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Mathematical Proof of Collatz Conjecture

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Abstract : Lothar Collatz introduced Collatz Conjecture in 1937. No one succeeded in proving this conjecture. In this article a convincing mathematical proof is introduced. Initially it is proved that for every natural number n in N={1,2,3,..}, the set An exists where An ={x/x is a term in Hailstone sequence starting with n}. Later it is proved thatthe intersections of An and Am is not empty foreverynatural number n≠m, m,n >1. Then it is observed that the countable intersection of all An contains A_0= {1}.This observation brings the conclusion that for all Hailstone sequences starting with any positive integer n ,there exists a a term 1 in the Hailstone sequence.This conclusion implies that for any positive integern, the Hailstone sequence starting with n eventually ends in 1.

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1. INTRODUCTION

Lothar Collatz introduced Collatz Conjecture in 1937.Collatz Conjecture explains a sequence that eventually ends in 1.The sequence is called Hailstone sequence which is defined as " Start with any positive integer , if that integer is odd number then triple it and add one to get the next term, but if the selected number is even divide it by 2.Continuing this procedure , whatever be the selected number at the end of the sequence the sequence reaches 1. Using computer the conjecture is tested till 5 x10^{18} and scientists believe that this conjecture might be true for all natural numbers. But nobody has succeeded to bring a proof which can convince the conjecture is correct or not. In this article a convincing mathematical proof is introduced. Initially it is proved that for every natural number *n in N ={1,2,3,..}*, the set *An* exists where *An* is the set that consists the numbers in Hailstone sequence starting with n. Later it is proved that $Am \cap An \neq \emptyset, \forall m, n \; where \; m \neq n$.
Then it is proved that the countable intersection of all *An* contains A_0 = {1}.Thisproof brings the conclusion that all Hailstone sequences starting with any positive integer*n*, eventually ends in 1.

2.PRELIMINARIES

Definition 2.1. Hailstone sequence
Hailstone sequence corresponding to a positive integer *n* is a sequence $\{a_i\}$ where a_i is obtained as the value applied to *n* recursively *i* times $a_i = f^{\,i}(n), n \in \{1,2,3,4, \dots\}$ and i=0,1,2,.. where$f^{\,0}(n) = n$ and for i > 0,

$$f^{\,i}(n) = \begin{cases} \frac{n}{2}, if \;\; n \; is \; even \\ 3n+1, if \; n \; is \; odd \end{cases}$$

2.2. Collatz Conjecture

Statement : For any positive integer $n \in N$, the Hailstone sequence starting with *n* eventually ends in 1.

Remark1: Some authors consider the set of natural numbers N={0,1,2,3,..} .In this article the set of natural numbers N is considered as N ={1,2,3,…} since Indian students follow the definition N ={1,2,3..} from school level and author is from India.

3.PROBLEM THAT DISCUSSED IN THIS ARTICLE AND SOLUTION

For any positive integer $n \in N$, the Hailstone sequence starting with *n* eventually ends in 1. Though this statement is tested and seemed true, a convincing mathematical proof is not introduced by any one since1937.So this statement remains as a conjecture

Solution to the Problem- Mathematical Proof of Collatz Theorem
It is proved that for all Hailstone sequences starting with any natural number *n*, there exists a natural number *i* such that there exists a term $a_i = f^i(n) = 1$.This proves collatz conjecture.
Theorem 1 .$\forall n \in N$, Anexistswhere *An* is the set that consists the numbers in Hailstone sequence starting with n.
Theorem 2. $Am \cap An \neq \emptyset, \forall m,n \in N, \quad m \neq n.$
Corollary 3.$\bigcap_{n=1}^{\infty} An = A_2 \supset A_0 = \{1\}$, $n \in N$

4.PROOF OF THEOREMS

Theorem 1.$\forall n \in N$, $An\ exists$.

Proof. The set *An* consists the numbers $a_i$where a_i is obtained as the value applied to *n* recursively *i* times $a_i = f^i(n)$, $n \in N$

As per definition,$f^0(n) = n$ and for i > 0,*and* $f^i(n) = \begin{cases} \frac{n}{2}, if\ \ n\ is\ even \\ 3n+1, if\ n\ is\ odd \end{cases}$
It is clear that for every $n \in N$, $f^i(n)$is a natural number and so a_i exists. Hence An exists $\forall n \in N$.

Remark1: The above proof never implies that *An* must contain 1 or *An* must be finite. The proof conveys that *An* exists and the elements in *An* are positive integers.

Theorem 2 . $Am \cap An \neq \emptyset, \forall m,n \in N, \quad m \neq n.$

To prove theorem 2, first we shall prove the following lemmas.
Lemma 2.1 :*For any odd number* $p > 1$, *the number* $(3p+1)$ *is even and* $(3p+1) > p$.
Lemma 2.2:*For any odd number* $p > 1$, *If* $\frac{(3p+1)}{2}$ *is odd then* $\frac{(3p+1)}{2} > p$
Lemma 2.3 :*For any odd number* $p > 1$, *If* $\frac{(3p+1)}{2^i}$ *is odd then* $\frac{(3p+1)}{2^i} < p$ *where* $i > 1$
Lemma 2.4 : *If m is an even number then it is a term of either the sequence* $\{2^u\}$
or the sequence$\{(2k+1)^v 2^u\}$where u,v, $k \in N$

5.PROOF OF LEMMAS

Lemma 2.1:*For any odd number* $p > 1$, *the number* $(3p+1)$ *is even and* $(3p+1) > p$.

