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Strong β-$\mathcal{H}$-Open Sets In Generalized Topological Space

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Abstract - *The aim of this paper is to characterize and discuss the properties of strong β-$\mathcal{H}$-open sets in a generalized topological space (X,λ) with a hereditary class $\mathcal{H}$.*

Keywords - *Hereditary class, Pre-$\mathcal{H}$-open set, $\mathcal{H}_R$-closed set, Semi-$\mathcal{H}$-open set, $Semi^* - \mathcal{H}$-closed set, Strong β-$\mathcal{H}$-open set.*

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I. INTRODUCTION

Let X a nonempty set. A nonempty subfamily λ of ℘(X) is called a generalized topology on X [1] if Ø ∈ λ and λ is closed under arbitrary union. The pair (X, λ) is called generalized topological space. A GTS (X, λ) is said to be strong if X ∈ λ. Elements of λ are called λ -open sets and the complement of a λ -open set is called a λ -closed set. The largest λ -open set contained in a subset A of X is denoted by $int_\lambda(A)$ [2] and is called the λ -interior of A. The smallest λ -closed set containing A is called the λ - closure of A and is denoted by $cl_\lambda(A)$ [2]. A subset A is said to be λ -dense if $cl_\lambda(A) = X$. A generalized topology (X, λ) is said to be a quasi topology [4] on X if M, N ∈ λ implies M ∩ N ∈ λ. A hereditary class $\mathcal{H}$ is a nonempty collection of subset of X such that A ⊂ B, B∈ $\mathcal{H}$ implies A ∈ $\mathcal{H}$[2]. For each subset A of X, a subset $A^*(\mathcal{H})$ or simply A^* of X is defined by A^* = {x∈ X | $M \cap A \notin \mathcal{H}$ for every M ∈ λ containing x} [2]. Let (X, λ) be a generalized topological space and $\mathcal{H}$ be a hereditary class of subset of X if $cl^*_\lambda(A) = X$ then A is called λ* - dense. If $\mathcal{H}$ is said to be λ-codense if λ∩ $\mathcal{H}$ = {Ø} [2] and is said to be strongly λ-codense [2] if M, N ∈ λ and M ∩ N ∈ $\mathcal{H}$, then M ∩ N = Ø. Every strongly λ-codense hereditary class is λ -codense but the converse is not true [2]. If $cl^*_\lambda(A) = A \cup A^*$ for every subset A of X, with respect to λ and a hereditary class $\mathcal{H}$ of subsets of X, then $\lambda^* = \{A \subset X / cl^*_\lambda(X - A) = X - A\}$ is a generalized topology [2]. Elements of λ* are called λ*-open sets and the complement of a λ*-open set is called a λ*-closed set. $Int^*_\lambda(A)$ *is the interior of* A *in* (X,λ^*). Let (X, λ) be a generalized topological space and H be a hereditary class of subsets of X. If cl^*_λ(A) = X, then A is called λ*-dense.

Lemma. 1.1[3].

Let (X,λ) be generalized topological space and $\mathcal{H}$ be a hereditary class of subsets of X. If $E, F \subset X$ then the following hold

- (i) If $E \subset F$ then $E^* \subset F^*$
- (ii) $(E^*)^* = E^*$ For every $E \subset X$.
- (iii) $E \subset F \subset X$. implies that $cl^*_\lambda(E) \subset cl^*_\lambda(F)$.
- (iv) $(E \cup E^*)^* \subset E^*$ *for every* $E \subset X$
- (v) $(E \cup F)^* = E^* \cup F^*$
- (vi) $\lambda \subset \lambda^*$.
- (vii) *If* $\beta = \{N - H : N \in \lambda, H \in \mathcal{H}\}$ *is a base for* λ^*.

Definition 1.2[3].

