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Fair Restrained Dominating Set in the Cartesian Product and Lexicographic Product of Graphs

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Abstract *- In this paper, we characterize the fair restrained dominating set in the Cartesian product of two graphs and give some important properties of the lexicographic product of two graphs.*

Keywords: *Cartesian, fair dominating set, fair restrained dominating set, lexicographic, restrained dominating set*

I. INTRODUCTION

Domination in graphs has been a huge area of research in graph theory. Let G be a simple connected graph. A subset S of a vertex set $V(G)$ is a dominating set of G if, for every vertex $v \in V(G)\backslash S$, there exists a vertex $x \in S$ such that xv is an edge of G. The domination number $\gamma(G)$ of G is the smallest cardinality of a dominating set S of G. Domination in graphs was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [1]. Some related studies on domination in graphs are found in [2,3,4,5,6,7,8,9,10,11,12,13,14].

One variant of domination in graphs is a fair domination in graph [15]. A dominating subset S of $V(G)$ is a fair dominating set of G if all the vertices not in S are dominated by the same number of vertices from S, that is, $|N(u) \cap S| = |N(v) \cap S|$ for every two distinct vertices u and v from $V(G) \setminus S$ and a subset S of $V(G)$ is a k-fair dominating set in G if for every vertex $v \in V(G) \setminus S$, $|N(v) \cap S| = k$. The minimum cardinality of a fair dominating set of G, denoted by $\gamma_{fd}(G)$, is called the fair domination number of G. A fair dominating set of cardinality $\gamma_{fd}(G)$ is called γ_{fd}-set. A related paper on fair domination in graphs is found in [16,17,18,19,20]. Another variant of domination in a graph is the restrained domination number in a graph. This was introduced by Telle and Proskurowski [21,22] indirectly as a vertex partitioning problem. Moreover, a restrained dominating set can be found in [23,24,25,26,27,28,29,30,31]. One practical application of restrained domination is that of prisoners and guards. Here, each vertex not in the restrained dominating set corresponds to a position of a prisoner, and every vertex in the restrained dominating set corresponds to a position of a guard. To effect security, each prisoner's position is observed by a guard's position. To protect the rights of prisoners, each prisoner's position is seen by at least one other prisoner's position. To be cost effective, it is desirable to place as few guards as possible.

A fair dominating set $S \subseteq V(G)$ is a fair restrained dominating [16,25] set if every vertex not in S is adjacent to a vertex in S and a vertex in $V(G)\backslash S$. The fair restrained domination number, $\gamma_{frd}(G)$ of G is the minimum cardinality of a fair restrained dominating set of G. A fair restrained dominating set of cardinality $\gamma_{frd}(G)$ is called a γ_{frd}-set. In this paper, we characterize the fair restrained dominating sets of the Cartesian product of two graphs. For general concepts, the reader may to [32].

II. RESULTS

Note that the set $S = V(G)$ is a fair dominating [15] since every vertex in $V(G) \setminus S = \emptyset$ vacuously satisfies the desired property. Similarly, every graph G has a restrained dominating set [22], since $S = V(G)$ is such a set.

Consider the graph $G \cong K_1 + \overline{K}_n$ where n is a positive integer (Figure 1).

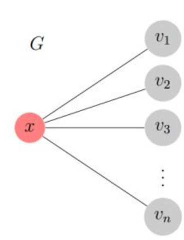

Figure 1: A graph $G \cong K_1 + \overline{K}_n$.

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Then $V(K_1) = \{x\}$ is the γ_{fd}-set of G. However, G does not contain a nontrivial restrained dominating set. Hence, $V(G)$ must be the γ_r-set of G. Since $V(G)$ is also a fair dominating set, it follows that $S = V(G)$ is the γ_{frd}-set of G.

Remark 2.1 Let S be a subset of $V(G)$. A fair restrained dominating set S is a fair dominating set and a restrained dominating set of a connected graph G.

The Cartesian product $G \boxdot H$ of two graphs G and H is the graph with $V(G \boxdot H) = V(G) \times V(H)$ and $(u,u')(v,v') \in E(G \boxdot H)$ if and only if either $uv \in E(G)$ and $u' = v'$ or $u = v$ and $u'v' \in E(H)$.

The following results are needed for the characterization of the Cartesian product of two connected graphs.

Lemma 2.2 Let G and H be connected graphs of orders $m \geq 3$ and $n \geq 3$ respectively. If $S = V(G) \times S_H$, where S_H is a fair dominating set of H, then S is a fair restrained dominating set of $G \boxdot H$.

