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Solving Fractional Delay Integro-Differential Equations by Chebyshev Wavelets

S. C. Shiralashetti[*1], B. S. Hoogar[*2]

[*1] *Department of Mathematics, Karnatak University, Dharwad - 580003, Karnataka, India*

[*2] *Department of Mathematics, S. D. M. College of Engineering and Technology, Dharwad-580002, Karnataka, India*

Abstract — *In this article, an efficient numerical technique based on second kind Chebyshev wavelets is proposed to solve a class of fractional delay integro-differential equations. The operational matrix of fractional integration is used to convert the equation under investigation into a system of algebraic equations which can be solved easily. Illustrative examples are included to demonstrate the high accuracy and applicability of the method. In addition, the numerical results are compared with exact solutions and other existing methods confirm that present technique is more efficient.*

Keywords — *Fractional calculus; Chebyshev wavelets; Delay Fredholm integro-differential equations; Numerical solution.*

I. INTRODUCTION

In recent years, many important problems in fluid mechanics, visco-elasticity, biology, oscillation theory, airfoil theory and other branches of science & engineering are involving fractional derivatives and integrals to an arbitrary order (real or complex) called as fractional calculus[1].Fractional calculus is a generalisation of ordinary differentiation & integration through arbitrary order. During last three decades, fractional calculus attracted by many scientist and researchers due its numerous applications in real life science and engineering problems. Some applications of fractional calculus are electrolyte polarization [2], optics and signal processing [3], circuit systems [4], probability and statistics [5], plasma physics, image processing and neutral network [6].

The problems containing delay integro-differential equations of fractional order are complex in nature and cannot be solved easily. Due to this complex nature and non-local issues of these problems development of approximate and numerical techniques play important role.

There have been few different techniques developed for solving fractional integro-differential equations, such as CAS wavelet method [7], Legendre wavelets method [8], variational iteration method and Homotopy perturbation method [9], Adomian decomposition method [10] and Taylor expansion method [11]. The methods based on wavelets are more efficient.

From last two decades, wavelet theory has been applied in many branches of science and engineering. It permits exact representation of functions. The properties of wavelets result into more accurate solutions of difficult problems.

In this paper, we consider the fractional integro-differential equation as

$$\begin{cases} D^{\alpha} f(t) + (J) \times f(t) + \tilde{K}(t, f(t)) = g(t), a \le t \le b \\ f(t) = \varphi(t) \\ f^{(i-1)}(\mu) = \varphi^{(i-1)}(t), t \in [\mu - \tau, \mu], i = 1,2,3,\ldots,N \end{cases} \tag{1}$$

where $J = \sum_{i=1}^{\alpha} J_0^{\omega_i}$ with $\omega_i \in R^T, i = 1,2,3,\ldots,\alpha$ for integer α and ω_i be the Riemann-Liouville fractional integer of $\omega_i > 0$. The function $\tilde{K}(t, f(t))$ is given by

$$\tilde{K}(t, f(t)) = \int_a^b k(t,s) f(t-\tau) dt \tag{2}$$

For delay fractional Fredholm integro-differential equation, we have

$$\tilde{K}\left(t, f\left(t\right)\right)=\int_{a}^{t}k\left(t,s\right)f\left(t-\tau\right)dt \qquad (3)$$

In the available literature, we noticed that there are few numerical techniques for the solution of fractional integral and differential equations based on wavelet theory [12].In the present work, we have used Chebyshev wavelets of second kind and its operational matrix for solving delay integro-differential equation of fractional order.

The remaining part of this article is organized as follows: In section 2, the basic definitions and mathematical preliminaries of Chebyshev wavelets and function approximation is given. The operational matrix of fractional integration using Chebyshev wavelets is obtained in section 3. In section 4, the method of solution is given. Some examples are included in section 5, to provide evidence of efficiency of present method. Lastly, the conclusion is given in section 6.

II. PRELIMINARIES OF CHEBYSHEV WAVELETS

This section gives some basic definitions and properties of Chebyshev wavelets of second kind.

A. Chebyshev wavelet of second kind

The wavelets are usually constructed by their polynomials. In the same way, the Chebyshev wavelets are constructed from the second kind Chebyshev polynomials which have important properties and are applicable in many fields.