Proof. Trivial .

$$Since\ p\ is\ odd,\ p = 2k+1, \text{ where } k \in N,$$
$$p < (3p+1) = 3(2k+1)+1 = 6k + 4 = 2(3k+2)$$

Lemma 2.2:*For any odd number* $p > 1$, *If* $\frac{(3p+1)}{2}$ *is odd then* $\frac{(3p+1)}{2} > p$

Proof.

Since p is odd, $p = (2k+1)$, where $k \in N, \frac{(3p+1)}{2} = (3k+2)$.

$3k+2 = 3\frac{(p-1)}{2} + 2 = 1.5p - 1.5 + 2 = 1.5p + .5 > p.$

Lemma 2.3 :*For any odd number* $p > 1$, *If* $\frac{(3p+1)}{2^i}$ *is odd then* $\frac{(3p+1)}{2^i} < p$ *where* $i > 1$

Proof.

Since p is odd, $p = 2k+1$, where $k \in N$, *and* $\frac{(3p+1)}{2} = (3k+2)$

It is obvious that $1.5k < 2k$ *for all* $k \in N$.

i.e, $(1.5k + 1) < (2k+1)$, *where* $k \in N$

i.e, $\frac{(3k+2)}{2} < (2k+1)$, *where* $k \in N$

i.e, $\frac{(3k+2)}{2^i} < (2k+1) = p$, *where* i , $k \in N$

i.e, $\frac{(3k+2)}{2^{i-1}} < (2k+1) = p$, *where* $i > 1$, $k \in N$

i.e, $\frac{(3p+1)}{2^i} = \frac{(3k+2)}{2^{i-1}} < p$, *where* $i > 1$, $k \in N$

i.e , *If* $\frac{(3p+1)}{2^i}$ *is odd* or even *then* $\frac{(3p+1)}{2^i} < p$ *where* $i > 1$

Hence , *If* $\frac{(3p+1)}{2^i}$ *is odd then* $\frac{(3p+1)}{2^i} < p$ *where* $i > 1$

Notes : 1. The Lemma 2.3 holds if $\frac{3p+1}{2^i}$ is even .

Notes : 2. The Lemma 2.3 holds for $p = 1$.

Corollary 1: From the above proofs and the definitions of $f^i(n)$ and *An*, *we* shall observe the following inequalities and sub set relations.

If $p > 1$ is an odd number

2.3.1 $A_{3p+1} \subset A_p$

2.3.2 *If* $\frac{(3p+1)}{2}$ *is odd then* $\frac{(3p+1)}{2} > p$ *and* $A_{\frac{(3p+1)}{2}} \subset A_{(3p+1)} \subset A_p$

2.3.3 *If* $\frac{(3p+1)}{2^i}$ *is odd then* $\frac{(3p+1)}{2^i} < p$ *where* $i > 1$ *and*

$$A_{\frac{(3p+1)}{2^i}} \subset A_{\frac{(3p+1)}{2^{i-1}}} \subset .... \subset A_{\frac{(3p+1)}{2}} \subset A_{3p+1} \subset A_p$$

Let $p > 1$ be any odd numbers in *N*, then the relations 2.3.2 and 2.3.3 imply that there exist some odd number q holding any of the following inequalities.

$(i)\qquad q = \frac{(3p+1)}{2} > p$

$(ii)\qquad p = \frac{(3q+1)}{2} > q$

$(iii)\qquad q = \frac{(3p+1)}{2^i} < p$

$(iv)\qquad p = \frac{(3q+1)}{2^i} < q$

Corollary 2:

Let p be any odd number .Then atleast any one of the following cases will hold.

Case 1: There exist some odd number q such that $Ap \subset Aq$.
Case 2: There exist some odd number q such that $Aq \subset Ap$.
Case 3: There exists an even number k such that $A_k \subset Ap$.

Define a relation *R* on the set {*An*}, where $n \in N$such that $Ap\ R\ Aq$ iff $Ap \sqsubseteq Aq$. Now *R* defines a partial order relation since it is reflexive, anti symmetric and transitive .Now({*An*}, *R*) is a partially ordered set.

Lemma 2.3.1: The minimum element in a partially ordered set is unique.

Proof : Suppose there are two minimum elements *Ap* and *Aq*.
Since *Ap* is minimum $Ap \sqsubseteq Aq$
Since *Aq* is also minimum $Aq \sqsubseteq Ap$.

Hence *Ap* = *Aq*. That means the minimum element is unique.

