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Teleparallel Killing Vectors of Non-Static Spherically Symmetric Space-Times

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Abstract

Killing symmetry equations in both coordinate formalism and tetrad formalism form a system of coupled partial differential equations and require decoupling before integration. This discourse discovers the method of minimum partial differentiation and applies it to decouple teleparallel Killing equations for a non-static spherically symmetric space-time involving functions $R(t)$ *and* $\lambda(r)$. *This method together with the method of separation of variables helps to decouple the basic system of teleparallel Killing equations and integrate. It finds that the function* $R(t)$ *remains arbitrary for teleparallel Killing vectors where as for the function* $\lambda(r)$ *there arise two cases. For one case the function* $\lambda(r) = -\ln r$ *and for other it is not. It shows eight teleparallel Killing vector fields for the first case and seven teleparallel Killing vector fields for the second.*

Keywords: *Torsion fields, Teleparallel Killing vector fields, Weitzenbock connection, Method of minimum partial differentiation*

I. INTRODUCTION

The well-known Einstein's general theory of relativity (GR) assumes a spin-less particle, with generalized coordinates, follow the geodesic of the underlying space-time. The generalized coordinate transformation formalism (CF) of this theory discovers Levi-Civita connection $\Gamma^{\rho}{}_{\mu\nu}$ symmetric in lower two indices. Thus, it discovers that the presence of gravitation field produces a curvature in space-time and the effect of torsion vanishes from the very beginning [1]. For the sake of argument let us assume the connection coefficient $\Gamma^{\rho}{}_{\mu\nu}$ is anti-symmetric i.e. $\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu} \neq 0$ then it easy to prove that the quantity $T^{*\rho}{}_{\nu\mu} = \Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu}$ now becomes a tensor [2]. However, this quantity disappears in GR because of its assumption.

An alternate to CF is tetrad formalism (TF). In this formalism, a set of four axes called tetrad

$$\gamma_m = \{\gamma_0, \quad \gamma_1, \quad \gamma_2, \quad \gamma_3\} \tag{1}$$

are attached to each point x^{μ} of space-time. Further, at each point of space-time there are a local set of coordinates ξ^m associated with the tetrad frame

$$\xi^m = \{\xi^0, \xi^1, \xi^2, \xi^3\} \tag{2}$$

Unlike the coordinate x^{μ} of the background geometry, the local coordinates ξ^m do not extend beyond the local frame at each point. The condition for orthonormality is the scalar product of these axes constitute the Minkowski metric $\eta_{mn} = (-1, \quad +1, \quad +1, \quad +1)$ and the scalar space-time distance ds^2 is

$$ds^2 = \gamma_{mn}\, d\xi^m\, d\xi^n \tag{3}$$

where $\gamma_m \cdot \gamma_n = \gamma_{mn}$ is a symmetric matrix called tetrad metric.

The transformation matrix between the tetrad frame (TF) and the coordinate frame (CF) is $h_m{}^{\mu}$ (the tetrad components) and the matrix inverse of it is $h^m{}_{\mu}$ (the inverse tetrad components), so that

$$h^m{}_{\mu}\, h_m{}^{\nu} = \delta^{\nu}_{\mu} \qquad \text{and} \qquad h^m{}_{\mu}\, h_n{}^{\mu} = \delta^m_n \tag{4}$$

In TF, the rising and lowering of indices are performed through tetrad metric γ_{mn} and it's inverse γ^{mn}. This is similar to CF where the rising and lowering of indices are performed by the Riemannian metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. Comparing the scalar space-time distance (3) for orthonormal tetrads with the scalar space-time distance of CF, the metric tensor $g_{\mu\nu}$ is found to be [3]

$$g_{\mu\nu} = \eta_{mn}\, h^m{}_{\mu}\, h^n{}_{\nu} \tag{5}$$

In order to distinguish coordinate frame and tetrad frame with their indices we will follow the Andrew's convention where Latin dummy indices label tetrad frame $(m, n, p, \cdots = 0,1,2,3)$ and Greek dummy indices label coordinate frame $(\mu, \nu, \rho, \cdots = 0,1,2,3)$.