Proof: Consider $S = V(G) \times S_H$ where S_H is a fair dominating set of H. Let $x \in V(G)$ and $u, v \in V(H) \setminus S_H$. Then $|N_H(u) \cap S_H| = |N_H(v) \cap S_H|$ since S_H is a fair dominating set of H. Further,

$$(x,u),(x,v) \in V(G) \times (V(H) \setminus S_H) = \big(V(G) \times V(H)\big) \setminus (V(G) \times S_H) = V(G \boxdot H) \setminus S,$$

and

$$\begin{aligned} &|N_H(u) \cap S_H| = |N_H(v) \cap S_H| \\ \Rightarrow\ &|\{x\} \times (N_H(u) \cap S_H)| = |\{x\} \times (N_H(v) \cap S_H)| \\ \Rightarrow\ &|\big(\{x\} \times N_H(u)\big) \cap (\{x\} \times S_H)| = |\big(\{x\} \times N_H(v)\big) \cap (\{x\} \times S_H)| \\ \Rightarrow\ &|\Big(N_{G\boxdot H}(x,u)\Big) \cap (V(G) \times S_H)| = |\Big(N_{G\boxdot H}(x,v)\Big) \cap (V(G) \times S_H)| \\ \Rightarrow\ &|N_{G\boxdot H}(x,u) \cap S| = |N_{G\boxdot H}(x,v) \cap S| \end{aligned}$$

This implies that S is a fair dominating set of $G \boxdot H$.

Case 1. If $(x,u)(x,v) \in E(G \boxdot H)$ then there exists $(x,w) \in S$ such that $(x,u)(x,w) \in E(G \boxdot H)$ since S is a fair dominating set of $G \boxdot H$.

Case 2. If $(x,u)(x,v) \notin E(G \boxdot H)$ then there exists $(x,z) \in S$ such that $(x,u)(x,z),(x,v)(x,z) \in E(G \boxdot H)$ since S is a fair dominating set of $G \boxdot H$. Further, since G is nontrivial connected graph, there exists $y \in V(G)$ distinct from $x \in V(G)$ such that $xy \in E(G)$. Thus, $(x,u)(y,u),(x,v)(y,v) \in E(G \boxdot H)$.

In either case, S is a restrained dominating set of $G \boxdot H$. Hence, by Remark 2.1, S is a fair restrained dominating set of $G \boxdot H$. ■

Lemma 2.3 Let G and H be connected graphs of orders $m \geq 3$ and $n \geq 3$ respectively. If $S = S_G \times V(H)$, where S_G is a fair dominating set of G, then S is a fair restrained dominating set of $G \boxdot H$.

Proof: Similar to Lemma 2.2. ■

Lemma 2.4 Let G and H be connected graphs of orders $m \geq 4$ and $n \geq 4$ respectively. Then S is a fair restrained dominating set of $G \boxdot H$ if $S = (V(G) \times S_H) \cup (S_G \times (V(H) \setminus S_H))$ where S_G and S_H are fair dominating sets of G and H respectively and the subgraph $\langle V(H) \setminus S_H \rangle$ does not contain an isolated vertex.

Proof: Consider $S = (V(G) \times S_H) \cup (S_G \times (V(H) \setminus S_H))$ where S_G and S_H are fair dominating sets of G and H respectively. Clearly, $V(G) \setminus S_G \neq \emptyset$ and $V(H) \setminus S_H \neq \emptyset$. Let $(x,u),(x,v) \in V(G \boxdot H) \setminus S$. Then

$$\begin{aligned} (x,u),(x,v) \in\ V(G \boxdot H) &= \big(V(G) \times V(H)\big) \setminus [(V(G) \times S_H) \cup (S_G \times (V(H) \setminus S_H))] \\ &= \big(V(G) \times V(H)\big) \setminus (V(G) \times S_H) \setminus (S_G \times (V(H) \setminus S_H)) \\ &= V(G) \times (V(H) \setminus S_H) \setminus (S_G \times (V(H) \setminus S_H)) \\ &= (V(G) \setminus S_G) \times (V(H) \setminus S_H) \end{aligned}$$

