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Further results on the reverse order law for the Core inverse in C^*-algebras

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Abstract

In this paper we investigated some equivalent condition of the reverse order law for the Core inverse in C^*-algebra extending some well-known results to more general settings.

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1 Introduction

In this paper we study necessary and sufficient conditions for the reverse order law for the core inverse in C^*-algebra.

The core inverse for a complex matrix were introduced by Baksalary and Trenkler [1]. let $\mathbb{A} \in M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the ring of all $n \times n$ complex matrices. A matrix $X \in M_n(\mathbb{C})$ is called core inverse of A, if it satisfies $AX = P_A$ and $R(X) \subseteq R(A)$, where $R(A)$ denotes the column space f A, and P_A is the orthogonal projector onto $R(A)$. And if such a matrix exists, then it is unique (and denoted by $A^{\oplus}$).

2 Preliminaries

Definition 2.1. [6] Let $\mathscr{R}$ be a ring a with involution $* : \mathscr{R} \to \mathscr{R}$ is an anti-isomorphism which satisfies $(a^*)^* = a$, $(a+b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathscr{R}$, $\mathscr{R}$ is called $a^* - ring$ if $\mathscr{R}$ is a ring with involution. An element $a \in \mathscr{R}$ is said to be Moore-Penrose invertible if the following equations:

(i) $axa = a.$ $\{1\}\, inverse$
(ii) $xax = x.$ $\{2\}\, inverse$
(iii) $(ax)^* = ax.$ $\{3\}\, inverse$
(iv) $(xa)^* = xa$ $\{4\}\, inverse$

Definition 2.2. [13] An element a is Hermitian if $a^* = a$, and a is called an idempotent if $a^2 = a$. A Hermitian idempotent is said to be a projection.

Definition 2.3. [1] Let $A \in M_{n\times n}$. A matrix $A^{\oplus} \in M_{n\times n}$ satisfying $(i) AA^{\oplus} = P_A$ and $(ii) R(A^{\oplus}) \subseteq R(A)$ is called core inverse of A.

Definition 2.4. [1] The core inverse of $a \in \mathscr{R}$ is the element $x \in \mathscr{R}$ which satisfies

$$(1)\ axa = a \qquad (2)\ xax = x \qquad (3)\ (ax)^* = ax \qquad (6)\ xa^2 = a\ (7)\ ax^2 = x$$

The element x is unique if it exist and is denoted by $a^{\oplus}$.

Definition 2.5. [1] Let $a \in \mathscr{R}$. An element $a^{\oplus} \in \mathscr{R}$ satisfying

$$aa^{\oplus}a = a \qquad a^{\oplus}R = aR \quad \text{and} \quad Ra^{\oplus} = Ra^*.$$

is called core inverse of a.

Definition 2.6. [13] An elements a is said to be normal $aa^{\oplus} = a^{\oplus}a$.

Definition 2.7. [13] An elements a is said to be invertible if $ab = ba = I$.

Theorem 2.8. *[13] For any $a \in \mathscr{R}^{\oplus}$, the following is satisfied:*
(a) $(a^{\oplus})^{\oplus} = a$;
(b) $(a^*)^{\oplus} = (a^{\oplus})^*$;
(c) $(a^*a)^{\oplus} = a^{\oplus}(a^{\oplus})^*$;
(d) $(aa^*)^{\oplus} = (a^{\oplus})^*a^{\oplus}$;
(e) $a^* = a^{\oplus}aa^* = a^*aa^{\oplus}$;
(f) $a^{\oplus} = (a^*a)^{\oplus}a^* = a^*(aa^*)^{\oplus}$;
(g) $(a^*)^{\oplus} = a(a^*a)^{\oplus} = (aa^*)^{\oplus}a$;
Notice that if $a = a^* \in R^{\oplus}$ then $aa^{\oplus} = a^{\oplus}a$ (meaning that a is an EP element of $\mathscr{R}$).

Lemma 2.9. [13] If $a \in \mathscr{R}^{\oplus}$, then $aa^*a \in \mathscr{R}^{\oplus}$ and $(aa^*a)^{\oplus} = a^{\oplus}(a^*)^{\oplus}a^{\oplus}$.
The following result is a consequence of a direct computation.