The directed derivative ∂_m is a 4-vector in TF defined as $\partial_m = h_m{}^{\mu} \dfrac{\partial}{\partial x^{\mu}}$ and it is independent of the choice of coordinates as it has tetrad index only and no coordinate index. Unlike the derivatives in CF, the directed derivatives in TF do not commute. Based on the commutation of the directed derivatives in TF, two kinds of tetrads holonomic and non-holonomic are defined. Among the various modifications of Einstein's theory, an alternative and equivalent formulation of GR is the Teleparallel Gravity (TG). Teleparallel equivalent of General Relativity (TEGR) relies on a global flat space-time (zero curvature) with a non-vanishing torsion. In TEGR, non-holonomic tetrad is chosen to describe the gravitation. This theory is different from GR in the sense that it attributes torsion responsible for the acceleration of the universe in place of curvature and uses the Weitzenbook connection in place of the Levi-Civita connection. The TEGR for a given a nontrivial tetrad defines the connection as

$$\tilde{\Gamma}^k{}_{mn} \overset{\text{def.}}{=} h^k{}_{\mu} \frac{\partial h_m{}^{\mu}}{\partial x^n} \tag{6}$$

The connection defined by (6) is called Weitzenbock connection, it is anti-symmetric in lower two indices and the quantity

$$T^k{}_{mn} = \tilde{\Gamma}^k{}_{mn} - \tilde{\Gamma}^k{}_{nm} \tag{7}$$

is defined as torsion tensor. With the definition (7) it is easy to prove that the quantity $T^k{}_{mn}$ of equation (7) is non-zero but the curvature tensor vanishes. Thus, in TEGR**,** curvature vanishes in place of torsion [4-7].

Symmetries play a key role in modern physics. The symmetries of curved space-time are determined by existence of Killing vectors (KVs). Let there is a vector $K^{\rho}(x^{\mu})$ at every point x^{μ} of space-time in CF. Formulating with the help of Lie derivative, the condition for the metric $g_{\mu\nu}$ to remain unchanged or to be symmetric under translation in the direction of this vector K^{ρ} is given by following Killing equation

$$g_{\mu\nu,\rho} K^{\rho} + g_{\rho\nu} K^{\rho}{}_{,\mu} + g_{\mu\rho} K^{\rho}{}_{,\nu} = K_{\nu;\mu} + K_{\mu;\nu} = 0 \tag{8}$$

where ";" (semi-colon) represents covariant derivative and "," (comma) represents partial derivative [8]. Thus, symmetry in space is present if and only if equation (8) is satisfied. For a given metric $g_{\mu\nu}$, equation (8) is a system of partial differential equations determining the vector field $K^{\rho}(x^{\mu})$; if it has no solution, then space has no symmetry. Vector K^{ρ}, which is a solution of the equation (8) is Killing vector [8-9].

Sharif and Jamil (2008) obtained the condition of Killing symmetry in TEGR for a given metric g_{mn}. They established the TP version of Lie derivative of a second rank covariant and contravariant tensor along the vector field K^{m} (called TP Killing vectors) and obtained the TP Killing equations for metric tensor g_{mn} as follow

$$g_{mn,p} K^{p} + g_{pn} K^{p}{}_{,m} + g_{mp} K^{p}{}_{,n} + K^{p}\left(g_{kn} T^{k}_{mp} + g_{mk} T^{k}_{np}\right) = 0 \tag{9}$$

where $T^{k}{}_{mn}$ is torsion tensor given by (7). Equation (9) is a system of partial differential equations determining the TP Killing vectors $K^{p}(x^{\mu})$; if it has no solution, then space has no TP symmetry [10].

