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Mathematical Modelling of Convective Transport of Dispersion in One-Dimensional Flow of Saturated and Unsaturated Porous Media

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Abstract *- The purpose of this paper is to study an analytical first order solution to the one-dimensional advection-dispersion equation with adsorption term* $C_0 e^{-\gamma t}$ *to study the transport of pollutant vary exponentially with time using a generalized integral transform method to investigate the transport of sorbing but otherwise non-reacting solutes in hydraulic homogenous but geochemically heterogeneous porous formations. The solution is derived under conditions of steady-state flow and arbitrary initial and inlet boundary conditions. The results obtained by this solution agree well with the results obtained by numerically inverting Laplace transform-generated solutions previously published in the literature. The solution is developed for a third or flux type inlet boundary condition, which is applicable when considering resident solute concentrations and a semi-infinite porous medium.*

Keywords — *Integral Transforms Method, Mathematical Modelling, One-Dimensional Flow, Porous Media.*

I. INTRODUCTION

Recent field evidence shows that the classical form of the convection-dispersion equation is inadequate for describing discipline-scale solute transport [Matheron and De Marsily (1980), Sposito et. al. (1986), Sudheendra (2012)] due, in part, to an apparent increase in the dispersity as a function of the travel distance. This has inspired research to develop other techniques for describing the sector transport behavior. Freeze and Cherry (1979) briefly discuss chemical adsorption and methods of measurement. Charbeneau (1981) has shown how the method of characteristics may simply be used in the analysis of linear or non-linear sorption and multi-component ion exchange. Finally, the transport solution from the dimensionless streamline and the breakthrough curve are combined to yield the pollutant break-through curve.

In many ground water problems, this added complexity makes no sense considering the fact that molecular diffusion is normally immaterial. neglecting molecular diffusion, Hunt (1998) established one-dimensional analytical solutions of a scale-based dispersion equation for instable flow with an immediate supply, and for stable flow with a continuous source. however, solutions acquired by Hunt (1998) are for infinite model areas, at the whereas few ground water problems require solutions for semi-infinite areas.

II. MATHEMATICAL MODEL

The Advection Dispersion Equations along with Initial and Boundary Conditions can be written as

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial z^2} - w\frac{\partial C}{\partial z} - \left(\frac{1-n}{n}\right)K_d C \tag{1}$$

Consider a semi-infinite porous medium in a unidirectional flow field in which the input tracer concentration is $C_0 e^{-\gamma t}$, where C_0 is a reference concentration and γ is a constant as show in figure 1. Originally, saturated flow of fluid of concentration, *C = 0*, taking place in the permeable media. At *t = 0*, the concentration of the higher surface is instantly reformed to $C = C_0 e^{-\gamma t}$