Lemma 2.3.2:A_5 is unique minimum element in partially ordered set ({*An*}, *R*) for a set of odd numbers (say P).
Proof : From the relation *R* ,definition of *An* , lemmas 2.2 to 2.3.1 ,corollary 1 and corollary 2 we get
Observation 1: By corollary 1, when $p = 3$, we get $A_5\ R\ A_3$.
Observation 2:The relation *R* ,definition of *An* , lemmas 2.2 to 2.3.1 ,corollary 1 and corollary 2, when applied to odd numbers, we get $A_5\ R\ A_{13}\ R\ A_{17}\ R\ A_{11}\ R\ A_7\ R\ A_9\ R$ …..
Observations 1 ,observation 2 and lemma 2.3.1 implies that A_5 is the unique minimum element in partially ordered set ({*An*}, *R*) for a set of odd numbers .Let *P* be that set of odd numbers in *N* for which A_5 is unique minimum element.

Then$\bigcap_{p\in P}^{\infty} Ap = A_5$_______Equation (1).
Let*Q* be the set of odd numbers in the set *N* – *P*. i.e, *Q* ={ x/ x is an odd number in *N*-*P*}

Lemma 2.4: If *m* is an even number then it is a term of either the sequence{ 2^u} or the sequence{$(2k+1)^v 2^u$} where u,v,k ∈ *N*.
Proof. The first sequence { 2^u} contains all even numbers that can be written as 2^u . Suppose *m* is an even number such that $m \neq 2^u$.Then $m = 2s$ where $s > 1$ and *s* is a natural number. If *s* is odd, then *m* is a term of the second sequence{$(2k+1)^v 2^u$}. If *s* is even, *s* can be written as product of powers of prime numbers .Since all prime numbers except 2 are odd, one factor of *s* is of the form (2k+1) , k ∈ *N*. Hence $m = 2s$ is a term of the second sequence {$(2k+1)^v 2^u$}. Hence If *m* is an even number then it is a term of either the sequence { 2^u} or the sequence {$(2k+1)^v 2^u$} whereu,v, k ∈ *N*.
Let S ={x/ x∈{ 2^u} or x∈{$(2k+1)^v 2^u$ }} . Now *N* = *SUQUP*. The sets *P* and *SUQ* is a partition for N.

Lemma 2.4.1: A_2 is unique minimum element in partially ordered set $(\{An\}, R))$ where $n \in SUQ$.
Proof : The definition of An and relation R implies that A_2 is included in all An where $n \in \{x / x = 2^u, u \in N\}$.
Also $A_{(2k+1)^v}\ R\ A_{(2k+1)^v 2^u}$ where u,v, k $\in N$.If $(2k+1)^v = p \in P$, then by lemma 2.3.2, $A_5\ R\ A_{(2k+1)^v}$ and by lemma 2.3 , $A_2\ R\ A_5$ Hence $A_2\ R\ A_{(2k+1)^v 2^u}$ .Suppose $(2k+1)^v = p \in Q$. By lemma 2.3, $A_4\ R\ A_1$, and $A_2\ R\ A_1$. For all odd $p \in Q$ where $p > 1$, by corollary 1 and corollary 2, there exists an odd q such that either $Aq\ R\ Ap$ or $Ap\ R\ Aq$. If $q \in P$, $A_2\ R\ A_5\ R\ Aq$ .Hence Ap includes A_2 .If $q \in Q$, without loss of generality suppose A_r, $r \in Q$ be the set such that $A_r = \cap A_{qi}$ for a set of $q_i \in Q$ *where* i= 1,2,3,…. .Now $3r + 1$ is even and except 1 there is no $q_i \in Q$ such that $A_{qi}\ R\ A_{3r+1}$ .Hence by corollary 1, Subset relation 2.3.3 we get $3r+1 \in \{x / x = 2^u, u \in N\}$.For all odd numbers in q in Q ,the number $3q+1$ is even and belongs to S.This implies that A_2 is the unique minimum element in partially ordered set $(\{An\}, R)$ where $n \in SUQ$.

Hence$\bigcap_{q \in SUQ}^{\infty} Aq = A_2$ ________ Equation (2)

From equations (1) and (2), $\forall m, n \in N = SUPUQ, m \neq n,\ Am \cap An \supset A_5 \cap A_2 = A_2$.
Hence $\forall m, n \in N, m \neq n,\ Am \cap An \neq \emptyset$.

Corollary 3.$\bigcap_{n=1}^{\infty} An = A_2 \supset A_0 = \{1\}, n \in N$.
Proof.
$\bigcap_{n=1}^{\infty} An = \bigcap_{p \in P}^{\infty} Ap \bigcap_{q \in SUQ}^{\infty} Aq = A_5 \cap A_2 = A_2 \supset A_0$.
This shows that the set A_0 is subset of all n , $n \in N$. Which implies that the element 1 belongs to all Hailstone sequences. Therefore, for all Hailstone sequences staring with n , $n \in N$,there exists a number i in N such that $a_i = f^{\,i}(n) = 1$. In other words all the Hailstone sequences staring with n , $n \in N$, contains the term 1. This proves the famous Collatz Conjecture.

Conclusion
Lothar Collatz introduced Collatz Conjecture in 1937. In this article Collatz Theorem is proved.

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