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TRANS-SASAKIAN MANIFOLD AND HYPERSURFACE

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Abstract. Object of present paper is to study of trans-Sasakian manifold with its hypersuface , considering some special condition satisfied by hypersurface. In addition , some theorems are given related to hypersurfaces of trans-Sasakian manifold and its curvature tensor with induced connection.

1. Introduction

Oubina [6] introduced the notion of trans-Sasakian manifolds which contains both the class of Sasakian and co-symplectic structures and are closely related to the locally conformal Kahler manifolds. A trans-Sasakian manifold of type $(0,0)$, $(\alpha,0)$ and $(0,\beta)$ are the cosymplectic , α-Sasakian and β-Kenmotsu manifold respectively. In 1972 , Chen and Yano introduced the notion of manifold of quasi-constant curvature [3]. Generalizing this notion , M.C. Chaki [7] introduced the idea of a manifold of generalized quasi-constant curvature. In this paper , section 2 contains definition and some of its relations on trans-Sasakian manifold. Section 3 and 4 deals with some properties of hypersurface of trans-Sasakian manifold together with its curvature tensor and theorems.

2. Trans-Sasakian Manifold

A $(2n+1)$ dimensional differentiable manifold M^{2n+1} is said to be an almost contact metric manifold [4] if it admits a $(1,1)$ tensor field ϕ, a contravariant vector field ξ , a 1-form η and a Riemannian metric g which satisfy

$$\phi^2 X = -X + \eta(X)\xi, \phi\xi = 0, \eta(\phi X) = 0 \tag{2.1}$$

$$g(\phi X, \phi Y) = -g(\phi X, \phi Y), \eta(X) = g(X,\xi), \eta(\xi) = 1 \tag{2.2}$$

$$g(\phi X, \phi Y) = -g(X,Y) - \eta(X)\eta(Y) \tag{2.3}$$

for all vector fields X , Y on M^{2n+1}.
An almost contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be trans-Sasakian Manifold [6], if $(M\times R, J, G)$ belongs to the class W_4 of the Hermitian manifolds, where J is the almost complex structure on $M \times R$ defined by

$$J(Z, f\frac{d}{dt}) = (\phi Z - f\xi, \eta(Z)\frac{d}{dt})$$

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for any vector field Z on M and smooth function f on $M \times R$ and G is the product metric on $M \times R$.
This may be defined by the condition [5]

$$(\nabla_X \phi)Y = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\phi X,Y)\xi - \eta(Y)\phi X] \tag{2.4}$$

where α, β are smooth functions on M^{2n+1} and we say that a structure trans-Sasakian structure of type (α, β). From Equation (2.4) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta[X - \eta(X)\xi] \tag{2.5}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.6}$$

In trans-Sasakian Manifold $M^{(2n+1)}(\phi, \xi, \eta, g)$ the following relation hold [9].

$$\begin{aligned} R(X,Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - (X\alpha)\phi Y - (X\beta)\phi^2(Y) \\ + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] + (Y\alpha)\phi X + (Y\beta)\phi^2 X \end{aligned} \tag{2.7}$$

$$\begin{aligned} \eta(R(X,Y,Z)) = (\alpha^2 - \beta^2)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ -2\alpha\beta[g(\phi X, Z)\eta(Y) - g(\phi Y, Z)\eta(X)] \\ -(Y\alpha)g(\phi X, Z) - (X\beta)[g(Y,Z) - \eta(Y)\eta(Z)] \\ + (X\alpha)g(\phi Y, Z) + (Y\beta)[g(X,Z) - \eta(Z)\eta(X)] \end{aligned} \tag{2.8}$$

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)[\eta(X)\xi - X] \tag{2.9}$$

$$S(X,\xi) = [2\eta(\alpha^2 - \beta^2) - \xi\beta]\eta(X) - (\phi(X) - \alpha) - (2n-1)(X\beta) \tag{2.10}$$

$$S(\xi,\xi) = 2n(\alpha^2 - \beta^2 - \xi\beta) \tag{2.11}$$

$$(\xi, \alpha) + 2\alpha\beta = 0 \tag{2.12}$$

$$\phi\xi = [2\eta(\alpha^2 - \beta^2 - \xi\beta)]\xi + \phi(grad\alpha) + (2n-1)(grad\beta) \tag{2.13}$$

for all vector fields X [1] .