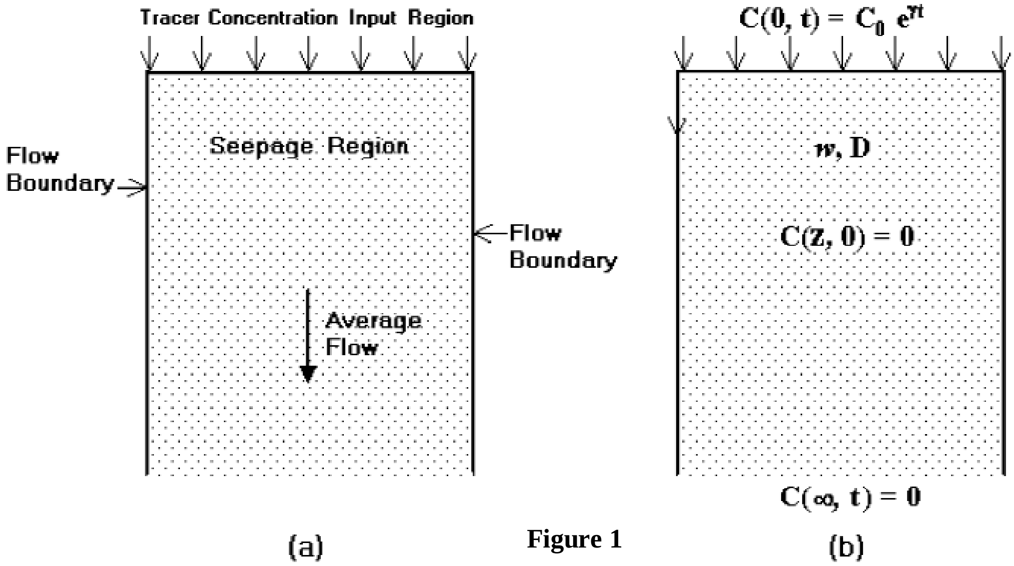


Figure 1

Thus, the suitable Boundary Conditions for the given model

$$\left.\begin{array}{ll} C(z,0)=0 & z\geq 0 \\ C(0,t)=C_0 e^{-\gamma t} & t\geq 0 \\ C(\infty,t)=0 & t\geq 0 \end{array}\right\}. \tag{2}$$

The problem is describe the concentration as a function of z and *t*,
To reduce equation (3) to a more conversant form, we take

$$C(z,\ t)=\Gamma(z,\ t)\,Exp\left[\frac{wz}{2D}-\frac{w^2 t}{4D}-\frac{K_d(1-n)\,t}{n}\right] \tag{3}$$

Substituting equation (3) into equation (1) gives

$$\frac{\partial \Gamma}{\partial t}=D\frac{\partial^2 \Gamma}{\partial z^2} \tag{4}$$

The Initial and Boundary Conditions (2) transform to

$$\left.\begin{array}{ll} \Gamma(0,t)=C_0\ Exp\left[\dfrac{w^2 t}{4D}+\dfrac{K_d(1-n)\,t}{n}-\gamma t\right] & t\geq 0 \\ \Gamma(z,0)=0 & z\geq 0 \\ \Gamma(\infty,\ t)=0 & t\geq 0 \end{array}\right\} \tag{5}$$

Equation (4) can be resolved for a time dependent arrival of the fluid at *z = 0*.
If $C = F\,(x,\ y,\ z,\ t)$, same has been explain the previous chapter, then the solution of the problem in which the surface is maintained at temperature $\phi(t)$ is

$$C=\int_0^t \phi(\tau)\frac{\partial}{\partial t}\,F(x,y,z,t-\tau)\,d\tau$$

Let us consider, the problem which C_0 is zero and the Boundary Condition is sustained at concentration unity. The Boundary Conditions are

$$\left.\begin{array}{ll}\Gamma(0,t)=0 & t\geq 0\\ \Gamma(z,0)=1 & z\geq 0\\ \Gamma(\infty,t)=0 & t\geq 0\end{array}\right\}$$

The Laplace transform of equation (4) is

$$L\left[\frac{\partial\Gamma}{\partial t}\right]=D\frac{\partial^2\Gamma}{\partial z^2}$$

Henceforth, it is condensed to an Ordinary Differential Equation

$$\frac{\partial^2\overline{\Gamma}}{\partial z^2}=\frac{p}{D}\overline{\Gamma} \tag{6}$$

The solution of the equation is $\overline{\Gamma}=Ae^{-qz}+Be^{qz}$ where, $q=\pm\sqrt{\frac{p}{D}}$.

The Boundary Condition as $z\to\infty$ requires that $B=0$ and Boundary Condition at $z=0$ requires that $A=\frac{1}{p}$ thus the specific
solution of the Laplace transform equation is

$$\overline{\Gamma}=\frac{1}{p}e^{-qz}$$

The transposal of the above function using table of Laplace transform. The outcome is

$$\Gamma=1-erf\left(\frac{z}{2\sqrt{Dt}}\right)=\frac{2}{\sqrt{\pi}}\int_{\frac{z}{2\sqrt{Dt}}}^{\infty}e^{-\eta^2}d\eta$$

Using Duhamel's theorem, we have

$$\Gamma=\int_0^t\phi(\tau)\frac{\partial}{\partial t}\left[\frac{2}{\sqrt{\pi}}\int_{\frac{z}{2\sqrt{D(t-\tau)}}}^{\infty}e^{-\eta^2}d\eta\right]d\tau$$