Thus, $(x,u),(x,v) \in (V(G) \setminus S_G) \times (V(H) \setminus S_H)$, implies that $x \in V(G) \setminus S_G$ and $u, v \in V(H) \setminus S_H$. Since S_H is a fair dominating set of H,

$$\begin{aligned} &|N_H(u) \cap S_H| = |N_H(v) \cap S_H| \\ \Rightarrow\ &|\{x\} \times (N_H(u) \cap S_H)| = |\{x\} \times (N_H(v) \cap S_H)| \\ \Rightarrow\ &|\big(\{x\} \times N_H(u)\big) \cap (\{x\} \times S_H)| = |\big(\{x\} \times N_H(v)\big) \cap (\{x\} \times S_H)| \\ \Rightarrow\ &|\Big(N_{G\boxdot H}(x,u)\Big) \cap (V(G) \times S_H)| = |\Big(N_{G\boxdot H}(x,v)\Big) \cap (V(G) \times S_H)| \\ \Rightarrow\ &\big|N_{G\boxdot H}(x,u) \cap S\big| = \big|N_{G\boxdot H}(x,v) \cap S\big|, \text{ since } (V(G) \times S_H) \subset S \end{aligned}$$

This implies that S is a fair dominating set of $G \boxdot H$.

Since $V(H) \setminus S_H \neq \emptyset$, let $u \in V(H) \setminus S_H$. Since $\langle V(H) \setminus S_H \rangle$ does not contain an isolated vertex, there exists $w \in V(H) \setminus S_H$ such that $uw \in E(H)$. Thus, $(x,u)(x,w) \in E(G \boxdot H)$ where $x \in V(G) \setminus S_G$. Since S_H is a fair dominating set of H, there exists $z \in S_H$ such that $uz \in E(H)$, that is, $(x,u)(x,z) \in E(G \boxdot H)$. Thus, for every $(x,u) \in V(G \boxdot H) \setminus S$, there exist $(x,z) \in S$ and $(x,v) \in V(G \boxdot H) \setminus S$ such that $(x,u)(x,z), (x,u)(x,w) \in E(G \boxdot H)$. Hence, S is a restrained dominating set of $G \boxdot H$.

Accordingly, S is a fair restrained dominating set of $G \boxdot H$. ■

Lemma 2.5 Let G and H be connected graphs of orders $m \geq 4$ and $n \geq 4$ respectively. Then S is a fair restrained dominating set of $G \boxdot H$ if $S = \left(S_G \times V(H)\right) \cup ((V(G) \setminus S_G) \times S_H)$ where S_G and S_H are fair dominating sets of G and H respectively and the subgraph $\langle V(G) \setminus S_G \rangle$ does not contain an isolated vertex.

Proof: Similar to Lemma 2.4. ■

The next result is the characterization of a fair restrained dominating set in the Cartesian product of two graphs.

Theorem 2.6 Let G and H be connected graphs of orders $m \geq 4$ and $n \geq 4$ respectively. Then a proper subset S of $V(G \boxdot H)$ is a fair restrained dominating set of $G \boxdot H$ if and only if one of the following statements is satisfied.

(i) $S = V(G) \times S_H$ where S_H is a fair dominating set of H.

(ii) $S = S_G \times V(H)$ where S_G is a fair dominating set of G.

(iii) $S = (V(G) \times S_H) \cup (S_G \times (V(H) \setminus S_H))$ where S_G and S_H are fair dominating sets of G and H respectively and the subgraph $\langle V(H) \setminus S_H \rangle$ does not contain an isolated vertex.

(iv) $S = \left(S_G \times V(H)\right) \cup ((V(G) \setminus S_G) \times S_H)$ where S_G and S_H are fair dominating sets of G and H respectively and the subgraph $\langle V(G) \setminus S_G \rangle$ does not contain an isolated vertex.

Proof: Suppose that a proper subset $S = V(G) \times S_H$ is a fair restrained dominating set of $G \boxdot H$. Let $S_H \subset V(H)$. Then $V(H) \setminus S_H \neq \emptyset$. Let $x \in V(G)$ and $u, v \in V(H) \setminus S_H$. Then

$$(x,u),(x,v) \in V(G) \times (V(H) \setminus S_H) = \left(V(G) \times V(H)\right) \setminus (V(G) \times S_H) = V(G \boxdot H) \setminus S.$$

Since S is a fair dominating set, for every distinct elements (x,u) and (x,v) of $V(G \boxdot H) \setminus S$, $|N_{G \boxdot H}\left((x,u)\right) \cap S| = |N_{G \boxdot H}\left((x,v)\right) \cap S|$. Thus, for every distinct elements u and v of $V(H) \setminus S_H$, $|N_H(u) \cap S_H| = |N_H(v) \cap S_H|$. By definition, S_H is a fair dominating set of H. This proves the statement (i).