A subset E of a generalized topological space (X, λ) with a hereditary class $\mathcal{H}$ is said to be

1. λ*-dense in itself if $E \subset E^*$
2. λ^*-closed if $E^* \subset E$

3. Strong-β-$\mathcal{H}$-open[10] if $E \subset cl_\lambda^*(int_\lambda(cl_\lambda^*(E))$,
4. Semi - $\mathcal{H}$ −open[3] if $E \subset cl_\lambda^*(int_\lambda(E))$
5. $semi^*$- $\mathcal{H}$ −open[3] if $E \subset cl_\lambda(int_\lambda^*(E))$,
6. pre - $\mathcal{H}$ −open[3] if $E \subset int_\lambda(cl_\lambda^*(E))$
7. $\mathcal{H}_R$-closed set[11] if $E \subset cl_\lambda^*(int_\lambda(E))$

The complement of a strong β- $\mathcal{H}$- open (resp., Semi - $\mathcal{H}$ −open, $semi^*$ - $\mathcal{H}$ −open, pre - $\mathcal{H}$ −open,) set is said to a strong β- $\mathcal{H}$-closed (resp., Strong-β- $\mathcal{H}$ -closed, Semi - $\mathcal{H}$ −closed, $semi^*$ - $\mathcal{H}$ −closed, pre - $\mathcal{H}$ −closed,) set. The largest pre-$\mathcal{H}$-open set contained in E, denoted by $p\mathcal{H}int_\lambda(E)$, is called the pre-$\mathcal{H}$-interior of E. The following lemma 1.3 will be useful in the sequel.

Lemma. 1.3.

Let (X, λ) be a quasi topological space, $\mathcal{H}$ be a hereditary class of subsets of X and $E \subset X$ then p$\mathcal{H}int_\lambda(E)$ = $E \cap int_\lambda(cl_\lambda^*(E))$.

Proof.

Since, $E \cap int_\lambda(cl_\lambda^*(E)) \subset int_\lambda(cl_\lambda^*(E)) = int_\lambda(int_\lambda(cl_\lambda^*(E)))$=$int_\lambda\left(cl_\lambda^*(E) \cap int_\lambda(cl_\lambda^*(E))\right) \subset int_\lambda(cl_\lambda^*\left(E \cap int_\lambda(cl_\lambda^*(E))\right))$, $E \cap int_\lambda(cl_\lambda^*(E))$ is a pre-$\mathcal{H}$-open set contained in E and so $E \cap int_\lambda(cl_\lambda^*(E)) \subset p\mathcal{H}int_\lambda(E)$. Since p$\mathcal{H}int_\lambda(E)$ is pre-$\mathcal{H}$-open, $p\mathcal{H}int_\lambda(E) \subset int_\lambda\left(cl_\lambda^*(p\mathcal{H}int_\lambda(E))\right) \subset int_\lambda(cl_\lambda^*(E))$ and so $p\mathcal{H}int_\lambda(E) \subset E \cap int_\lambda(cl_\lambda^*(E))$. Hence p$\mathcal{H}int_\lambda(E)$=$E \cap int_\lambda(cl_\lambda^*(E))$.

Theorem 1.4

Let (X, λ) be a generalized topological space, $\mathcal{H}$ be a hereditary class of subsets of X and $E \subset X$. Then, E is a strong $\beta - \mathcal{H}$-open if and only if $cl_\lambda^*(E)$ is $\mathcal{H}_R$-closed set.

Proof.

Let E be a strong $\beta - \mathcal{H}$-open set. Then, we have $E \subset cl_\lambda^*(int_\lambda(cl_\lambda^*(E)))$ and so $cl_\lambda^*(E) \subset cl_\lambda^*\left(cl_\lambda^*\left(int_\lambda(cl_\lambda^*(E))\right)\right) = cl_\lambda^*\left(int_\lambda(cl_\lambda^*(E))\right) \subset cl_\lambda^*(E)$ which implies that $cl_\lambda^*(E) = cl_\lambda^*\left(int_\lambda(cl_\lambda^*(E))\right)$. Conversely, let $cl_\lambda^*(E) = cl_\lambda^*\left(int_\lambda(cl_\lambda^*(E))\right)$, since $E \subset cl_\lambda^*(E)$ for every subset E of X. Therefore, by using hypothesis, we have $E \subset cl_\lambda^*\left(int_\lambda(cl_\lambda^*(E))\right)$. This shows that E is strong $\beta - \mathcal{H}$-open set.

Theorem 1.5.

Let (X, λ) be a generalized topological space, $\mathcal{H}$ be a hereditary class of subsets of X and $E \subset X$ then the following are equivalent.