Since $e^{-\eta^2}$ is a constant function, , which gives

$$\frac{2}{\sqrt{\pi}}\frac{\partial}{\partial t}\int_{\frac{z}{2\sqrt{D(t-\tau)}}}^{\infty}e^{-\eta^2}d\eta=\frac{z}{2\sqrt{\pi D}(t-\tau)^{3/2}}Exp\left[\frac{-z^2}{4D(t-\tau)}\right]$$

The solution to the problem is

$$\Gamma=\frac{z}{2\sqrt{\pi D}}\int_0^t\phi(\tau)Exp\left[\frac{-z^2}{4D(t-\tau)}\right]\frac{d\tau}{(t-\tau)^{3/2}} \tag{7}$$

Putting $\mu=\frac{z}{2\sqrt{D(t-\tau)}}$ then the equation (7) becomes

$$\Gamma=\frac{2}{\sqrt{\pi}}\int_{\frac{z}{2\sqrt{Dt}}}^{\infty}\phi\left(t-\frac{z^2}{4D\mu^2}\right)e^{-\mu^2}d\mu\quad. \tag{8}$$

Since $\phi(t) = C_0 \; Exp\left(\frac{w^2 t}{4D} + \frac{K_d(1-n)t}{n} - \gamma t \right)$ the particular solution of problem be written as

$$\Gamma(z,t) = \frac{2C_0}{\sqrt{\pi}} Exp\left(\frac{w^2 t}{4D} + \frac{K_d(1-n)t}{n} - \gamma t \right) \left\{ \int_0^\infty Exp\left(-\mu^2 - \frac{\varepsilon^2}{\mu^2} \right) d\mu - \int_0^\alpha Exp\left(-\mu^2 - \frac{\varepsilon^2}{\mu^2} \right) d\mu \right\} \tag{9}$$

where, $\alpha = \frac{z}{2\sqrt{D\,t}}$ and $\varepsilon = \sqrt{\left(\frac{w^2}{4D} + \frac{K_d(1-n)}{n} - \gamma \right)} \; \left(\frac{z}{2\sqrt{D}} \right)$.

III. EVALUATION OF THE INTEGRAL SOLUTION

The integration of 1st term of equation (9) gives

$$\int_0^\infty Exp\left(-\mu^2 - \frac{\varepsilon^2}{\mu^2} \right) d\mu = \frac{\sqrt{\pi}}{2} e^{-2\varepsilon} . \tag{10}$$

For suitability the 2nd integral can be stated in terms of error function
Noticing that

$$-\mu^2 - \frac{\varepsilon^2}{\mu^2} = -\left(\mu + \frac{\varepsilon}{\mu} \right)^2 + 2\varepsilon = -\left(\mu - \frac{\varepsilon}{\mu} \right)^2 - 2\varepsilon .$$

The 2nd integral of equation (9) becomes

$$I = \int_0^\alpha Exp\left(-\mu^2 - \frac{\varepsilon^2}{\mu^2} \right) d\mu = \frac{1}{2} \left\{ e^{2\varepsilon} \int_0^\alpha Exp\left[-\left(\mu + \frac{\varepsilon}{\mu} \right)^2 \right] d\mu + e^{-2\varepsilon} \int_0^\alpha Exp\left[-\left(\mu - \frac{\varepsilon}{\mu} \right)^2 \right] d\mu \right\} \tag{11}$$

Then the system of reducing integral to a tabularized function is the same for both integrals in the right side of equation (11), only the 1st term is considered. Let $a = \varepsilon / \mu$ and the integral can be stated as

$$I_1 = e^{2\varepsilon} \int_0^\alpha Exp\left[-\left(\mu + \frac{\varepsilon}{\mu} \right)^2 \right] d\mu$$

$$= -e^{2\varepsilon} \int_{\varepsilon/\alpha}^\infty \left(1 - \frac{\varepsilon}{a^2} \right) Exp\left[-\left(\frac{\varepsilon}{a} + a \right)^2 \right] da + e^{2\varepsilon} \int_{\varepsilon/\alpha}^\infty Exp\left[-\left(\frac{\varepsilon}{a} + a \right)^2 \right] da . \tag{12}$$