Next, suppose that a proper subset $S = S_G \times V(H)$ is a fair restrained dominating set of $G \boxdot H$. Let $S_G \subset V(G)$. Then $V(G) \setminus S_G \neq \emptyset$. Let $x, y \in V(G) \setminus S_G$ and $u \in V(H)$. Then

$$(x,u),(y,u) \in (V(G) \setminus S_G) \times V(H) = \left(V(G) \times V(H)\right) \setminus \left(S_G \times V(H)\right) = V(G \boxdot H) \setminus S.$$

Since S is a fair dominating set, for every distinct elements (x,u) and (y,u) of $V(G \boxdot H) \setminus S$, $|N_{G \boxdot H}\left((x,u)\right) \cap S| = |N_{G \boxdot H}\left((y,u)\right) \cap S|$. Thus, for every distinct elements x and y of $V(G) \setminus S_G$, $|N_G(x) \cap S_G| = |N_G(y) \cap S_G|$. By definition, S_G is a fair dominating set of G. This proves the statement (ii).

Now, suppose that $S = (V(G) \times S_H) \cup (S_G \times (V(H) \setminus S\_H))$ is a fair restrained dominating set of $G \boxdot H$. Then $V(G) \times S_H$ must be a fair dominating set of $G \boxdot H$ (otherwise, S is not a fair dominating set of $G \boxdot H$). Since $V(G) \times S_H$ is a fair dominating set of $G \boxdot H$, S_H is a fair dominating set of $V(H)$ by statement (i). Let $x, y \in V(G) \setminus S_G$ and $u \in V(H) \setminus S_H$. Then

$$\begin{aligned} (x,u),(y,u) &\in (V(G) \setminus S_G) \times (V(H) \setminus S_H) \\ &= V(G) \times (V(H) \setminus S_H) \setminus (S_G \times (V(H) \setminus S_H)) \\ &= \left(V(G) \times V(H)\right) \setminus (V(G) \times S_H) \setminus (S_G \times (V(H) \setminus S_H)) \\ &= \left(V(G) \times V(H)\right) \setminus [(V(G) \times S_H) \cup (S_G \times (V(H) \setminus S_H))] \\ &= V(G \boxdot H) \setminus S \end{aligned}$$

Thus, $(x,u),(y,u) \in V(G \boxdot H) \setminus S$. Since S is a fair dominating set, for every distinct elements (x,u) and (y,u) of $V(G \boxdot H) \setminus S$, $|N_{G \boxdot H}\left((x,u)\right) \cap S| = |N_{G \boxdot H}\left((y,u)\right) \cap S|$. Thus, for every distinct elements x and y of $V(G) \setminus S_G$, $|N_G(x) \cap S_G| = |N_G(y) \cap S_G|$. By definition, S_G is a fair dominating set of G.

Since S is a restrained dominating set of $G \boxdot H$, for every $(x,u) \in V(G \boxdot H) \setminus S$, there exists, say $(x,v) \in V(G \boxdot H) \setminus$

S such that $(x,u)(x,v) \in E(G \boxdot H$.This means that $uv \in E(H)$ where $u,v \in V(H) \setminus S_H$. Since, u is an arbitrary element of $V(H) \setminus S_H$ it follows that the subgraph $\langle V(H) \setminus S_H \rangle$ does not contain an isolated vertex. This proves the statement (iii).

The proof of statement (iv) is similar to the proof of statement (iii).

For the converse, suppose that statement (i) is satisfied. Using the Lemma 2.2, S is a fair restrained dominating set of $G \boxdot H$. Similarly, if statement (ii), (iii), or (iv) is satisfied, then using the Lemma 2.3, Lemma 2.4, or Lemma 2.5, S is a fair restrained dominating set of $G \boxdot H$. ■

The following result is an immediate consequence of Theorem 2.6.

Corollary 2.7 Let G and H be nontrivial connected graphs of orders $m \geq 4$ and $n \geq 4$ respectively. Then $\gamma_{frd}(G \boxdot H) = min\{m \cdot \gamma_{fd}(H), \gamma_{fd}(G) \cdot n\}$.

Proof: Suppose that $S = V(G) \times S_H$ where S_H is a fair dominating set of H. Then S is a fair restrained dominating set of $G \boxdot H$ by Theorem 2.6(i). This implies that $\gamma_{frd}(G \boxdot H) \leq |S| = |V(G) \times S_H| = |V(G)| \cdot |S_H| = m \cdot |S_H|$ for all fair dominating set S_H. Hence, $\gamma_{frd}(G \boxdot H) \leq m \cdot \gamma_{sfd}(H)$.