In order to answer the question of Killing symmetry it is required to solve the corresponding coupled partial differential equations (8) or (9) whichever is the case. One common technique for solving equations for Killing vectors or TP Killing vectors is to decouple unknowns before integration.

The paper is organized as follows. Section 2 discovers the method of minimum partial differentiation. Section 3 provides the system of TP Killing equations for a non-static spherically symmetric space-time. Section 4 provides solution of the TP Killing equations of section 3 using method of section 2. The last section presents the result.

II. METHOD OF MINIMUM PARTIAL DIFFERENTIATION

One of the methods for solution of an ordinary differential equation (ODE) is the "method of differentiation". The idea of this method is to differentiate the given ODE with respect to the independent variable to result in a new ODE that may sometimes factor. One may find several possible solutions by considering each factor of this new equation equal to zero. However, the general solution of each term must then be used in the original equation, possibly to constraint some of the parameters. According to Deniel Zwillinger (1989), this method is applicable to nonlinear ODE [11].

The Killing equation (8) is system of partial differential equations (PDE). Stephani H. (1996) has applied a technique similar to the "method of differentiation" for the solution of Killing equations (8) for Minkowski space in Cartesian coordinates. His idea is to differentiate partially "all the given Killing equations with respect to all the independent variables" to result in new PDE's. He then combines and integrates the resulting equations for Killing vectors. Finally, substitutes the solutions to original PDEs for possible constraint of the parameters [12]. The technique of Stephani H. (1996) is similar to the "method of differentiation" but instead of ordinary derivatives, it applies partially derivatives therefore we may call it "method of partial differentiation" (PD).

The Literatures [13-20] discuss Killing symmetry through (8) of some static and non-static space-time using PD similar to Stephani (1996) but they have applied partial differentiation intuitively to a minimum number of times. Mushtaq (1997) [20] argued that it is not necessary to differentiate "all the given Killing equations with respect to all the independent variables" instead "minimum number of partial differentiation" can result the same. The method of [13-20] will be referred hare as the "method of minimum partial differentiation" (MPD).

III. BASIC TELEPARALLEL KILLING EQUATIONS FOR NON-STATIC SPACE-TIMES

The idea of symmetry of space-time with torsion only is given in [10]. Finding the TP version of Lie derivative and the corresponding Killing equations it evaluates the TP KVs of the Einstein universe. The TP KVs of static spherically symmetric and the Friedmann space-time has evaluated in [21]. The TP KVs of the Schwarzschild space-time is calculated in [22] through non-othogonal tetrads. The TP KVs of Bianchi type VIII and IX space-times are calculated and classified according to their TP KVs in [23]. It further shows that the TP KVs for Bianchi type VIII and IX are different from that in general relativity. The TP KVs of Kantowski-Sachs space-time are explored in [24] using non-diagonal tetrad. The authors of the article [24] claim of using direct integration technique to solve the related system of coupled partial differential equations. Where as at page 497 they differentiate the related differential equation with respect to selected independent variable thus decouples the function before integration and substitute back in equations for possible solution. Again the direct integration method is claimed in [25] for classifying the cylindrically symmetric static space-times according to their teleparallel homothetic vector fields. Although the authors of the article [25] have avoided the calculation details but it is easy to check that direct integration without decoupling the related functions is not possible. Therefore, the appropriate name for the method of [24-25] is MPD.

Using the procedure of [10] and MPD technique, in this communication, we find TP KVs for the following non-static space-times in spherical coordinates $\left(x^0, \quad x^1, \quad x^2, \quad x^3\right)$ labeled by $(t, \quad r, \quad \theta, \quad \phi)$

$$ds^2 = -dt^2 + R^2(t)\left[e^{\lambda(r)}dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right] \tag{10}$$

Therefore

$$g_{00} = -1, \quad g_{11} = R^2(t)\, e^{\lambda(r)}, \quad g_{22} = R^2(t)\, r^2, \quad g_{33} = R^2(t)\, r^2 \sin^2\theta \tag{11}$$