Further, let, $\beta = \left(\frac{\varepsilon}{a} + a \right)$ in the $\beta = \frac{\varepsilon}{a} + a$ 1st term of the above equation, then

$$I_1 = -e^{2\varepsilon} \int_{\alpha + \frac{\varepsilon}{\alpha}}^\infty e^{-\beta^2} d\beta + e^{2\varepsilon} \int_{\frac{\varepsilon}{\alpha}}^\infty Exp\left[-\left(\frac{\varepsilon}{a} + a \right)^2 \right] da . \tag{13}$$

Similar evaluation of the 2nd integral of equation (11) gives

$$I_2 = e^{-2\varepsilon} \int_{\varepsilon/\alpha}^\infty Exp\left[-\left(\frac{\varepsilon}{a} - a \right)^2 \right] da - e^{-2\varepsilon} \int_{\varepsilon/\alpha}^\infty Exp\left[-\left(\frac{\varepsilon}{a} - a \right)^2 \right] da .$$

Again substituting $-\beta = \frac{\varepsilon}{a} - a$ into the first term, the result is

$$I_2 = e^{-2\varepsilon}\int_{\frac{\varepsilon}{\alpha}-\alpha}^{\infty} e^{-\beta^2}\,d\beta - e^{-2\varepsilon}\int_{\varepsilon/\alpha}^{\infty} Exp\left[-\left(\frac{\varepsilon}{a}-a\right)^2\right]da\,.$$

Noticing that

$$\int_{\varepsilon/\alpha}^{\infty} Exp\left[-\left(a+\frac{\varepsilon}{a}\right)^2+2\varepsilon\right]da = \int_{\varepsilon/\alpha}^{\infty} Exp\left[-\left(\frac{\varepsilon}{a}-a\right)^2-2\varepsilon\right]da$$

Substitution into equation (11) gives

$$I = \frac{1}{2}\left(e^{-2\varepsilon}\int_{\frac{\varepsilon}{\alpha}-\alpha}^{\infty} e^{-\beta^2}\,d\beta - e^{2\varepsilon}\int_{\alpha+\frac{\varepsilon}{\alpha}}^{\infty} e^{-\beta^2}\,d\beta\right). \tag{14}$$

Thus, equation (9) becomes

$$\Gamma(z,t) = \frac{2C_0}{\sqrt{\pi}} Exp\left(\frac{w^2t}{4D}+\frac{K_d(1-n)\,t}{n}-\gamma t\right)\left\{\frac{\sqrt{\pi}}{2}e^{-2\varepsilon}-\frac{1}{2}\left[e^{-2\varepsilon}\int_{\frac{\varepsilon}{\alpha}-\alpha}^{\infty} e^{-\beta^2}\,d\beta - e^{2\varepsilon}\int_{\alpha+\frac{\varepsilon}{\alpha}}^{\infty} e^{-\beta^2}\,d\beta\right]\right\} \tag{15}$$

Though, by definition,

$$e^{2\varepsilon}\int_{\alpha+\frac{\varepsilon}{\alpha}}^{\infty} e^{-\beta^2}\,d\beta = \frac{\sqrt{\pi}}{2}e^{2\varepsilon}\; erfc\left(\alpha+\frac{\varepsilon}{\alpha}\right)$$

also,

$$e^{-2\varepsilon}\int_{\frac{\varepsilon}{\alpha}-\alpha}^{\infty} e^{-\beta^2}\,d\beta = \frac{\sqrt{\pi}}{2}e^{-2\varepsilon}\left(1+erf\left(\alpha-\frac{\varepsilon}{\alpha}\right)\right).$$

Equation (15) in terms of error functions, we have

$$\Gamma(z,t) = \frac{C_0}{2} Exp\left(\frac{w^2t}{4D}+\frac{K_d(1-n)t}{n}-\gamma t\right)\left[e^{2\varepsilon}erfc\left(\alpha+\frac{\varepsilon}{\alpha}\right)+e^{-2\varepsilon}erfc\left(\alpha-\frac{\varepsilon}{\alpha}\right)\right] \tag{16}$$

Hence, Substitute in Equation (3) the solution is

$$\frac{C}{C_0} = \frac{1}{2}Exp\left[\frac{wz}{2D}-\gamma t\right]\left[e^{-2\varepsilon}erfc\left(\alpha-\frac{\varepsilon}{\alpha}\right)+e^{2\varepsilon}erfc\left(\alpha+\frac{\varepsilon}{\alpha}\right)\right]$$

