{(b_5 + b_6)[b_8(1-\hat{\lambda}) - b_7]}{b_5}$ and $b_8(1-\hat{\lambda}) > b_7$

Proof: Let the variational matrix at the point $E_2(\hat{u}_s, \hat{u}_I, 0, 0)$ be $J(E_2)$. At $b_4 = b_4^*$, the system has a zero eigen value $\lambda_{2u_s} = 0$ and the variational matrix $J(E_2)$ with $b_4 = b_4^*$ becomes

$$\hat{J}(E_2) = J_{E_2}(b_4 = b_4^*) = \begin{bmatrix} 1-2\hat{u}_s-(1+b_1)\hat{u}_I & -(1+b_1)\hat{u}_s & 0 & -\dfrac{b_2}{(1-\lambda)} \\ b_1\hat{u}_I & 0 & 0 & 0 \\ 0 & 0 & -(b_5+b_6) & \dfrac{(b_5+b_6)(b_8(1-\lambda)-b_7)}{b_5(1-\lambda)} \\ 0 & 0 & b_5 & \dfrac{b_7}{(1-\lambda)}-b_8 \end{bmatrix}$$

Let $X^{[2]} = \left(X_1^{[2]}, X_2^{[2]}, X_3^{[2]}, X_4^{[2]}\right)^T$ be the eigen vector corresponding to the eigen value $\lambda_{2u_s} = 0$. Thus $\left(\hat{J}_{E_2} - \lambda_{2u_s} I\right) X^{[2]} = 0$ gives

$$X^{[2]} = \left(0, -\frac{b_2}{(1+b_1)(1-\lambda)\hat{u}_s} X_4^{[2]}, -\frac{b_7 - b_8(1-\lambda)}{b_5(1-\lambda)} X_4^{[2]}, X_4^{[2]}\right)^T \text{ where } X_4^{[2]} \neq 0.$$

Let $W^{[2]} = \left(W_1^{[2]}, W_2^{[2]}, W_3^{[2]}, W_4^{[2]}\right)^T$ be the eigen vector associated with the eigen value $\lambda_{2u_s} = 0$ of the matrix $\left[\hat{J}_{E_2}\right]^T$. Then by solving the matrix $\left(\left[\hat{J}_{E_2}\right]^T - \lambda_{2u_s} I\right) W^{[2]} = 0$ we get

$$W^{[2]} = \left(0, 0, \frac{b_5}{(b_5+b_6)} W_4^{[2]}, W_4^{[2]}\right)^T \text{ where } W_4^{[2]} \neq 0.$$

Now, $\dfrac{\partial F}{\partial b_4} = F_{b_4}(Y, b_4) = \left(\dfrac{\partial F_1}{\partial b_4}, \dfrac{\partial F_2}{\partial b_4}, \dfrac{\partial F_3}{\partial b_4}, \dfrac{\partial F_4}{\partial b_4}\right) = \left(0, 0, \dfrac{u_s v_2}{u_s(1-\lambda) + K_1 v_2}, 0\right)^T$.

Hence $\dfrac{\partial F}{\partial b_4}(E_2, b_4^*) = (0,0,0,0)^T$ and so $\left[W^{[2]}\right]^T \dfrac{\partial F}{\partial b_4}(E_2, b_4^*) = 0$.

Therefore, according to Sotomayor's theorem, Saddle node bifurcation cannot occur. But the first condition for the Transcritical bifurcation is satisfied.

$$\text{Now, } \left[W^{[2]}\right]^T DF(E_2, b_4^*) X^{[2]} = \left(0, 0, \frac{b_5}{(b_5+b_6)} W_4^{[2]}, W_4^{[2]}\right)^T \begin{bmatrix} 0 \\ 0 \\ \dfrac{X_4^{[2]}}{(1-\lambda)} \\ 0 \end{bmatrix} = \frac{b_5}{(b_5+b_6)} X_4^{[2]} W_4^{[2]} \neq 0.$$

Now the second condition for Transcritical bifurcation is satisfied.