Wavelets constitute a family of various functions formed by dilation and translation of a single function $\psi\left(t\right)$ called as mother wavelet. The family of continuous wavelets is given by the following equation with variation of dilation parameter 'a' and translation parameter 'b' as,

$$\psi_{a,b}\left(t\right)=\left(\left|a\right|^{-1/2}\right)\psi\left[\frac{t-b}{a}\right] \text{ with } a,b\in R \;\&\; a\neq 0$$

When these parameters a & b are restricted as $a=\left(a_0\right)^{-k}, b=\eta b_0\left(a_0\right)^{-k}$, $a_0>0 \;\&\; b_0>0$ then we get the family of discrete wavelets as

$$\psi_{m,n}\left(t\right)=\left(\left|a_0\right|^{\frac{m}{2}}\right)\psi\left[a_0^m t-\eta b_0\right] \text{ where } m,n\in z$$

Here $\psi_{m,n}\left(t\right)$ forms wavelet basis for $L^2\left(R\right)$. Also $\psi_{m,n}\left(t\right)$ forms an orthonormal basis when $a_0=2$ and $b_0=1$.

The second kind Chebyshev wavelets $\psi_{n,m}\left(t\right)$ defined on the interval $0\leq t<1$ and have four arguments $\psi_{n,m}\left(t\right)=\psi\left(k,n,m,t\right)$ where $n=1,2,3,...,2^{k-1}$, k is a positive integer, '*m*' is degree of Chebyshev polynomials and 't' is normalized time. The $\psi_{n,m}\left(t\right)$ is defined as,

$$\psi_{n,m}\left(t\right)=\begin{cases}\dfrac{2^{\frac{k}{2}+1}}{\sqrt{\pi}}V_m\left[2^{k+1}-2\left(n\right)-1\right], t\in\left[\dfrac{n}{2^k},\dfrac{n+1}{2^k}\right]\\ 0, \qquad\qquad\qquad \text{otherwise}\end{cases} \qquad (4)$$

where $V_m\left(t\right)$ is the Chebyshev polynomial of second kind of degree of m and is given from the following equation

$$V_m\left(t\right)=\frac{\left[sin\left(m+1\right)\phi\right]}{sin\phi}, t=cos\left(\phi\right) \quad \text{with } -1\leq t\leq 1 \text{ and } \phi\in\left[0,\pi\right]$$

These second kind Chebyshev wavelets $\psi_{n,m}\left(t\right), m=0,1,2,3,...,M-1$ forms orthonormal basis with weight function $\omega_{n,k}\left(t\right)=\omega\left(2^{k+1}t-2n-1\right)$ for $L^2_{\omega_{n,k}}$ over the interval $\left[0,1\right]$ in which $\omega\left(t\right)=\left(1-t^2\right)^{\frac{1}{2}}$.

The starting few polynomials are given by $V_0\left(t\right)=1$, $V_1\left(t\right)=2t$, $V_{m+1}\left(t\right)=2t\times V_m\left(t\right)-V_{m-1}\left(t\right)$ with $m=1,2,3,...,$

B. Chebyshev wavelet function approximation

By using the orthonormality property of the second kind Chebyshev wavelets, the function $f(t)$ defined over $0 \le t < 1$ can be expanded as

$$f(t) \cong \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{m,n} \psi_{m,n}(t) \tag{5}$$

The above equation having infinite sets when truncated, then it can also be written as

$$f(t) \cong \sum_{n=0}^{2^k-1} \sum_{m=0}^{\infty} C_{m,n} \psi_{m,n}(t) = [C]^T \psi(t) \tag{6}$$

where C and $\psi(t)$ are the column vectors with $\hat{m} = 2^k M$ and are given by

$$C = \left[C_{0,0}, C_{0,1}, \ldots, C_{0,(M-1)}, C_{1,0}, C_{1,1}, \ldots, C_{1,(M-1)}, \ldots, C_{2^k-1,0}, C_{2^k-1,1}, \ldots, C_{2^k-1,(M-1)}\right]^T \tag{7}$$

and

$$\psi(t) = \left[\psi_{0,0}(t), \psi_{0,1}(t), \ldots, \psi_{0,(M-1)}(t), \psi_{1,0}(t), \psi_{1,1}(t), \ldots, \psi_{1,(M-1)}(t), \ldots, \psi_{2^k-1,0}(t), \psi_{2^k-1,1}(t), \ldots, \psi_{2^k-1,(M-1)}(t)\right]^T \tag{8}$$