Re-substituting for ε and α gives

$$\frac{C}{C_0} = \frac{1}{2}\,Exp\left[\frac{wz}{2D}-\gamma t\right]$$

$$\left[Exp\left[\frac{\sqrt{w^2n+4D(1-n)K_d-4Dn\gamma}}{2D\sqrt{n}}z\right]\cdot erfc\left[\frac{z+\sqrt{w^2n+4D(1-n)K_d-4Dn\gamma}}{2\sqrt{Dnt}}t\right]+\right.$$

$$\left.\left[Exp\left[-\frac{\sqrt{u^2n+4D(1-n)K_d-4Dn\gamma}}{2D\sqrt{n}}z\right]\cdot erfc\left[\frac{z-\sqrt{u^2n+4D(1-n)K_d-4Dn\gamma}}{2\sqrt{Dnt}}t\right]\right.\right. \tag{17}$$

IV. RESULTS AND DISCUSSIONS

In this paper, we accomplish that the mathematical results were advanced for forecasting the feasible concentration of a given liquified constituent in unidirectional ejection flows through semi-infinite, homogeneous, and isotropic permeable media subject to basis concentration that vary exponentially with time. The terms take into account the pollutants as well as mass transmission from the liquid to the solid stage because of adsorption. For instant dispersion and adsorption of a solute, the dispersion system is measured to be adsorbing at a rate related to its concentration.

The major limits of analytical techniques are that the applicability is only for simple problems. The geometry of the problem must be consistent. The belongings of the soil inside the area taken into consideration should be homogeneous in the sub area.

The analytical process is relatively more elastic than the usual form of different processes for one-dimensional transference model. Figures 3 to 12 denotes the concentration contours v/s distance inside the adsorbing media for exclusive values of time and permeability n . it's seen that for a static velocity w, dispersal coefficient D and distribution coefficient Kd, C/C0 decreases with depth as permeability n decreases because of the distributive coefficient Kd and if time increases the concentration decreases for distinct depth and decay

Figures 4 to 12 represent the break-through-Curves for C/C0, and is maximum on the surface z=0 and reduces to reaches zero on the depth of a 100 meters. With an increase in most of the contaminants get absorbed by the solid surface and thereby retarding the movements of the contaminants as evident from the graphs. maximum of the pollutants are diminished within the unsaturated area itself and therefore the risk of ground water being polluted is minimize

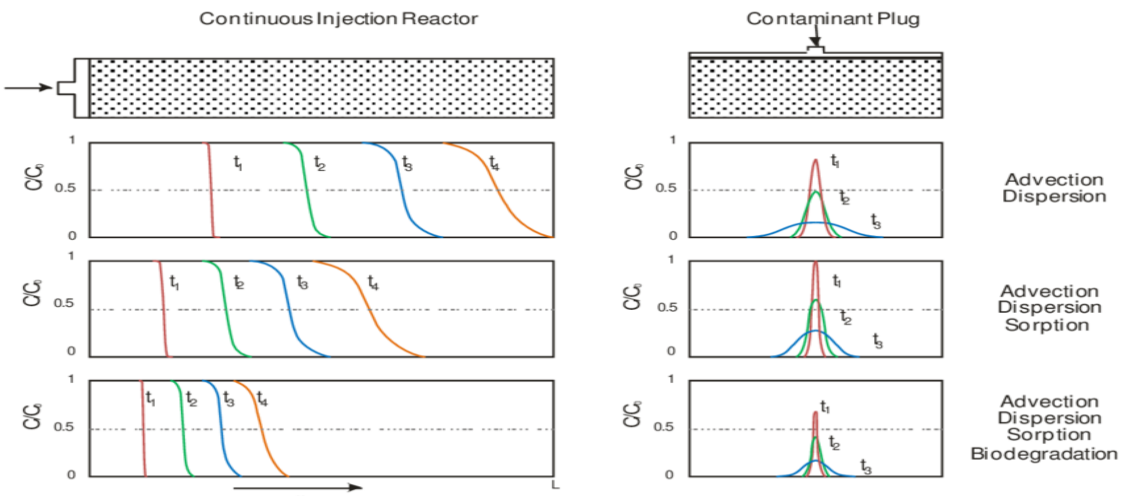


Figure (2): **Effects of various processes on the break through curve for a continuous injection reactor and Ground water contamination**