The tetrad components and inverse tetrad components for the non-static space-time (10) from equation (5) applying (11) are

$$h_m{}^{\mu} = \left(1, \quad R(t)\, e^{\lambda(r)/2}, \quad R(t)\, r, \quad R(t)\, r \sin\theta\right) \tag{12}$$

and

$$h^m{}_{\mu} = \left(1, \quad R^{-1}(t)\, e^{-\lambda(r)/2}, \quad R^{-1}(t)\, r^{-1}, \quad R^{-1}(t)\, r^{-1} (\sin\theta)^{-1}\right) \tag{13}$$

Using equations (12-13) in equation (6), we obtain the following non-vanishing Weitzenbock connection coefficients

$$\tilde{\Gamma}^1_{10} = \tilde{\Gamma}^2_{20} = \tilde{\Gamma}^3_{30} = H(t), \qquad \tilde{\Gamma}^2_{21} = \tilde{\Gamma}^3_{31} = \frac{1}{r}, \qquad \tilde{\Gamma}^1_{11} = \frac{\lambda'}{2}, \qquad \tilde{\Gamma}^3_{32} = \cot\theta \tag{14}$$

where

$$H(t) = \frac{\dot{R}(t)}{R(t)} \quad \text{and} \quad \dot{R}(t) = \frac{d\,R(t)}{d\,t} \tag{15}$$

Utilizing (14) in (7), we find following non-vanishing torsion tensor components

$$T^{1}_{10} = T^{2}_{20} = T^{3}_{30} = -H(t), \qquad T^{2}_{21} = T^{3}_{31} = \frac{-1}{r}, \qquad T^{3}_{32} = -\cot\theta \tag{16}$$

Substituting equations (11) and (16) in equation (9) for the TP Killing vector field $K^{p} = \left(K^{0}, \quad K^{1}, \quad K^{2}, \quad K^{3}\right)$, we find following basic coupled partial differential equations

$$K^{0}_{,0} = 0 \tag{17}$$

$$K^{1}_{,1} = -\frac{\lambda'}{2} K^{1} \tag{18}$$

$$K^{2}_{,2} = 0 \tag{19}$$

$$K^{3}_{,3} = 0 \tag{20}$$

$$r\,K^{2} + r^{2}\,K^{2}{}_{,1} + e^{\lambda}\,K^{1}{}_{,2} = 0 \tag{21}$$

$$r\sin^{2}\theta\,K^{3} + r^{2}\sin^{2}\theta\,K^{3}{}_{,1} + e^{\lambda}\,K^{1}{}_{,3} = 0 \tag{22}$$

$$\sin\theta\,\cos\theta\,K^{3} + \sin^{2}\theta\,K^{3}{}_{,2} + K^{2}{}_{,3} = 0 \tag{23}$$

$$R^{2}\,H\,e^{\lambda}\,K^{1} + R^{2}\,e^{\lambda}\,K^{1}{}_{,0} - K^{0}{}_{,1} = 0 \tag{24}$$

$$R^{2}\,H\,r^{2}\,K^{2} + R^{2}\,r^{2}\,K^{2}{}_{,0} - K^{0}{}_{,2} = 0 \tag{25}$$

$$R^{2}\,H\,r^{2}\sin^{2}\theta\,K^{3} + R^{2}\,r^{2}\sin^{2}\theta\,K^{3}{}_{,0} - K^{0}{}_{,3} = 0 \tag{26}$$