Suppose that $S = S_G \times V(H)$ where S_G is a fair dominating set of G. Then S is a fair restrained dominating set of $G \boxdot H$ by Theorem 2.6(ii). This implies that $\gamma_{frd}(G \boxdot H) \leq |S| = |S_G \times V(H)| = |S_G| \cdot |V(H)| = |S_G| \cdot n$ for all fair dominating set S_G. Hence, $\gamma_{frd}(G \boxdot H) \leq \gamma_{fd}(G) \cdot n$. Thus, $\gamma_{frd}(G \boxdot H) \leq min\{m \cdot \gamma_{fd}(H), \gamma_{fd}(G) \cdot n\}$.

Now, Let S^o be a γ_{frd}-set of $G \boxdot H$. Then $|S^o| = min\{|S|: S$ is a fair restrained dominating set of $G \boxdot H\}$. Consider the following cases.

Case 1. Consider that $|S^o| \leq m \cdot \gamma_{fd}(H)$. If $|S^o| = m \cdot \gamma_{fd}(H)$, then $|S^o| = min\{m \cdot \gamma_{fd}(H), \gamma_{fd}(G) \cdot n\}. If$ $|S^o| < m \cdot \gamma_{fd}(H)$, then consider the next case.

Case 2. Consider that $|S^o| \leq \gamma_{fd}(G) \cdot n$. If $|S^o| = \gamma_{fd}(G) \cdot n$, then $|S^o| = min\{m \cdot \gamma_{fd}(H), \gamma_{fd}(G) \cdot n\}$. If $|S^o| < \gamma_{fd}(G) \cdot n$, then consider the next case.

Case 3. Consider that $|S^o| < m \cdot \gamma_{fd}(H)$ and $|S^o| < \gamma_{fd}(G) \cdot n$. Then $|S\text{^}o| < min\{m \cdot \gamma_{fd}(H), \gamma_{fd}(G) \cdot n\}$.

Suppose that $|S^o| = (m-1)\gamma_{fd}(H)$. Let $x \in V(G) \setminus S_G$ where $S_G \subset V(G)$ and let $a \in V(H) \setminus S_H$ where S_H is a fair dominating set of H. Then $(x,a) \in V(G \boxdot H) \setminus S^o$ and $(x,a)(u,v) \notin E(G \boxdot H)$ for all $(u,v) \in S^o$. Thus, S^o is not a dominating set of $G \boxdot H$ contradict to the fact that S^o is a dominating set of $G \boxdot H$. Hence, $|S^o| \neq (m-1)\gamma_{fd}(H)$. Similarly, if S_H is not a dominating set of H, then S^o is not a dominating set of $G \boxdot H$, a contradiction. Moreover, using the same arguments, if $|S^o| = \gamma_{fd}(G)(n-1)$, then S^o is not a dominating set of $G \boxdot H$, a contradiction. Hence, $|S^o| \neq \gamma_{fd}(G)(n-1)$. If S_G is not a dominating set of G, then S^o is not a dominating set of $G \boxdot H$, a contradiction. Thus, $|S^o|$ is not lesser than $\{m \cdot \gamma_{fd}(H), \gamma_{fd}(G) \cdot n\}$, that is, $|S^o| \geq \{m \cdot \gamma_{fd}(H), \gamma_{fd}(G) \cdot n\}$. Consequently,

$\gamma_{frd}(G \boxdot H) = |S^o| \geq min\{m \cdot \gamma_{fd}(H), \gamma_{fd}(G) \cdot n\}$. Hence, $\gamma_{frd}(G \boxdot H) = min\{m \cdot \gamma_{\{fd\}(H)}, \gamma_{fd}(G) \cdot n\}$. ■

The lexicographic product of two graphs G and H is the graph $G[H]$ with vertex-set $V(G[H]) = V(G) \times V(H)$ and edge-set $E(G[H])$ satisfying the following conditions: $(x,u)(y,v) \in E(G[H])$ if and only if either $xy\backslash in\, E(G)$ or $x = y$ and $uv \in E(H)$.

The following result shows some properties of a fair restrained dominating set in the lexicographic product of two graphs.

Theorem 2.8 Let $G = P_n = [v_1, v_2, \ldots, v_n]$, $n \geq 3$ and $H = K_3 = [u_1, u_2, u_3]$ where $n \geq 4$. A proper subset S of $V(G[H])$ is a fair restrained dominating set if one of the following statement is satisfied.