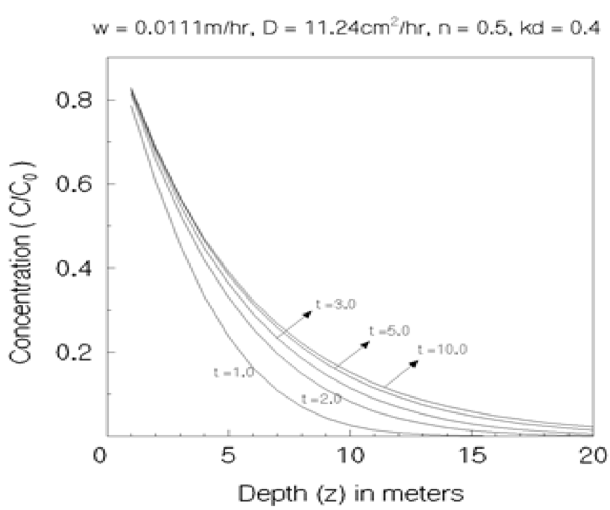


Figure (3)

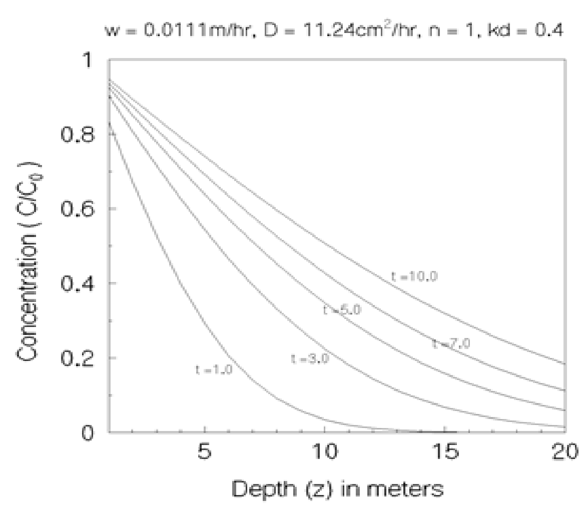


Figure (4)

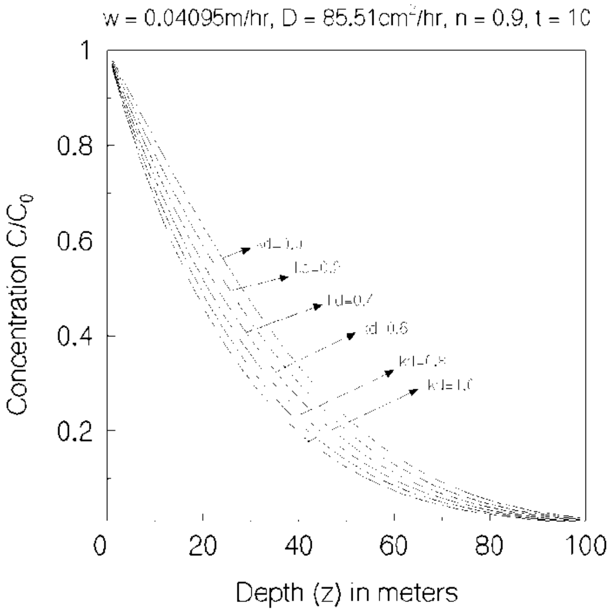


Figure (5)

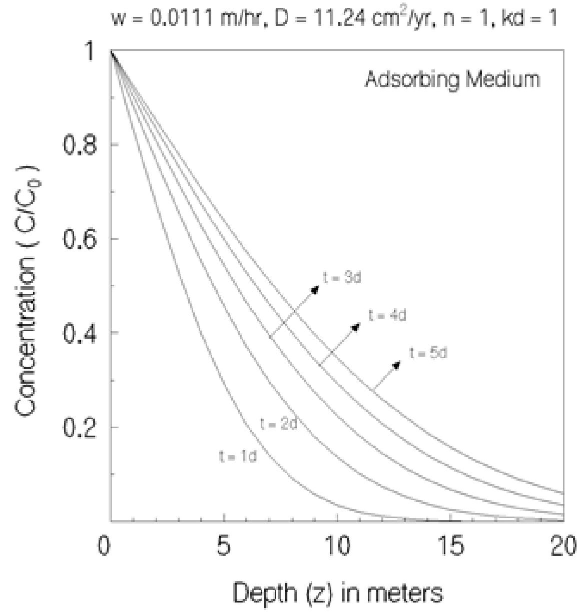


Figure (6)