$$D^2 F_{b_4}(E_2, b_4^*)(X^{[2]}, X^{[2]}) = \begin{bmatrix} -2[X_1^{[2]}]^2 - 2(1+b_1) X_1^{[2]} X_2^{[2]} + \dfrac{2 b_2 K_1}{(1-\lambda)^2} [X_4^{[2]}]^2 \\ 2 b_1 X_1^{[2]} X_2^{[2]} \\ -\dfrac{2 b_4 K_1}{(1-\lambda)^2} [X_4^{[2]}]^2 \\ -\dfrac{2 b_7 K_1}{(1-\lambda)^2} [X_4^{[2]}]^2 \end{bmatrix}$$

$$[W^{[2]}]^T D^2 F_{b_4}(E_2, b_4^*)(X^{[2]}, X^{[2]}) = -\frac{2 K_1 b_8}{(1-\lambda)} W_4^{[2]} [X_4^{[2]}]^2 \neq 0$$

Thus, according to Sotomayor's theorem, the system experience Transcritical bifurcation at the point $E_2(\hat{u}_s, \hat{u}_I, 0, 0)$ when the parameter b_4 crosses the value b_4^*.

Also, the system does not experience Pitchfork bifurcation at E_2 when $b_4 = b_4^*$.

C. Local bifurcation Analysis near $E_3(\hat{\hat{u}}_s, 0, \hat{\hat{v}}_1, \hat{\hat{v}}_2)$

Theorem 3: If the parameter value b_3 passes through the value $b_3^* = b_3$ where $b_3^* = b_1 \hat{\hat{u}}_s$, then the system (1) at the infection free equilibrium point $E_3(\hat{\hat{u}}_s, 0, \hat{\hat{v}}_1, \hat{\hat{v}}_2)$ possesses:

- No saddle-node bifurcation
- Transcritical bifurcation
- No Pitch-fork bifurcation

if $\psi_2 \neq 0$.

Proof: Let the variational matrix at the point $E_3(\hat{\hat{u}}_s, 0, \hat{\hat{v}}_1, \hat{\hat{v}}_2)$ be $J(E_3)$. At $b_3 = b_3^*$, the system has a zero eigen value $\lambda_{3u_s} = 0$ and the variational matrix $J(E_3)$ with $b_3 = b_3^*$ becomes

$$\hat{J}(E_3) = J_{E_3}(b_3 = b_3^*) = \begin{bmatrix} 1 - 2\hat{\hat{u}}_s - \dfrac{b_1 K_1 \hat{\hat{v}}_2^2}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1 \hat{\hat{v}}_2\right]^2} & -(1+b_1)\hat{\hat{u}}_s & 0 & -\dfrac{b_2 \hat{\hat{u}}_s^2 (1-\hat{\lambda})}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1 \hat{\hat{v}}_2\right]^2} \\ 0 & 0 & 0 & 0 \\ \dfrac{b_4 K_1 \hat{\hat{v}}_2^2}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1 \hat{\hat{v}}_2\right]^2} & 0 & -(b_5+b_6) & \dfrac{b_4 \hat{\hat{u}}_s^2 (1-\hat{\lambda})}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1 \hat{\hat{v}}_2\right]^2} \\ \dfrac{b_7 K_1 \hat{\hat{v}}_2^2}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1 \hat{\hat{v}}_2\right]^2} & 0 & b_5 & \dfrac{b_7 \hat{\hat{u}}_s^2 (1-\hat{\lambda})}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1 \hat{\hat{v}}_2\right]^2} - b_8 \end{bmatrix}$$

Let $X^{[3]} = \left(X_1^{[3]}, X_2^{[3]}, X_3^{[3]}, X_4^{[3]}\right)^T$ be the eigen vector corresponding to the eigen value $\lambda_{3u_s} = 0$. Thus $\left(\hat{J}_{E_3} - \lambda_{3u_s} I\right) X^{[3]} = 0$ gives $X^{[3]} = \left(\psi_1 X_4^{[3]}, \psi_2 X_4^{[3]}, \psi_3 X_4^{[3]}, \psi_4 X_4^{[3]}\right)^T$ where $X_4^{[3]} \neq 0$ and