From these notations, the above equation can be written as

$$f(t) \cong \sum_{i=1}^{m} C_{i,j} \psi_{i,j}(t) = C^T \psi(t) \tag{9}$$

By using the collocation points as

$$t_r = \frac{[2i-1]}{2^k M} \text{ where } i = 1,2,3,\ldots,2^{k-1}M$$

We, now define the second kind Chebyshev wavelet matrix $Q_{m \times m}$ as

$$Q_{m \times m} = \left[\psi\left(\tfrac{1}{2m}\right), \psi\left(\tfrac{3}{2m}\right), \psi\left(\tfrac{5}{2m}\right), \ldots, \psi\left(\tfrac{2m-1}{2m}\right)\right] \quad \text{, with} \quad m = 2^{k-1}M \tag{10}$$

To illustrate the above matrix, for $M = 3$ and $K = 2$ the second kind Chebyshev wavelet can be expressed as

$$Q_{6 \times 6} = \begin{bmatrix} \frac{7979}{5000} & \frac{7979}{5000} & \frac{7979}{5000} & 0 & 0 & 0 \\ -\frac{21277}{10000} & 0 & \frac{21277}{10000} & 0 & 0 & 0 \\ \frac{3103}{2500} & -\frac{7979}{5000} & \frac{3103}{2500} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{7979}{5000} & \frac{7979}{5000} & \frac{7979}{5000} \\ 0 & 0 & 0 & -\frac{21277}{10000} & 0 & \frac{21277}{10000} \\ 0 & 0 & 0 & \frac{3103}{2500} & -\frac{7979}{5000} & \frac{3103}{2500} \end{bmatrix}$$

III. CHEBYSHEV WAVELET OPERATIONAL MATRIX OF THE FRACTIONAL INTEGRATION

Now, we obtain the fractional integration operational matrix by using second kind Chebyshev wavelets as follows:
In equation (8), the integration of the vector $\psi(t)$ can be expressed as

$$\int_0^t \psi(\tau)\,d\tau \approx P\psi(t) \tag{11}$$

It can also be re-written as,

$$\left[I^{\beta}\psi_m\right](t) \cong P^{\beta}_{m\times m}\psi_m(t) \tag{12}$$

Where the matrix $P^{\beta}_{m\times m}$ is second kind Chebyshev wavelet fractional integration operational matrix

Since, we have $$\psi_m(t) = Q_{m\times m}D_m(t) \tag{13}$$

where $D_m(t) \cong \left[d_0(t), d_1(t), d_2(t), \ldots, d_{m-1}(t)\right]^T$

The Adem K. et al [14] used the BPOM [Block pulse operational matrix for the fractional integration H^{β} as

$$[I^{\beta}D_m](t) \cong H^{\beta}.D_m(t) \tag{14}$$

where,

$$H^{\beta} = \frac{1}{m^{\beta}}\frac{1}{\overline{)\beta+2}}\begin{bmatrix} 1 & \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{m-1} \\ 0 & 1 & \mu_1 & \mu_2 & \cdots & \mu_{m-2} \\ 0 & 0 & 1 & \mu_1 & \cdots & \mu_{m-3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \mu_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{15}$$

From equations (13) and (14), the equation (12) can be expressed as

$$\begin{aligned} \left[I^{\beta}\psi_m\right](t) &\cong \left[I^{\beta}Q_{m\times m}D_m\right](t) \\ &= Q_{m\times m}\left(I^{\beta}D_m\right)(t) \end{aligned}$$

$$\left[I^{\beta}\psi_m\right](t) \cong Q_{m\times m}H^{\beta}D_m(t) \tag{16}$$

By equation (12) and (16), we have

$$P^{\beta}_{m\times m}\psi_m(t) = P^{\beta}_{m\times m}Q_{m\times m}D_m(t)$$

$$P^{\beta}_{m\times m}\psi_m(t) = Q_{m\times m}H^{\beta}D_m(t) \tag{17}$$

Therefore the fractional integration operational matrix $P^{\phi}_{m\times m}$ by using second kind Chebyshev wavelet is

$$P^{\beta}_{m\times m} = Q_{m\times m}H^{\beta}Q^{-1}_{m\times m} \tag{18}$$

We use this operational matrix for the solution of fractional order delay integro-differential equation in modified from as per the requirement based on the delay term.