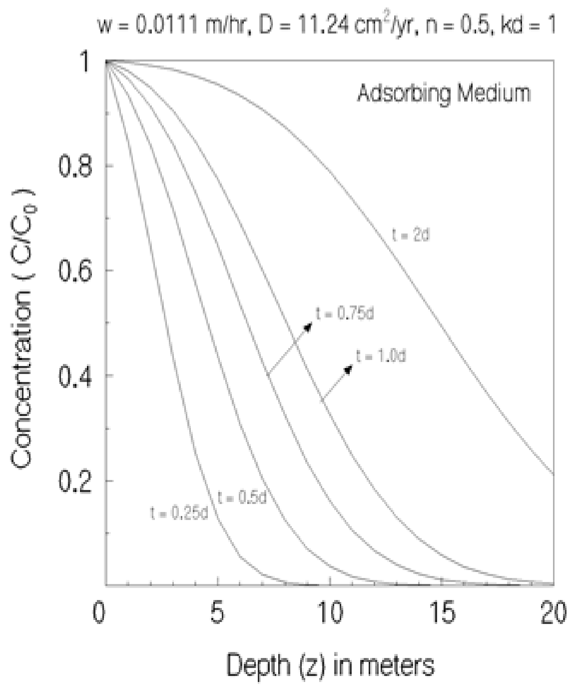
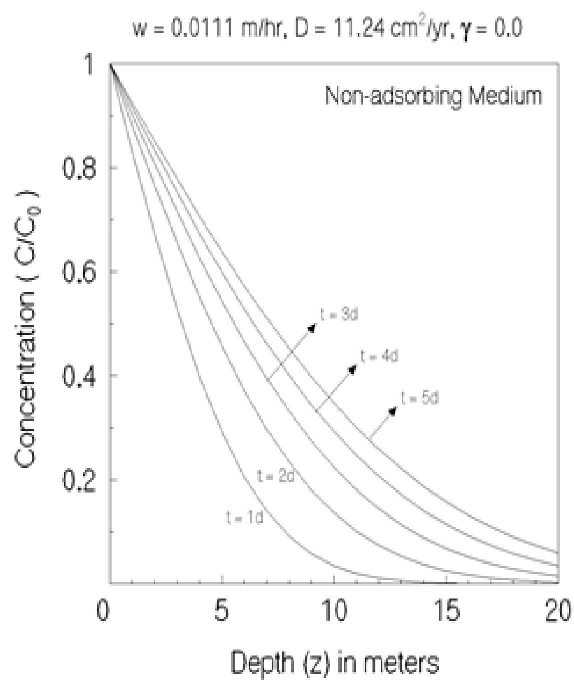


Figure (7)



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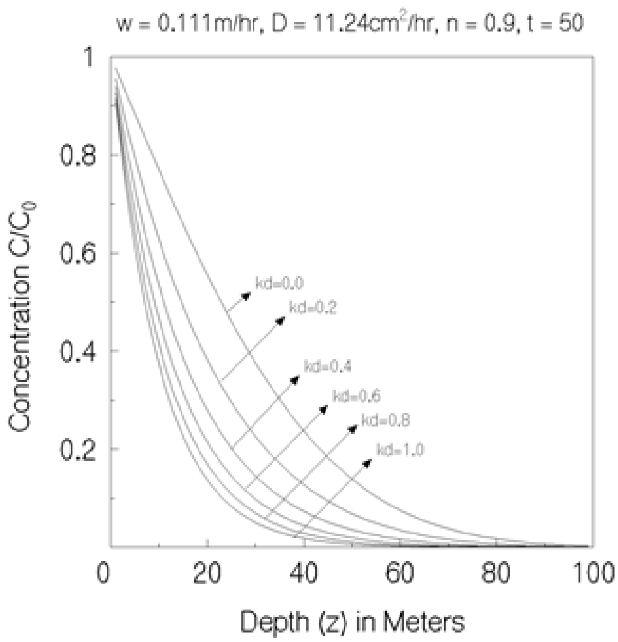