3. Hypersurface of Trans-Sasakian Manifold

Let M^{2n+1} trans-Sasakian manifold with Riemannian metric g and Riemmanian connection ∇. Also let $\overline{M}$ be a hypersurface of M^{2n+1}. Let for a hypersurface $\overline{M}$ of M^{2n+1} we have[8]

$$(\nabla_X \eta)Y = (\nabla_Y \eta)X, \tag{3.1}$$

$$g(\eta(Y), Z) = H(Y, Z) \tag{3.2}$$

for all X, $Y \in \overline{M}$.
By (2.6) , we have

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y)$$

Also, by equation (2.2) we get ,

$$\alpha(g(\phi X, Y)) = 0 \tag{3.3}$$

Let ∇ be Riemannian connection on trans -Sasakian manifold M^{2n+1} and $\overline{\nabla}$ be induced connection on hepersurface $\overline{M}$ of trans-Sasakian manifold M^{2n+1} , then by Gauss-Wiengarten equations , we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{3.4}$$

$$\overline{\nabla}_X \xi = \nabla_X \xi + h(X,Y) \tag{3.5}$$

$$g(h(X,Y),N) = g(A_N X,Y), \tag{3.6}$$

for all X, $Y \in \overline{M}$, where h is the second fundamental form on $\overline{M}$ and A is shape operator[2]. So we have the following results:

Theorem 3.1. *If M^{2n+1} is trans-Sasakian manifold and $\overline{M}$ be a hypersurface of trans -Sasakian manifold M^{2n+1} , then $\overline{M}$ is β-Sasakian manifold if $g(\phi X,Y) \neq 0$.*

Proof. By equations (2.2) ,(2.6) , (3.1) and (3.2), we got the result. □

Theorem 3.2. *The hypersurface $\overline{M}$ of a trans-Sasakian manifold M^{2n+1} have invariant structure iff $g(\phi X,Y) = 0$, $\alpha \neq 0$,$\beta \neq 0$.*

Proof. Let $\overline{M}$ be a hypersurface of trans-Sasakian manifold M^{2n+1} , then by equation (3.3) , let us take $\alpha \neq 0$,$\beta \neq 0$, this implies

$$g(\phi X,Y) = 0,$$

for all X, Y in $\overline{M}$.
$\Rightarrow$ Since $\overline{M}$ is a hypersurface of trans-Sasakian manifold M^{2n+1}. Conversely, if $g(\phi X,Y) = 0 \Rightarrow g(X,\phi Y) = 0$.

□

Theorem 3.3. *If $\overline{M}$ is a hypersurface of M^{2n+1} and if $\overline{M}$ is invariant then we have*

$$(a)(\nabla_X \phi)Y = \alpha[g(X,Y)\xi - \eta(Y)] - \beta\eta(Y)\phi X \tag{3.7}$$

and

$$(b)(\nabla_X \eta)Y = \beta g(\phi X,\phi Y) \tag{3.8}$$

Proof. By equations (2.4) , (2.6) and (3.1),we got results (a) and (b). □

Theorem 3.4. *If $\overline{M}$ is hypersurface of a trans-Sasakian manifold M^{2n+1} and if $\alpha g(\phi X,Y) = 0$, then hypersurface $\overline{M}$ is co-symplectic if*

$$g(\phi X,Y)\xi - \eta(y)\phi X = 0$$

and

$$g(\phi X,\phi Y) = 0 \tag{3.9}$$

or

$$\begin{aligned} g(X,Y)\xi - \eta(Y)\phi X &= 0, \\ \eta(Y)\phi X &= 0, \end{aligned} \tag{3.10}$$

for all X ,$Y \in M$.

Proof. By equations (2.4) , (2.6) and (3.3) , we have the results (3.9) and (3.10). □

Theorem 3.5. *Let $\overline{M}$ is hypersurface of a trans-Sasakian manifold M^{2n+1}, then*

$$\begin{aligned} (\overline{\nabla}_X \phi)Y = {} & \alpha[g(X,Y)\xi - \eta(X)Y] + \beta[\eta(X)\phi Y - g(\phi Y,X)\xi] \\ & - \phi(\nabla_X Y) - h(X,Y) - g(\phi Y,h(X,\xi))\xi, \end{aligned} \tag{3.11}$$

and

$$\overline{\nabla}_X \eta)Y = \alpha g(X,\phi Y) + \beta[g(X,Y) - \eta(X)\eta(Y)] + \eta(h(X,Y)) + g(Y,h(X,\xi)) \tag{3.12}$$

for all $X,Y \in \overline{M}$.

Proof. With the help of equations (2.2) , (2.3) ,(3.4) and (3.5), we got both results. □

Theorem 3.6. *On hypersurface $\overline{M}$ of trans-Sasakian manifold M^{2n+1} , we have*

$$(i)(\overline{\nabla}_X\phi)Y = (\nabla_X\phi)Y$$

if

$$\eta(X)\phi Y + \eta(Y)\phi X = 0$$

and

$$\phi(\nabla_X Y) + h(X,Y) + g(\phi Y, h(X,\xi))\xi = 0,$$

for all X, $Y \in M$.

$$(ii)(\nabla_X\eta)Y = (\overline{\nabla}_X\eta)Y.$$

if

$$h(X,Y) + g(h(X,\xi),Y)\xi = 0$$

Proof. With the help of equations (2.4) , (3.11) and (3.12) ,we got both results. □