IV. METHOD OF SOLUTION

Consider the fractional delay integro-differential given by equation (1).We approximate the fraction $g(t)$ by using second kind Chebyshev wavelet as

$$g(x) \cong S^{\mathrm{T}}\psi(t) \tag{19}$$

Also, it can be written as

$$E^{\beta}f(t) \cong C^{\mathrm{T}}\psi(t) \tag{20}$$

By using the given conditions, it can be expressed as,

$$f(t) \cong C^{\mathrm{T}} P^{\beta}_{m\times m} \psi(t) \tag{21}$$

Using equation (10) in the above equation (18), we get

$$f(t) \cong C^{\mathrm{T}} P^{\beta}_{m\times m} Q_{m\times m} D_m(t) \tag{22}$$

Let, $C^{\mathrm{T}} P^{\beta}_{m\times m} Q_{m\times m} = [b_0, b_1, b_2, ..., b_{m-1}] = B$ (23)

Also, by using equations (10) and (11), we get

$$f(t) \cong AD_m(t)$$

By using the properties, we have

$$\begin{aligned}
\left[f(t)\right]^2 &= \left[BD_m(t)\right]^2 \\
&= \left[b_0 d_0(t) + b_1 d_1(t) + ... + b_{m-1} d_{m-1}(t)\right]^2 \\
&= b_0^2 d_0(t) + b_1^2 d_1(t) + ... + b_{m-1}^2 d_{m-1}(t) \\
&= \left[b_0^2, b_1^2, b_2^2, ..., b_{m-1}^2\right] D_m(t) \\
&= \tilde{B}_2 D_m(t)
\end{aligned} \tag{24}$$

By using induction property, we get

$$\left[f(t)\right]^q \cong \left[b_0^q, b_1^q, b_2^q, ..., b_{m-1}^q\right] D_m(t) = \tilde{B}_q D_m(t) \tag{25}$$

where $\tilde{B}_q = \left[b_0^q, b_1^q, \tilde{b}_2^q, ..., b_{m-1}^q\right]$ for $q > 0$.

Since $g(x,t) = \left[\psi(x)\right]^{\mathrm{T}} G\left[\psi(t)\right]$ and from equations (13) & (25), we have

$$\begin{aligned}
\int_{t=0}^{1} g(x,t)\left[f(t)\right]^q dt &= \int_{t=0}^{1} \left[\psi(x)\right]^T G\left[\psi(t)\right] D_m^T(x) B_q^T dt \\
&= \int_{t=0}^{1} \left[\psi(x)\right]^T G Q_{m\times m} D_m(t) D_m^T(x) B_q^T dt \\
&= \left[\psi(x)\right]^T G Q_{m\times m} \int_{t=0}^{1} D_m(t) D_m^T(x) B_q^T dt
\end{aligned} \tag{26}$$

Using the disjointness & orthogonality properties of block pulse function and simplifying the above equation's integral part, we have

$$\begin{aligned}
\int_{t=0}^{1} D_m(t) D_m^T(t) B_q^T dt &= \int_{t=0}^{1} \left\{ \begin{bmatrix} d_0(t) & 0 & \cdots & 0 \\ 0 & d_1(t) & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & d_{m-1}(t) \end{bmatrix} \begin{bmatrix} b_0^q \\ b_1^q \\ \vdots \\ b_{m-1}^q \end{bmatrix} \right\} dt \\
&= \int_{t=0}^{1} \left[b_0^q d_0(t), b_1^q d_1(t), ..., b_{m-1}^q d_{m-1}(t)\right]^T dt \\
&= \frac{1}{m}\left[b_0^q d_0(t), b_1^q d_1(t), ..., b_{m-1}^q d_{m-1}(t)\right]^T \\
&= \frac{1}{m}\hat{B}_q
\end{aligned}$$

Now, equation (26) becomes,

$$\int_{t=0}^{1} g(x,t)\times[f(t)]^{q}\,dt \cong \frac{1}{m}\psi^{T}(x)GQ_{m\times m}\tilde{B}_{q} \tag{27}$$

Next, we substitute these equations (16), (17), (4) and (24) in equation (1), we get

$$[\psi(t)]^{T}C-\mu\frac{1}{m}[\psi(t)]^{T}GQ_{m\times m}\tilde{B}_{q}\cong[\psi(t)]^{T}H \tag{28}$$