Figure (9)

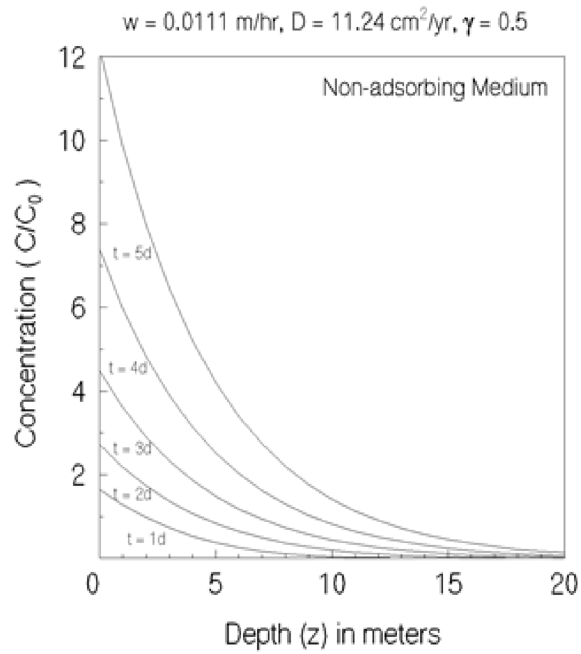

Figure (10)

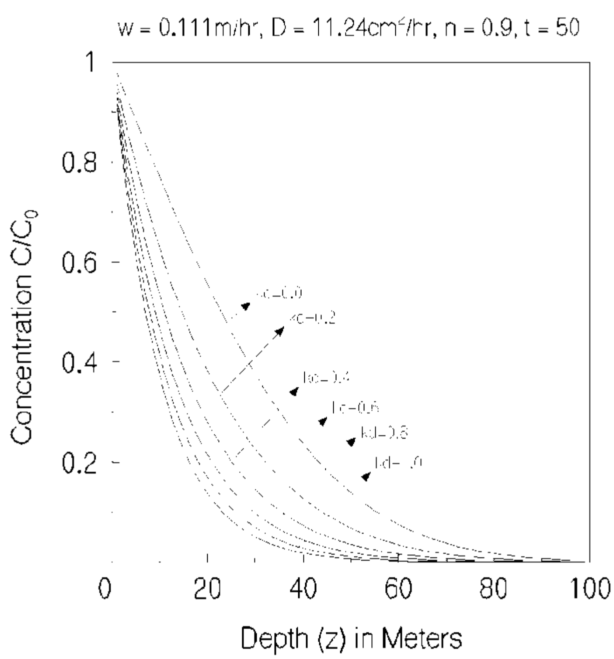

Figure (11)

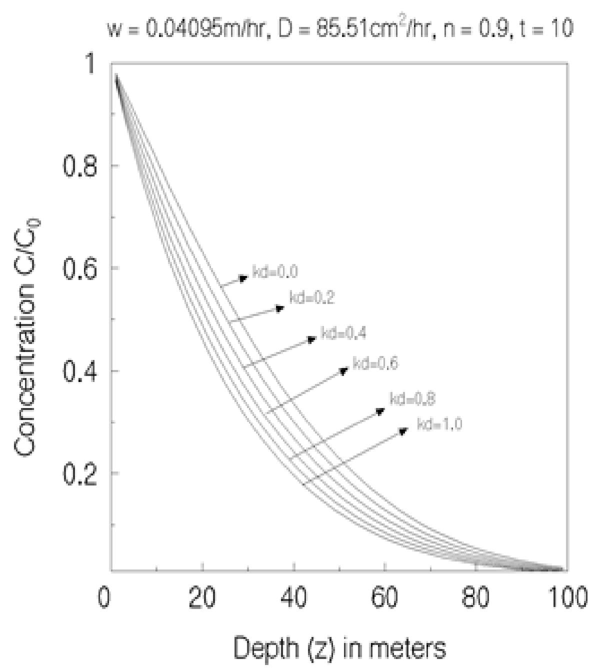

Figure (12)

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