IV. DECOUPLING AND SOLUTIONS OF BASIC EQUATIONS

On direct partial integration, the equations (17-20) provides

$$K^{0} = A(r,\theta,\phi) \tag{27}$$

$$K^{1} = e^{-\lambda/2} B(t,\theta,\phi) \tag{28}$$

$$K^{2} = C(t,r,\phi) \tag{29}$$

$$K^{3} = D(t,r,\theta) \tag{30}$$

where A, B, C, D are functions due to partial integration. Applying the method of MPD, the method of separation of variables and integration, we have

$$K^{0} = \left(\alpha_{1}\,\theta\,\phi + \alpha_{2}\,\theta + \alpha_{3}\,\phi + \alpha_{4}\right)\int e^{\lambda/2}\,dr + \left(C_{12}\,\theta\,\phi + C_{22}\,\theta + C_{32}\,\phi + C_{42}\right) \tag{31}$$

and

$$K^1 = \left(\alpha_1 \theta \phi + \alpha_2 \theta + \alpha_3 \phi + \alpha_4\right)\frac{e^{-\lambda/2}}{R}\int \frac{1}{R} dt$$

$$+\left(C_{11} \theta \phi + C_{21} \theta + C_{31} \phi + C_{41}\right)\frac{e^{-\lambda/2}}{R} \qquad (32)$$

where α_i are separation constants and C_{i1} and C_{i2} are constants of integration and $i \in \{1, 2, 3, 4\}$. Now in order to determine K^2 and K^3 we apply the method of MPD on equations (17-26) and utilizing equations (27-32), we obtain following constraint equations

$$C_{j2} + \alpha_j \left(\int e^{\lambda/2} dr - 2 r e^{\lambda/2}\right) = 0 \qquad (33)$$

where $j \in \{1, 2, 3\}$. Equation (33) leads to following two cases

Case I: $C_{j2} = 0$, and $\left(\int e^{\lambda/2} dr - 2 r e^{\lambda/2}\right) = 0$ (34)

Case II: $C_{j2} = 0$, and $\left(\int e^{\lambda/2} dr - 2 r e^{\lambda/2}\right) \neq 0$ (35)

The case I keeps $R(t)$ arbitrary, implies $\lambda = \ln\left(\frac{c}{r}\right)$ and decoupling of K^2 and K^3 with the method of MPD requires

$$c = 1, \quad \alpha_1 = \alpha_3 = 0, \quad C_{11} = C_{31} = 0 . \qquad (36)$$

Therefore, the TP KVs of the metric (10), for $\lambda = -\ln r$ are following

$$K^0 = 2\left(\alpha_2 \theta + \alpha_4\right)\sqrt{r} + C_{42} \qquad (37)$$

$$K^1 = \left(\alpha_2 \theta + \alpha_4\right)\frac{\sqrt{r}}{R}\int \frac{1}{R} dt + \left(C_{21} \theta + C_{41}\right)\frac{\sqrt{r}}{R} \qquad (38)$$

$$K^2 = \frac{2}{r^{3/2} R}\left(\alpha_2 \int \frac{1}{R} dt + C_{21}\right) + \frac{\left(\beta_1 \phi + \beta_2\right)}{r R} \qquad (39)$$

$$K^3 = \frac{\left(\beta_3 - \beta_1 \ln \tan \theta / 2\right)}{R r \sin \theta} \qquad (40)$$

β_j, $j \in \{1, 2, 3\}$ are constant of integrations.

This leads to the space-time

$$ds^2 = -dt^2 + R^2(t)\left[\frac{1}{r} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2\right] \qquad (41)$$

having following eight generators of the TP KVs

$$K_{(1)} = \frac{\partial}{\partial t}$$

$$K_{(2)} = \frac{\sqrt{r}}{R(t)}\frac{\partial}{\partial r}$$

$$K_{(3)} = \frac{1}{r\,R(t)}\frac{\partial}{\partial \theta}$$

$$K_{(4)} = \frac{1}{r\,R(t)\sin\theta}\frac{\partial}{\partial \varphi}$$

$$K_{(5)} = \frac{\varphi}{r\,R(t)}\frac{\partial}{\partial \theta} - \frac{\ln\tan\theta/2}{r\,R(t)\sin\theta}\frac{\partial}{\partial \varphi}$$

$$K_{(6)} = \frac{\theta\sqrt{r}}{R(t)}\frac{\partial}{\partial r} + \frac{2}{r^{3/2}\,R(t)}\frac{\partial}{\partial \theta}$$