Multiplying above equation (28) by $\psi(t)$ on both sides, then we have by the orthonormality property of second kind Chebyshev wavelet integration over the interval $[0,1]$ as,

$$C-\mu\frac{1}{m}GQ_{m\times m}\tilde{B}_{q}=H \tag{29}$$

This is a nonlinear system containing algebraic equations. Solving this nonlinear system of algebraic equations, we obtain the numerical solution of equation (1) in accordance with equation (25). For the delay term τ the same procedure is carried with the modification in the variables.

V. NUMERICAL EXAMPLES

The purpose of this section is to demonstrate the accuracy and applicability of the present technique.

Example 1. Consider the delay fractional Fredholm integro-differential equation *[12]*

$$\frac{dy}{dt}+J_{0}^{1/2}y(t)-\int_{x=0}^{1}y\left(x-\frac{1}{3}\right)dx=\frac{5\sqrt{\pi}}{16}t^{3}+\frac{16}{15\sqrt{\pi}}t^{\frac{5}{2}}+\frac{5}{2}t^{\frac{3}{2}}+2t-K,\quad 0\le t\le 1 \tag{30}$$

subjected to the initial condition $y(t)=0,\quad -\frac{1}{3}\le t\le 0$.

where $K=\frac{8}{81}+\frac{16\sqrt{6}}{567}$ and $J=\sum_{i=1}^{\alpha}J_{0}^{\omega_i}$ with $\omega_i\in R^{+}, i=1,2,3,\ldots,\alpha$ for a integer α and ω_i be the Riemann-Liouville fractional integral of order $\omega_i>0$.

The exact solution of the above equation (30) is $y(t)=t^{2}+t^{\frac{5}{2}}$. This equation (30) can be solved by using the present technique which is described in previous section. The computational results are displayed in table 1 and shown in Fig.1.As we see the results in table 1 and Fig.1, it is clear that the numerical solution converges to the exact solution.

Table 1. Exact and numerical solutions of example 1.

t	Exact solution	Numerical solution
0	0.00000000000	0.000000000000
0.1	0.01316227766	0.013169925011
0.2	0.05788854382	0.057887923951
0.3	0.13929503017	0.139294903018
0.4	0.26119288512	0.261191988521
0.5	0.42677669529	0.426769723492
0.6	0.63885480092	0.638853984215
0.7	0.89996341300	0.899959341985
0.8	1.21243340224	1.212429638213
0.9	1.57843347142	1.578429832605
1.0	2.00000000000	1.999942659872

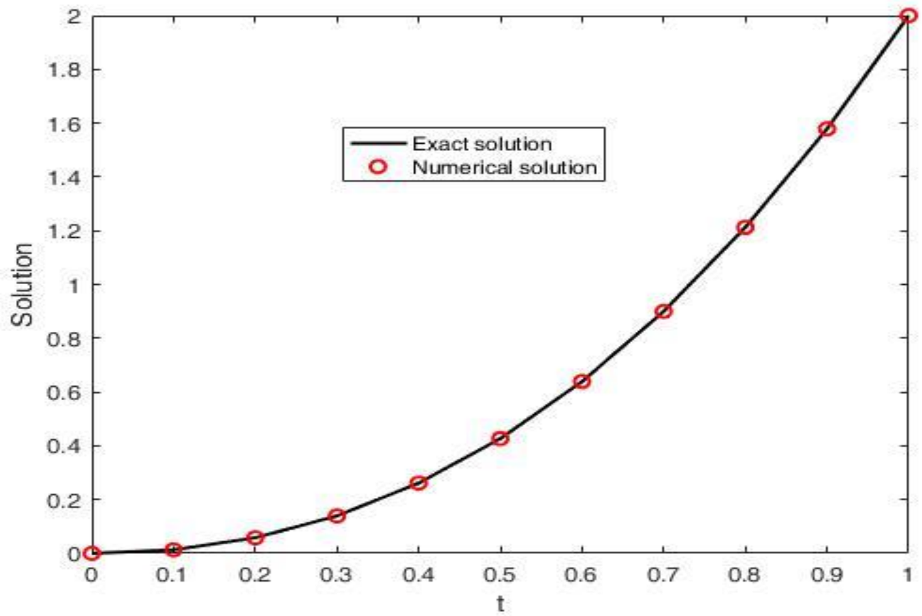

Fig.1 Graph of exact and numerical solution at various values of t of example 1.