(i) $S = S_G \times V(H)$ where S_G is a fair dominating set of G.

(ii) $S = S_G \times S_H$ where S_G is a fair dominating set of G, $\langle V(G) \setminus (S_G \setminus \{v_1, v_n\}) \rangle$ does not contain an isolated vertex, and

$a)$ $S_H = \{u_2\}$, or

$b)$ $S_H = V(H) \setminus \{u_2\}$.

Proof: Let $G = P_n = [v_1, v_2, \ldots, v_n]$, $n \geq 3$ and $H = K_3 = [u_1, u_2, u_3]$ where $n \geq 4$. Suppose that statement (i) is satisfied. Then $S = S_G \times V(H)$ where S_G is a fair dominating set of G. Let $v, w \in V(G) \setminus S_G$. Then $|N_G(v) \cap S_G| = |N_G(w) \cap S_G|$. Let $u \in V(H)$. Then $(v,u), (w,u) \in (V(G) \setminus S_G) \times V(H) = (V(G) \times V(H)) \setminus (S_G \times V(H)) = V(G[H]) \setminus S$.

Thus, $(v,u), (w,u) \in V(G[H]) \setminus S$. Since S_G is a fair dominating set of G,

$$|N_G(v) \cap S_G| = |N_G(w) \cap S_G|$$
$$\Rightarrow |(N_G(v) \cap S_G) \times V(H)| = |(N_G(w) \cap S_G) \times V(H)|$$

$\Rightarrow |(N_G(v) \times V(H)) \cap (S_G \times V(H))| = |(N_G(w) \times V(H)) \cap (S_G \times V(H))|$

$\Rightarrow |N_{G[H]}(v,u) \cap S| = |N_{G[H]}(w,u) \cap S|$

This implies that S is a fair dominating set of $G[H]$.

Now, let $v \in V(G) \setminus S_G$ and $u,z \in V(H)$ such that $uz \in E(H)$. Then $(v,u),(v,z) \in V(G[H]) \setminus S$ such that $(v,u)(v,z) \in E(G[H])$. Since S_G is a fair dominating set of G. there exists $x \in S_G$ such that $vx \in E(G)$. Thus, $(v,u)(x,u) \in E(G[H])$. Hence, for every $(v,u) \in V(G[H]) \setminus S$, there exist $(x,u) \in S$ and $(v,z) \in V(G[H]) \setminus S$ such that $(v,u)(x,u),(v,u)(v,z) \in E(G[H]) \setminus S$. By definition, S is a restrained dominating set of $G[H]$. This implies that S is a fair restrained dominating set of $G[H]$.

Suppose that statement (ii) is satisfied. Then $S = S_G \times S_H$ where S_G is a fair dominating set of G and $\langle V(G) \setminus (S_G \setminus \{v_1,v_n\})\rangle$ does not contain an isolated vertex.

Case 1. If $S_H = \{u_2\}$, then let $v,w \in V(G) \setminus S_G$ and

$$(v,u_2),(w,u_2) \in (V(G) \setminus S_G) \times S_H = V(G) \times S_H) \setminus (S_G \times S_H) \subseteq (V(G) \times V(H)) \setminus (S_G \times S_H) = V(G[H]) \setminus S.$$

Thus, $(v,u_2),(w,u_2) \in V(G[H]) \setminus S$. Since S_G is a fair dominating set of G, Then $|N_G(v) \cap S_G| = |N_G(w) \cap S_G|$. This implies that

$|N_G(v) \cap S_G| = |N_G(w) \cap S_G|$

$\Rightarrow |(N_G(v) \cap S_G) \times \{u_2\}| = |(N_G(w) \cap S_G) \times \{u_2\}|$

$\Rightarrow |(N_G(v) \times \{u_2\}) \cap (S_G \times \{u_2\})| = |(N_G(w) \times \{u_2\}) \cap (S_G \times \{u_2\})|$

$\Rightarrow |N_{G[H]}(v,u_2) \cap (S_G \times S_H)| = |N_{G[H]}(w,u_2) \cap (S_G \times S_H)|$

$\Rightarrow |N_{G[H]}(v,u_2) \cap S| = |N_{G[H]}(w,u_2) \cap S|$

If $v \in V(G) \setminus (S_G \setminus \{v_1,v_n\})$, then there exists $v' \in V(G) \setminus (S_G \setminus \{v_1,v_n\})$ such that $vv' \in E(G)$ since $\langle V(G) \setminus (S_G \setminus \{v_1,v_n\})\rangle$ does not contain an isolated vertex. Thus, there exists $(v',u') \in V(G[H]) \setminus S$ such that $(v,u')(v',u') \in E(G[H])$ where $u' \in V(H) \setminus S_H$. Similarly