$$K_{(7)} = 2\sqrt{r}\frac{\partial}{\partial t} + \frac{\sqrt{r}}{R(t)}\int\frac{dt}{R(t)}\frac{\partial}{\partial r}$$

$$K_{(8)} = 2\theta\sqrt{r}\frac{\partial}{\partial t} + \frac{\theta\sqrt{r}}{R(t)}\int\frac{dt}{R(t)}\frac{\partial}{\partial r} + \frac{2}{r^{3/2}\,R(t)}\int\frac{dt}{R(t)}\frac{\partial}{\partial \theta} \tag{42}$$

Similarly, the case II keeps $R(t)$ arbitrary, $\lambda \neq -\ln r$ and decoupling of K^2 and K^3 with the method of MPD requires

$$\alpha_1 = \alpha_2 = \alpha_3 = 0, \quad C_{11} = C_{31} = 0\,, \tag{43}$$

and provides following TP KVs of the metric (10)

$$K^0 = \alpha_4 \int e^{\lambda/2} dr + C_{42} \tag{44}$$

$$K^1 = \alpha_4 \frac{e^{-\lambda/2}}{R}\int\frac{1}{R}dt + \left(C_{21}\theta + C_{41}\right)\frac{e^{-\lambda/2}}{R} \tag{45}$$

$$K^2 = -\frac{C_{21}}{r\,R}\int\frac{e^{\lambda/2}}{r}dr + \frac{(\beta_1\phi + \beta_2)}{r\,R} \tag{46}$$

$$K^3 = \frac{(\beta_4 - \beta_1\ln\tan\theta/2)}{R\,r\sin\theta} \tag{47}$$

β_4 , is constant of integrations.

This leads to space-times having following seven generators of the TP KVs

$$K_{(1)} = \frac{\partial}{\partial t}$$

$$K_{(2)} = \frac{1}{r\,R(t)}\frac{\partial}{\partial \theta}$$

$$K_{(3)} = \frac{1}{r\,R(t)\sin\theta}\frac{\partial}{\partial\varphi}$$

$$K_{(4)} = \frac{e^{-\lambda/2}}{R(t)}\frac{\partial}{\partial r}$$

$$K_{(5)} = \frac{\varphi}{r\,R(t)}\frac{\partial}{\partial\theta} - \frac{\ln\tan\theta/2}{r\,R(t)\sin\theta}\frac{\partial}{\partial\varphi}$$

$$K_{(6)} = \frac{\theta\,e^{-\lambda/2}}{R(t)}\frac{\partial}{\partial r} + \frac{1}{r\,R(t)}\int\frac{e^{\lambda/2}\,dr}{r}\frac{\partial}{\partial\theta}$$

$$K_{(7)} = \int e^{\lambda/2}\,dr\,\frac{\partial}{\partial t} + \frac{e^{-\lambda/2}}{R(t)}\int\frac{dt}{R(t)}\frac{\partial}{\partial r} \tag{48}$$

V. CONCLUSION

This communication discovers the method of "minimum partial differentiation". The idea of this method is to differentiate partially "the given system of PDE minimum number of times with respect to minimum number of the independent variables" to result in new PDE's that may sometimes factor. One may find several possible solutions by considering each factor of this new equation equal to zero. However, the general solution of each term must then be used in all the original partial differential equations, possibly to constraint some of the parameters.

Applying the method of minimum partial differentiation it finds TP Killing vector field of a non-static spherically symmetric space-time involving two functions $R(t)$ and $\lambda(r)$. For arbitrary $R(t)$ and two cases for $\lambda(r)$, it shows that TP Killing symmetry is present. In one case $\lambda = -\ln r$ and for other it is not. There are seven generators of TP Killing vectors for the case $\lambda \neq -\ln r$ and for $\lambda = -\ln r$ there are eight generators of TP Killing vectors.

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