Example 2. Consider the following linear delay fractional integro-differential equation *[13]*

$$\frac{d^2y}{dt^2}+D^{0.7}y(t)+D^{0.5}y(t)+D^{0.5}y(t-1)=\int_{x=-1}^{1}(3x-3t)y(x)dx+f(t), \quad 0\le t\le 1 \tag{31}$$

with initial conditions $y(0)=1, y'(0)=0$

where $f(t)=2+(5.3333)t-(2.2563)(t)^{\frac{1}{2}}+(1.7142)(t)^{1.3}+(3.0087)(t)^{\frac{3}{2}}$.

The equation (31) has the exact solution as $y(t)=t^2+1$. The approximate solutions obtained by applying the present scheme are listed in the following table 2 over the interval $[0,1]$. The solutions obtained are converging with the exact solutions and are shown in the Fig.2.

Table 2. The exact, obtained approximate solutions and absolute errors of example 2.

t	Exact solution	Approximate solution	Absolute error
0	1.00000000	1.00004572	4.572×10^{-5}
0.1	1.01000000	1.01021524	2.1524×10^{-4}
0.2	1.04000000	1.04001522	1.522×10^{-5}
0.3	1.09000000	1.09004519	4.519×10^{-5}
0.4	1.16000000	1.16005416	5.416×10^{-5}
0.5	1.25000000	1.25001523	1.523×10^{-5}
0.6	1.36000000	1.36000541	5.410×10^{-6}
0.7	1.49000000	1.49005653	5.653×10^{-5}
0.8	1.64000000	1.64001528	1.528×10^{-5}
0.9	1.81000000	1.81000254	2.540×10^{-6}
1.0	2.00000000	2.00000021	2.100×10^{-7}

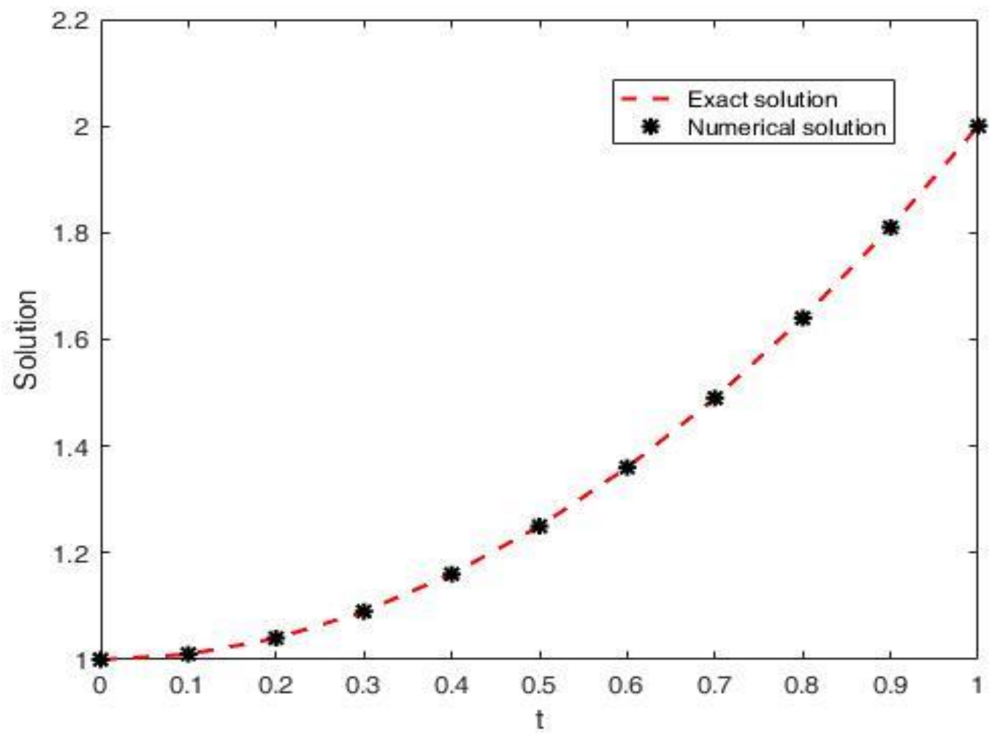

Fig.2 Graph of exact and numerical solution of example 2 at various values of t.