$|N_G(v) \cap S_G| = |N_G(v') \cap S_G|$

$\Rightarrow |(N_G(v) \cap S_G) \times \{u'\}| = |(N_G(v') \cap S_G) \times \{u'\}|$

$\Rightarrow |(N_G(v) \times \{u'\}) \cap (S_G \times \{u'\})| = |(N_G(v') \times \{u'\}) \cap (S_G \times \{u'\})|$

$\Rightarrow |N_{G[H]}(v,u') \cap (S_G \times \{u'\})| = |N_{G[H]}(v',u') \cap (S_G \times \{u'\})|$

Since, $|N_{G[H]}(v,u') \cap (S_G \times \{u'\})| = |N_{G[H]}(v,u') \cap (S_G \times \{u_2\})| = |N_{G[H]}(v,u') \cap (S_G \times S_H|$

and $|N_{G[H]}(v',u') \cap (S_G \times \{u'\})| = |N_{G[H]}(v',u') \cap (S_G \times \{u_2\})| = |N_{G[H]}(v',u') \cap (S_G \times S_H|$,

it follows that $|N_{G[H]}(v,u') \cap S| = |N_{G[H]}(v',u') \cap S|$ for all $u' \in V(H) \setminus S_H$

This implies that S is a fair dominating set of $G[H]$.

Now, let $v \in V(G) \setminus S_G$. Since $\langle V(G) \setminus (S_G \setminus \{v_1,v_n\})\rangle$ does not contain an isolated vertex, there exists $v' \in V(G) \setminus (S_G \setminus \{v_1,v_n\})$ such that $vv' \in E(G)$. Thus, there exists $(v',u_2) \in V(G[H]) \setminus S$ such that $(v,u_2)(v',u_2) \in E(G[H])$ where $u \in V(H)$. Since S_G is a fair dominating set of G. there exists $x \in S_G$ such that $vx \in E(G)$. Thus, $(v,u_2)(x,u_2) \in E(G[H])$. Hence, for every $(v,u_2) \in V(G[H]) \setminus S$, there exist $(x,u_2) \in S$ and $(v',u_2) \in V(G[H]) \setminus S$ such that $(v,u_2)(x,u_2),(v,u_2)(v',u_2) \in E(G[H]) \setminus S$. By definition, S is a restrained dominating set of $G[H]$. This implies that S is a fair restrained dominating set of $G[H]$.

Case 2. If $S_H \neq \{u_2\}$, then $S_H = V(H) \setminus \{u_2\}$, a fair dominating set of H. Let $v,w \in V(G) \setminus S_G$ and $u \in S_H$. This implies that $(v,u),(w,u) \in (V(G) \setminus S_G) \times S_H = (V(G) \times S_H) \setminus (S_G \times S_H) \subseteq (V(G) \times V(H)) \setminus (S_G \times S_H) = V(G[H]) \setminus S$.

Thus, $(v,u),(w,u) \in V(G[H]) \setminus S$. Since S_G is a fair dominating set of G, Then $|N_G(v) \cap S_G| = |N_G(w) \cap S_G|$. This implies that

$|N_G(v) \cap S_G| = |N_G(w) \cap S_G|$

$\Rightarrow |(N_G(v) \cap S_G) \times S_H| = |(N_G(w) \cap S_G) \times S_H|$

$\Rightarrow |(N_G(v) \times S_H) \cap (S_G \times S_H)| = |(N_G(w) \times S_H) \cap (S_G \times S_H)|$

$\Rightarrow |(N_G(v) \times V(H)) \cap (S_G \times S_H)| = |(N_G(w) \times V(H)) \cap (S_G \times S_H)|$

$\Rightarrow |N_{G[H]}(v,u) \cap (S_G \times S_H)| = |N_{G[H]}(w,u) \cap (S_G \times S_H)|$