(i) E is strong $\beta - \mathcal{H}$-open set.
(ii) There exist a pre-$\mathcal{H}$-open set M such that $M \subset E \subset cl_\lambda^*(M)$.
(iii) $cl_\lambda^*(E)\ is\ \mathcal{H}_R - closed\ set.$

Proof.

$(i) \Rightarrow (ii)$. Let E be a strong $\beta - \mathcal{H}$-open set in X. Then, we have $E \subset cl_\lambda^*(int_\lambda(cl_\lambda^*(E)))$ and so $cl_\lambda^*(E) \subset cl_\lambda^*\left(int_\lambda(cl_\lambda^*(E))\right) \subset cl_\lambda^*(cl_\lambda^*(E)) \subset cl_\lambda^*(E)$ which implies that $cl_\lambda^*(E) = cl_\lambda^*\left(int_\lambda(cl_\lambda^*(E))\right)$. If $M = E \cap int_\lambda(cl_\lambda^*(E))$, then by lemma 1.3 $M =$ p$\mathcal{H}int_\lambda(E)$ and so M is pre-$\mathcal{H}$-open. Also, $cl_\lambda^*(M) = cl_\lambda^*\left(E \cap int_\lambda(cl_\lambda^*(E))\right) \supset cl_\lambda^*(E) \cap int_\lambda(cl_\lambda^*(E)) = int_\lambda(cl_\lambda^*(E))$ which implies that $cl_\lambda^*(M) \supset cl_\lambda^*(int_\lambda(cl_\lambda^*(E)))$ and so $cl_\lambda^*(M) \supset cl_\lambda^*(E)$. Since $M \subset E \subset cl_\lambda^*(E) \subset cl_\lambda^*(M)$.

$(ii) \Rightarrow (iii)$. Suppose there exists a pre-$\mathcal{H}$-open set M such that $M \subset E \subset cl_\lambda^*(M)$ then $cl_\lambda^*(E) = cl_\lambda^*(M) \subset cl_\lambda^*\left(int_\lambda\left(cl_\lambda^*(M)\right)\right) \subset cl_\lambda^*\left(int_\lambda\left(cl_\lambda^*(E)\right)\right) \subset cl_\lambda^*(E)$ and so $cl_\lambda^*(E) = cl_\lambda^*(int_\lambda(cl_\lambda^*(E)))$ which implies that $cl_\lambda^*(E)$ is $\mathcal{H}_R$-closed set.

$(iii) \Rightarrow (i)$. By theorem 1.4

Theorem 1.6.

Let (X, λ) be a generalized topological space, $\mathcal{H}$ be a hereditary class of subsets of X and $E \subset X$ then the following are equivalent.

(i) E is strong $\beta - \mathcal{H}$-open.
(ii) $cl_\lambda^*(E)\ semi - \mathcal{H}$-open
(iii) E is λ^*-dense in a $\mathcal{H}_R$-closed subspace of X.
(iv) E is λ^*-dense in a semi-$\mathcal{H}$-open subspace in X.

Proof.

$(i) \Rightarrow (ii)$. Suppose E is strong $\beta - \mathcal{H}$-open by theorem 1.5, $cl_\lambda^*(E)$ is $\mathcal{H}_R$-closed and since every $\mathcal{H}_R$-closed set is semi-$\mathcal{H}$-open, $cl_\lambda^*(E)$ semi-$\mathcal{H}$-open.

$(ii) \Rightarrow (iii)$. $cl_\lambda^*(E)\ is\ semi - \mathcal{H}$-open implies that $cl_\lambda^*(E) \subset cl_\lambda^*\left(int_\lambda\left(cl_\lambda^*(E)\right)\right) \subset cl_\lambda^*(E)$ which implies that $cl_\lambda^*(E)$ is $\mathcal{H}_R$-closed. Therefore, E is λ^*-dense in a $\mathcal{H}_R$-closed subspace of X.

$(iii) \Rightarrow (iv)$. Since every $\mathcal{H}_R$-closed set is semi-$\mathcal{H}$-open, the result follows.