Example 3. Consider another linear delay fractional integro-differential equation *[13]*

$$y^{\beta}(t)-t\frac{dy}{dt}+\left[t\times y(t)\right]-y'(t-1)+y(t-1)=f(t)+\int_{x=-1}^{1}(3x-2t)y(x)\,dx, \quad 0\le t\le 1 \tag{32}$$

subjected to the conditions $y(0)=1, y'(0)=0$

where $f(t)=2t^2+t\left[\sin(t)+\cos(t)\right]-\cos(t)+\sin(t-1)+\cos(t-1)+4t\sin(1).$

This delay fractional integro-differential equation at $\beta=2$ has the analytic solution as $y(t)=\cos(t)$. The computational results of this example are displayed for various values of β in the following table 3 and are shown in Fig.3. From this table 3 and Fig.3, it is clearly observed that the present scheme has good accuracy and converges to the exact solution.

Table 3. Computational results for various values of β of example 3.

t	Exact solution at $\beta=2$	Present method at $\beta=2$	Present method at $\beta=1.9$	Present method at $\beta=1.8$
0	1.00000000	1.00000120	1.00000214	1.00000218
0.1	0.99500416	0.99500415	0.99500335	0.99857512
0.2	0.98006657	0.98006591	0.98006581	0.98042569
0.3	0.95533648	0.95533647	0.95533623	0.95236214
0.4	0.92106099	0.92106098	0.92106521	0.92473768
0.5	0.87758256	0.87758245	0.87845293	0.87523906
0.6	0.82533561	0.82533554	0.83034528	0.83906854
0.7	0.76484218	0.76484216	0.77584215	0.78459212
0.8	0.69670671	0.69670668	0.71074512	0.73412586
0.9	0.62160996	0.62160984	0.63614258	0.68955421
1.0	0.54030230	0.54030221	0.57854123	0.61095684

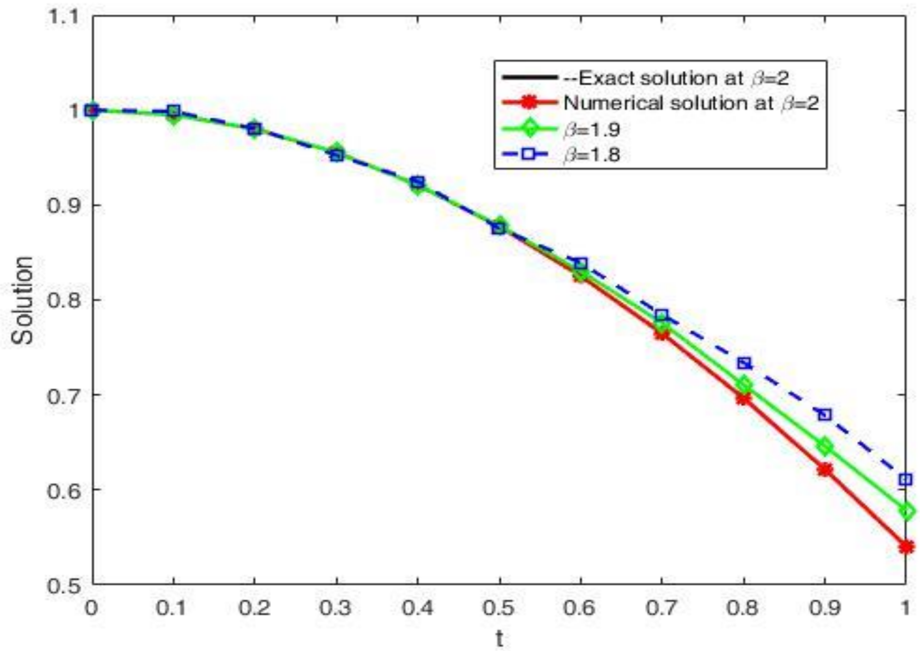

Fig. 3 Comparison of exact and approximate solutions for different values of β of example 3.