$$\psi_1 = \frac{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1 \hat{\hat{v}}_2\right]^2 b_8 (b_5 + b_6) - \hat{\hat{u}}_s^2 (1-\hat{\lambda}) \left[b_4 b_5 + b_7 (b_5 + b_6)\right]}{\left[b_4 b_5 + b_7 (b_5 + b_6)\right] K_1 \hat{\hat{v}}_2^2}$$

$$\psi_2 = -\frac{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1 \hat{\hat{v}}_2\right]^2 b_4 b_5 (1-\hat{\lambda})}{b_7 \left[b_4 b_5 + b_7 (b_5 + b_6)\right] K_1 \hat{\hat{v}}_2^2 (1+b_1) \hat{\hat{u}}_s} \left[1 - 2\hat{\hat{u}}_s - \frac{b_1 K_1 \hat{\hat{v}}_2^2}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1 \hat{\hat{v}}_2\right]^2}\right] +$$

$$\frac{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1 \hat{\hat{v}}_2\right]^2}{b_7 K_1 \hat{\hat{v}}_2^2 (1+b_1) \hat{\hat{u}}_s} \left\{ -\frac{b_2 b_7 K_1 (1-\hat{\lambda}) \hat{\hat{v}}_2^2 \hat{\hat{u}}_s^2}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1 \hat{\hat{v}}_2\right]^2} - \left[1 - 2\hat{\hat{u}}_s - \frac{b_1 K_1 \hat{\hat{v}}_2^2}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1 \hat{\hat{v}}_2\right]^2}\right] \left[\frac{b_7 \hat{\hat{u}}_s^2 (1-\hat{\lambda})}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1 \hat{\hat{v}}_2\right]^2} - b_8\right] \right\}$$

$$\psi_3 = \frac{b_4 b_8}{\left[b_4 b_5 + b_7 (b_5 + b_6)\right]}$$

$$\psi_4 = \psi_4$$

Let $W^{[3]} = \left(W_1^{[3]}, W_2^{[3]}, W_3^{[3]}, W_4^{[3]}\right)^T$ be the eigen vector associated with the eigen value $\lambda_{3u_s} = 0$ of the matrix $\left[\hat{J}_{E_3}\right]^T$. Then by solving the matrix $\left(\left[\hat{J}_{E_3}\right]^T - \lambda_{3u_s} I\right) W^{[3]} = 0$ we get $W^{[3]} = \left(0, W_2^{[3]}, 0, 0\right)^T$ where $W_2^{[3]} \neq 0$.

Now, $\dfrac{\partial F}{\partial b_3} = F_{b_3}(Y, b_3) = \left(\dfrac{\partial F_1}{\partial b_3}, \dfrac{\partial F_2}{\partial b_3}, \dfrac{\partial F_3}{\partial b_3}, \dfrac{\partial F_4}{\partial b_3}\right) = \left(0, -u_I, 0, 0\right)^T$.

Hence $\dfrac{\partial F}{\partial b_3}(E_3, b_3^*) = (0,0,0,0)^T$ and so $\left[W^{[3]}\right]^T \dfrac{\partial F}{\partial b_3}(E_3, b_3^*) = 0$.

Therefore, according to Sotomayor's theorem, Saddle node bifurcation cannot occur. But the first condition for the Transcritical bifurcation is satisfied. Now,

$$\left[W^{[3]}\right]^T DF(E_3,b_3^*)X^{[3]}=\left(0,W_2^{[3]},0,0\right)\begin{bmatrix}0\\ \psi_2 X_4^{[3]}\\ 0\\ 0\end{bmatrix}=\psi_2 X_4^{[3]}W_2^{[3]}\neq 0\,.$$

Now the second condition for Transcritical bifurcation is satisfied.

$$D^2F_b\ (E_3,b_3^*)(X^{[3]},X^{[3]})=\begin{bmatrix}-2(1+\eta_1)[X_1^{[3]}]^2-2(1+b_1)X_1^{[3]}X_2^{[3]}-2(b_1+b_2)\eta_2X_1^{[3]}X_4^{[3]}+2b_2\eta_3[X_4^{[2]}]^2\\ 2b_1X_1^{[3]}X_2^{[3]}\\ -2b_4\eta_1[X_1^{[3]}]^2+4b_4\eta_2X_1^{[3]}X_4^{[3]}-2b_4\eta_3[X_4^{[3]}]^2\\ -2b_7\eta_1[X_1^{[3]}]^2+4b_7\eta_2X_1^{[3]}X_4^{[3]}-2b_7\eta_3[X_4^{[3]}]^2\end{bmatrix}$$