$\Rightarrow |N_{G[H]}(v,u_2) \cap S| = |N_{G[H]}(w,u_2) \cap S|$

Let $v \in V(G) \setminus (S_G \setminus \{v_1, v_n\})$. Since $\langle V(G) \setminus (S_G \setminus \{v_1, v_n\})\rangle$ does not contain an isolated vertex, there exists $v' \in V(G) \setminus (S_G \setminus \{v_1, v_n\})$ such that $vv' \in E(G)$ Thus, there exists $(v', u_2) \in V(G[H]) \setminus S$ such that $(v, u_2)(v', u_2) \in E(G[H])$ where $u_2 \in V(H) \setminus S_H$. By similar arguments as shown earlier, $|N_{G[H]}(v, u_2) \cap S| = |N_{G[H]}(v', u_2) \cap S|$ where $u_2 \in V(H) \setminus S_H$. Hence, S is a fair dominating set of $G[H]$. Similarly, S is a restrained dominating set of $G[H]$. Accordingly, S is a fair restrained dominating set of $G[H]$.

The next result is an immediate consequence of Theorem 2.8.

Corollary 2.9 Let $G = P_n = [v_1, v_2, \ldots, v_n]$, $n \geq 3$ and $H = P_3 = [u_1, u_2, u_3]$ where $n \geq 4$. Then

$$\gamma_{frd}(\mathrm{G[H]}) = \begin{cases} \frac{n}{3} & if\ n = 0(mod\ 3), \\ \frac{n+2}{3} & if\ n = 1(mod\ 3) \\ \frac{n+1}{3} & if\ n = 2(mod\ 3) \end{cases}$$

Proof: Suppose that $S = S_G \times S_H$ where S_G is a fair dominating set of G, $\langle V(G) \setminus (S_G \setminus \{v_1, v_n\})\rangle$ does not contain an isolated vertex, and $S_H = \{u_2\}$. Then S is a fair restrained dominating set of $G[H]$ by Theorem 2.8. Thus, \$$\gamma_{frd}(G[H]) \leq |S|$.

Case 1. If $n = 0(mod\ 3)$, then $S_G = \{v_{3k-1}: k = 1,2, \ldots, \frac{n}{3}\}$ is a fair dominating set of G. Now, $|S| = |S_G \times S_H| = |S_G| \cdot |S_H| = \left(\frac{n}{3}\right) \cdot 1 = \frac{n}{3}$. Thus, $\gamma_{frd}(G[H]) \leq \frac{n}{3}$. Since for any fair dominating set S_G, $S_G \setminus \{v\}$ is not a dominating set of G when $n = 0(mod\ 3)$. It follows that S_G is a minimum fair dominating set of G. Hence, $S = S_G \times \{u_2\}$ is a minimum fair dominating set of $G[H]$, that is, $\frac{n}{3} = |S| = \gamma_{frd}(G[H]) \leq \frac{n}{3}$. Therefore, $\gamma_{frd}(G[H]) = \frac{n}{3}$, if $n = 0(mod\ 3)$.

Case 2. If $n = 1(mod\ 3)$, then $S_G = \{v_1, v_{3k-2}: k = 2,3, \ldots, \frac{n+2}{3}\}$ is a fair dominating set of G. $Now,$ $|S| = |S_G \times S_H| = |S_G| \cdot |S_H| = \left(\frac{n+2}{3}\right) \cdot 1 = \frac{n+2}{3}$. Thus, $\gamma_{frd}(G[H]) \leq \frac{n+2}{3}$. By similar arguments in Case 1, $\gamma_{frd}(G[H]) = \frac{n+2}{3}$, if $n = 1(mod\ 3)$.

Case 3. If $n = 2(mod\ 3)$, then $S_G = \{v_{3k-1}: k = 2,3, \ldots, \frac{n+1}{3}\}$ is a fair dominating set of G. Further, S_H is is a fair dominating set of H. Now, $|S| = |S_G \times S_H| = |S_G| \cdot |S_H| = \left(\frac{n+1}{3}\right) \cdot 1 = \frac{n+1}{3}$. Thus, $\gamma_{frd}(G[H]) \leq \frac{n+1}{3}$. Similarly, $\gamma_{frd}(G[H]) = \frac{n+1}{3}$, if $n = 2(mod\ 3)$.

By Case 1, Case 2, and Case 3, the desired results is prove.

III. CONCLUSION

In this paper, we extend the concept of the fair restrained domination in graphs. The fair restrained domination in the Cartesian product and lexicographic product of two connected graphs were characterized. Moreover, the fair restrained domination number of the Cartesian product and lexicographic product of two connected graphs were computed. This study may motivate researchers to work on fair restrained dominating set of other binary operations of two graphs. Other parameters involving fair restrained domination in graphs may also be explored. Finally, the characterization of a fair restrained domination in graphs and its bounds is also an extension of this study.

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