Example 4. Consider another fractional integro-differential equation with delay as *[13]*

$$D^{0.5}y(t-\lambda)=\frac{t}{12}+\int_{x=0}^{1} xy(x)dx+g(t), \quad t\in[1-\lambda,1] \tag{33}$$

subjected to the condition $y(0)=0$

The exact solution of (33) when $\lambda=0$ is $y(t)=t^2-t$ .The obtained results by applying the present scheme are displayed in table 4 and Fig. 4. In table 4, we compare the results with other method in [13].Also from this table 4, it is clear that the present technique has good accuracy with exact solution and other methods.

Table 4. Comparison of numerical results at various values of λ for example 4.

t	Exact solution at $\lambda=0$	Spectral collocation method *[13]* solution at			Present method solution at		
		$\lambda=0.01$	$\lambda=0.02$	$\lambda=0.03$	$\lambda=0.01$	$\lambda=0.02$	$\lambda=0.03$
0.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.2	-0.16000	-0.160769	-0.157172	-0.153576	-0.160012	-0.157254	-0.153498
0.4	-0.24000	-0.245119	-0.237903	-0.230698	-0.240193	-0.238541	-0.230125
0.6	-0.24000	-0.251869	-0.241074	-0.230300	-0.249511	-0.241254	-0.230298
0.8	-0.16000	-0.180557	-0.1662231	-0.151918	-0.180023	-0.165984	-0.151891
1.0	0.00000	-0.030929	-0.013094	0.0044010	-0.002984	-0.013009	0.0044009

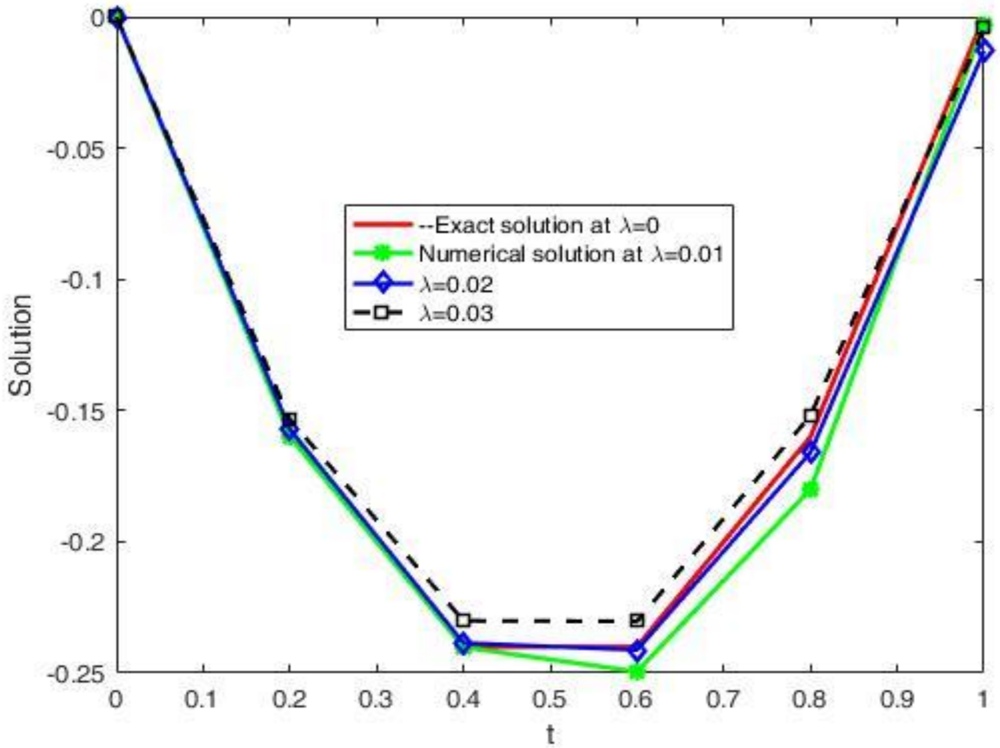

Fig. 4 Comparison of exact and numerical solutions at various values of λ of example 4.

VI. CONCLUSIONS

In this study, Chebyshev wavelets of second kind and its operational matrix of the fractional integration are used for the solution of fractional order delay integro-differential equations. The main advantage of this method is that converts considered equation into system of algebraic equations & can be solved easily. The achieved numerical results are compared with the exact solutions and with other existing methods to demonstrate the powerfulness of the present technique. Through examples, the high accuracy, computational efficiency and direct applicability of this method have been expressed.

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