$$[W^{[3]}]^T D^2F_{b_3}(E_3,b_3^*)(X^{[3]},X^{[3]})=\psi_2W_2^{[3]}\neq 0$$

Thus, according to Sotomayor's theorem, the system experiences Transcritical bifurcation at the point $E_3(\hat{\hat{u}}_s,0,\hat{\hat{v}}_1,\hat{\hat{v}}_2)$ when the parameter b_3 crosses the value b_3^*.

Also, the system does not experience Pitchfork bifurcation at E_3 when $b_3=b_3^*$.

D. Local Bifurcation analysis near $E_4\left(\tilde{u}_s,\tilde{u}_I,\tilde{v}_1,\tilde{v}_2\right)$

Theorem 4: If the parameter value b_8 passes through the value $b_8^*=b_8$, then the system at the axial equilibrium point $E_4\left(\tilde{u}_s,\tilde{u}_I,\tilde{v}_1,\tilde{v}_2\right)$ possesses:

- Saddle-node bifurcation
- No Transcritical bifurcation
- No Pitch-fork bifurcation

if $b_8^*=\left[\dfrac{b_4b_5}{b_5+b_6}+b_7\right]\dfrac{\tilde{u}_s^2(1-\tilde{\lambda})}{[\tilde{u}_s(1-\tilde{\lambda})+k_1\tilde{v}_2]^2}$.

Proof: The characteristic equation of the variational matrix $J(E_4)$ is $\lambda^4+H_1\lambda^3+H_2\lambda^2+H_3\lambda+H_4=0$ has a zero eigen value if and only if $H_4=0$ which yields E_4 a non-hyperbolic equilibrium point with $\lambda_{4u_s}=0$. Therefore, the variational matrix of the system at the equilibrium point E_4 with parameter $b_8=b_8^*$ is $\hat{J}(E_4)$

$$\hat{J}_{E_4}(b_8=b_8^*)=\begin{bmatrix}1-2\tilde{u}_s-(1+b_1)\tilde{u}_S-\dfrac{b_1K_1\tilde{v}_2^2}{\left[\tilde{u}_s(1-\tilde{\lambda})+K_1\tilde{v}_2\right]^2} & -(1+b_1)\tilde{u}_S & 0 & -\dfrac{b_2\tilde{u}_s^2(1-\tilde{\lambda})}{\left[\tilde{u}_s(1-\tilde{\lambda})+K_1\tilde{v}_2\right]^2}\\ b_1\tilde{u}_I & 0 & 0 & 0\\ \dfrac{b_4K_1\tilde{v}_2^2}{\left[\tilde{u}_s(1-\tilde{\lambda})+K_1\tilde{v}_2\right]^2} & 0 & -(b_5+b_6) & \dfrac{b_4\tilde{u}_s^2(1-\tilde{\lambda})}{\left[\tilde{u}_s(1-\tilde{\lambda})+K_1\tilde{v}_2\right]^2}\\ \dfrac{b_7K_1\tilde{v}_2^2}{\left[\tilde{u}_s(1-\tilde{\lambda})+K_1\tilde{v}_2\right]^2} & 0 & b_5 & -\dfrac{b_4b_5\tilde{u}_s^2(1-\tilde{\lambda})}{(b_5+b_6)\left[\tilde{u}_s(1-\tilde{\lambda})+K_1\tilde{v}_2\right]^2}\end{bmatrix}$$

Let $X^{[4]}=\left(X_1^{[4]},X_2^{[4]},X_3^{[4]},X_4^{[4]}\right)^T$ be the eigen vector corresponding to the eigen value $\lambda_{4u_s}=0$. Thus $\left(\hat{J}_{E_4}-\lambda_{4u_s}I\right)X^{[4]}=0$ which gives

$$X^{[4]}=\left(0,\frac{b_2\tilde{u}_s(1-\tilde{\lambda})}{\left[\tilde{u}_s(1-\tilde{\lambda})+K_1\tilde{v}_2\right]^2(1+b_1)}X_4^{[4]},\frac{b_4\tilde{u}_s^2(1-\tilde{\lambda})}{(b_5+b_6)\left[\tilde{u}_s(1-\tilde{\lambda})+K_1\tilde{v}_2\right]^2}X_4^{[4]},X_4^{[4]}\right)^T \text{ where } X_4^{[4]}\neq 0$$