$(iv) \Rightarrow (i)$. Let E is λ^*-dense in a semi-$\mathcal{H}$-open subspace in X then there exists a semi-$\mathcal{H}$-open set U such that $E \subset U \subset cl_\lambda^*(E)$. Now $cl_\lambda^*(E) \subset cl_\lambda^*(U) \subset cl_\lambda^*(E)$ implies that $cl_\lambda^*(E) = cl_\lambda^*(U)$ and so $cl_\lambda^*\left(int_\lambda\left(cl_\lambda^*(E)\right)\right) = cl_\lambda^*\left(int_\lambda\left(cl_\lambda^*(U)\right)\right) \supset cl_\lambda^*\left(int_\lambda(U)\right) = cl_\lambda^*(U) = cl_\lambda^*(E)$. Also, $cl_\lambda^*\left(int_\lambda\left(cl_\lambda^*(E)\right)\right) \subset cl_\lambda^*(E)$ and so $cl_\lambda^*(E) = cl_\lambda^*\left(int_\lambda\left(cl_\lambda^*(E)\right)\right)$. Hence, by theorem 1.4. E is a strong $\beta - \mathcal{H}$-open set.

Theorem 1.7.

Let (X, λ) be a generalized topological space, $\mathcal{H}$ be a hereditary class of subsets of X and $E, F \subset X$ such that $E \subset F \subset cl_\lambda^*(E)$. If E is strong $\beta - \mathcal{H}$-open set, then $cl_\lambda^*(E) = cl_\lambda^*(F)$ and hence F is a strong $\beta - \mathcal{H}$-open set.

Proof.

Let $E \subset F \subset cl_\lambda^*(E)$ and E is strong $\beta - \mathcal{H}$-open. Then, we have $cl_\lambda^*(E) \subset cl_\lambda^*(F) \subset cl_\lambda^*(E)$ which implies that $cl_\lambda^*(E) = cl_\lambda^*(F)$. On the other hand $E \subset cl_\lambda^*(int_\lambda(cl_\lambda^*(E)))$. By hypothesis, Since $E \subset F \subset cl_\lambda^*(E)$, we obtain that $F \subset cl_\lambda^*\left(cl_\lambda^*\left(int_\lambda\left(cl_\lambda^*(E)\right)\right)\right) = cl_\lambda^*\left(int_\lambda\left(cl_\lambda^*(E)\right)\right) = cl_\lambda^*\left(int_\lambda\left(cl_\lambda^*(F)\right)\right)$. Therefore, we have $F \subset cl_\lambda^*\left(int_\lambda\left(cl_\lambda^*(F)\right)\right)$ and this shows that F is a strong $\beta - \mathcal{H}$-open set.

Theorem 1.8.

Let (X, λ) be a generalized topological space, $\mathcal{H}$ be a hereditary class of subsets of X and $E, F \subset X$. If $E \subset F \subset cl_\lambda^*(E)$ and E is pre-$\mathcal{H}$-open set then F is strong $\beta - \mathcal{H}$-open.

Proof.

Let $E \subset F \subset cl_\lambda^*(E)$ and E is pre-$\mathcal{H}$-open. Then, we have $cl_\lambda^*(E) \subset cl_\lambda^*(F) \subset cl_\lambda^*(E)$ which implies that $cl_\lambda^*(E) = cl_\lambda^*(F)$. Since $F \subset cl_\lambda^*(E)$ and $E \subset int_\lambda(cl_\lambda^*(E))$ so $F \subset cl_\lambda^*\left(int_\lambda\left(cl_\lambda^*(E)\right)\right) = cl_\lambda^*\left(int_\lambda\left(cl_\lambda^*(F)\right)\right)$. This shows that F is strong $\beta - \mathcal{H}$-open set.

Theorem 1.9.

Let (X, λ) be a generalized topological space, $\mathcal{H}$ be a hereditary class of subsets of X and $E \subset X$. If E is both strong $\beta - \mathcal{H}$ −open and $semi^* - \mathcal{H}$-closed then E is semi-$\mathcal{H}$-open.

Proof.

If E is $semi^* - \mathcal{H}$-closed, then $int_\lambda(cl_\lambda^*(E) \subset E$ and so $int_\lambda(cl_\lambda^*(E)) \subset int_\lambda(E)$. Therefore, $cl_\lambda^*(int_\lambda(cl_\lambda^*(E))) \subset cl_\lambda^*(int_\lambda(E))$. Since E is strong $\beta - \mathcal{H}$-open, it follows that $E \subset cl_\lambda^*(int_\lambda(E))$ and so E is semi-$\mathcal{H}$-open.

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