Theorem 3.7. *If $\overline{M}$ invariant hypersurface of trans Sasakian manifold then if X and ξ are conjugate then X and Y are also conjugate in $\overline{M}$.*

Proof. The result followed by previous theorem. □

Theorem 3.8. *If $\overline{M}$ invariant hypersurface of trans Sasakian manifold M^{2n+1} then if X and ξ are conjugate then all X and Y are asymptotic in $\overline{M}$.*

Proof. The proof is trivial. □

4. Riemannian Curvature Tensor of Hypersuface with Connection E

If R is Riemannian Curvature in trans-Sasakian manifold M^{2n+1} given by

$$R(X,Y,Z) = \nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z$$

then the Riemannian curvature tensor$\overline{R}$ on hypersurface $\overline{M}$ is given by

$$\begin{aligned}\overline{R}(X,Y,Z) = R(X,Y,Z) + \alpha[g(A_Z Y,\phi X)\xi - g(A_Z X,\phi X)\xi + \phi X\eta(A_Z Y) - \eta(A_Z X)\phi Y]\\ -\beta[g(A_Z X,\phi^2 Y)\xi - g(A_Z Y,\phi^2 X)\xi - \eta(A_Z X)\phi^2 X] + [g((E_Y h)(X,\xi),Z)\\ -g((E_X h)(Y,\xi),Z)]\xi + \eta(h(X,\xi))\eta(A_Z Y)\xi - \eta(A_Z X)\eta(h(Y,\xi))\xi\\ +\eta(A_Z X)\phi^2 Y + \eta(A_Z[X,Y])\xi + [\eta(A_Z(\nabla_Y Z)) - \eta(A_Z(\nabla_X Y))]\xi\\ +[\eta(A_Y X)\eta(A_Z\xi) + \eta(A_X Y)\eta(A_Z\xi)]\xi]\end{aligned}$$

where

$$g(A_Z\nabla_X Y,\xi) = g(h(\nabla_X Y),N)$$
$$E_X Y = \nabla_X Y - g(h(X,\xi),Y)\xi,$$

and we have the following relations:

$$\overline{R}(X,\xi,Z) = R(X,\xi,Z) + \alpha[g(A_Z\xi,\phi X) - g(A_Z X,\phi X)\xi$$

$$
\begin{aligned}
&+\phi X\eta(A_Z\xi) - \eta(A_Z\phi X)] - \beta[-g(A_Z\xi, \phi^2 X)\xi - \eta(A_Z X)\phi^2 X] \\
&+[g((E_X h)(X,\xi), Z) - g((E_X h)(\xi,\xi), Z)]\xi + \eta(h(X,\xi))\eta(A_Z\xi)\xi \\
&- [\eta(A_Z(\nabla_\xi Z))]\xi + [\eta(A_\xi X)\eta(A_Z\xi) + \eta(A_X\xi)\eta(A_Z\xi)]\xi \qquad (4.1)
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{R}(\xi,\xi,Z) &= \eta(h(\xi,\xi))\eta(A_Z\xi)\xi - \eta(A_\xi\xi)\eta(h(\xi,\xi))\xi \\
&[\eta(A_Z(\nabla_\xi Z))] + [\eta(A_\xi\xi)\eta(A_Z\xi) + \eta(A_\xi\xi)\eta(A_Z\xi)]\xi \qquad (4.2)
\end{aligned}
$$

Theorem 4.1. *If ξ is conjugate along any $X \in \overline{M}$ of trans-Sasakian manifold M^{2n+1}, then*

$$\overline{R}(X,Y,Z) = R(X,Y,Z) + [g((E_Y h)(X,\xi), Z) - g((E_X h)(Y,\xi), Z)]\xi \qquad (4.3)$$

Proof. If ξ is conjugate along any $X \in \overline{M}$ that is on trans Sasakian manifolds M^{2n+1} then by conjugate property $h(X,\xi) = 0$, we got the equation (4.3). □

Theorem 4.2. *If M^{2n+1} is trans-Sasakian manifold and $\overline{M}$ is a hypersurface of M^{2n+1}, then*

$$
\begin{aligned}
\overline{R}(X,\xi,Z) = R(X,\xi,Z) &+ \alpha[g(A_Z\xi,\phi X) - g(A_Z X,\phi X)\xi + \phi X\eta(A_Z\xi) - \eta(A_Z\phi X)] \\
&- \beta[-g(A_Z\xi,\phi^2 X)\xi - \eta(A_Z X)\phi^2 X] \qquad (4.4)
\end{aligned}
$$

if

$$
\begin{aligned}
[g((E_X h)(X,\xi), Z) - g((E_X h)(\xi,\xi), Z)] &= -\eta(h(X,\xi))\eta(A_Z\xi) \\
+ [\eta(A_Z(\nabla_\xi Z))] &- [\eta(A_\xi X)\eta(A_Z\xi) - \eta(A_X\xi)\eta(A_Z\xi)]. \qquad (4.5)
\end{aligned}
$$

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