Let $W^{[4]}=\left(W_1^{[4]},W_2^{[4]},W_3^{[4]},W_4^{[4]}\right)^T$ be the eigen vector associated with the eigen value $\lambda_{4u_s}=0$ of the matrix $\left[\hat{J}_{E_4}\right]^T$. Then by solving the matrix $\left(\left[\hat{J}_{E_4}\right]^T-\lambda_{4u_s}I\right)W^{[4]}=0$ we get

$$W^{[4]} = \left(0, -\frac{K_1\tilde{v}^2\left[b_4b_5 + b_7(b_5+b_6)\right]}{b_1u_I(b_5+b_6)\left[\tilde{u}_s(1-\tilde{\lambda}) + K_1\tilde{v}_2\right]^2}W_4^{[4]}, \frac{b_5}{(b_5+b_6)}W_4^{[4]}, W_4^{[4]}\right)^T \text{ where } W_4^{[4]} \neq 0.$$

Now, $\frac{\partial F}{\partial b_8} = F_{b_8}(Y, b_8) = \left(\frac{\partial F_1}{\partial b_8}, \frac{\partial F_2}{\partial b_8}, \frac{\partial F_3}{\partial b_8}, \frac{\partial F_4}{\partial b_8}\right) = \left(0,0,0,-v_2\right)^T$.

Hence $\frac{\partial F}{\partial b_8}(E_8, b_8^*) = \left(0,0,0,\tilde{v}_2\right)^T$ and so $\left[W^{[4]}\right]^T \frac{\partial F}{\partial b_8}(E_4, b_8^*) = \tilde{v}_2 W_4^{[4]} \neq 0$.

Therefore, according to Sotomayor theorem neither Transcritical nor Pitchfork bifurcation cannot occur at E_4. But the first condition for the Saddle node bifurcation is satisfied.

$$D^2F_{b_8}(E_4, b_8^*)(X^{[4]}, X^{[4]}) = \begin{bmatrix} -2(1+\eta_1)[X_1^{[4]}]^2 - 2(1+b_1)X_1^{[4]}X_2^{[4]} - 2(b_1+b_2)\eta_2 X_1^{[4]}X_4^{[4]} + 2b_2\eta_3[X_4^{[4]}]^2 \\ 2b_1X_1^{[4]}X_2^{[4]} \\ -2b_4\eta_1[X_1^{[4]}]^2 + 4b_4\eta_2X_1^{[4]}X_4^{[4]} - 2b_4\eta_3[X_4^{[4]}]^2 \\ -2b_7\eta_1[X_1^{[4]}]^2 + 4b_7\eta_2X_1^{[4]}X_4^{[4]} - 2b_7\eta_3[X_4^{[4]}]^2 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix}$$

$$[W^{[4]}]^T D^2F_{b_8}(E_4, b_8^*)(X^{[4]}, X^{[4]}) = -\frac{K_1\tilde{v}^2\left[b_4b_5 + b_7(b_5+b_6)\right]}{b_1u_I(b_5+b_6)\left[\tilde{u}_s(1-\tilde{\lambda}) + K_1\tilde{v}_2\right]^2}W_4^{[4]}\xi_2 + \frac{b_5}{(b_5+b_6)}W_4^{[4]}\xi_3 + W_4^{[4]}\xi_4 \neq 0$$

Thus, by Sotomayor's theorem system (1) has a Saddle node bifurcation at the equilibrium point.

IV. CONCLUSION

Local bifurcation is observed at equilibrium points by fixing each time a distinct parameter as the threshold value such that the jacobian matrix at that equilibrium point has atleast one zero eigen value. We also observed Transcritical bifurcation at the equilibrium point E_1, E_2, E_3 and E_4. For the preferred bifurcation parameter we obtained Saddle node bifurcation at the positive equilibrium point.

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