Lemma 2.10. If $a \in \mathscr{R}^{\oplus}$, then aa^*aa^*, $a^*aa^*a \in R^{\oplus}$, $[(aa^*)^2]^{\oplus} = [(aa^*)^{\oplus}]^2 = [(a^*)^{\oplus}a^{\oplus}]^2$ and $[(a^*a)^2]^{\oplus} = [(a^*a)^{\oplus}]^2 = [a^{\oplus}(a^*)^{\oplus}]^2$.
Let $\mathscr{A}$ be a unital C^*-algebra. We state the following theorem.

3 Reverse order law for the Core inverse

In this section we present necessary and sufficient conditions such that the reverse order law for the core inverse holds.

Theorem 3.1. *Let A be a unital C^*-algebra, and let $a, b \in R^{\oplus}$. Then the following conditions are equivalent:*
(i) $ab \in \mathscr{A}^{\oplus}$ and $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$;
(ii) $ab,\ a^{\oplus}ab \in \mathscr{A}^{\oplus}$, $(ab)^{\oplus} = (a^{\oplus}ab)^{\oplus}a^{\oplus}$ and $(a^{\oplus}ab)^{\oplus} = b^{\oplus}a^{\oplus}a$;
(iii) $ab,\ abb^{\oplus} \in \mathscr{A}^{\oplus}$, $(ab)^{\oplus} = b^{\oplus}(abb^{\oplus})^{\oplus}$ and $(abb^{\oplus})^{\oplus} = bb^{\oplus}a^{\oplus}$;
(iv) $ab,\ a^*ab \in \mathscr{A}^{\oplus}$, $(ab)^{\oplus} = (a^*ab)^{\oplus}a^*$ and $(a^*ab)^{\oplus} = b^{\oplus}(a^*a)^{\oplus}$;
(v) $ab,\ abb^* \in \mathscr{A}^{\oplus}$, $(ab)^{\oplus} = b^*(abb^*)^{\oplus}$ and $(abb^*)^{\oplus} = (bb^*)^{\oplus}a^{\oplus}$;
(vi) $ab,\ a^*abb^* \in \mathscr{A}^{\oplus}$, $(ab)^{\oplus} = b^*(a^*abb^*)^{\oplus}a^*$ and $(a^*abb^*)^{\oplus} = (bb^*)^{\oplus}(a^*a)^{\oplus}$;
(vii) $ab,\ aa^*abb^*b \in \mathscr{A}^{\oplus}$, $(ab)^{\oplus} = b^*b(aa^*abb^*b)^{\oplus}aa^*$ and $(aa^*abb^*b)^{\oplus} = (bb^*b)^{\oplus}(aa^*a)^{\oplus}$;
(viii) $ab,\ (a^*a)^2(bb^*)^2 \in \mathscr{A}^{\oplus}$, $(ab)^{\oplus} = b^*bb^*[(a^*a)^2(bb^*)^2]^{\oplus}a^*aa^*$ and $[(a^*a)^2(bb^*)^2]^{\oplus} = [(bb^*)^2]^{\oplus}[(a^*a)^2]^{\oplus}$;
(ix) $(a^{\oplus})^*b \in \mathscr{R}^{\oplus}$, and $[(a^{\oplus})^*b]^{\oplus} = b^{\oplus}a^*$;
(x) $a(b^{\oplus})^* \in \mathscr{R}^{\oplus}$, and $[a(b^{\oplus})^*]^{\oplus} = b^*a^{\oplus}$;

proof $(i) \Rightarrow (ii)$: Since $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$.
Let $a = a^{\oplus}ab$, $x = (a^{\oplus}ab)^{\oplus}$, we get

$$\begin{aligned}
axa &= a^{\oplus}ab(a^{\oplus}ab)^{\oplus}a^{\oplus}ab \\
&= a^{\oplus}ab(b^{\oplus}a^{\oplus}a)a^{\oplus}ab \\
&= a^{\oplus}(abb^{\oplus}a^{\oplus}ab) \\
&= a^{\oplus}ab \\
&= a \\
xax &= (a^{\oplus}ab)^{\oplus}a^{\oplus}ab(a^{\oplus}ab)^{\oplus} \\
&= b^{\oplus}a^{\oplus}a(a^{\oplus}ab)b^{\oplus}a^{\oplus}a \\
&= (b^{\oplus}a^{\oplus}abb^{\oplus}a^{\oplus})a \\
&= b^{\oplus}a^{\oplus}a \\
&= x \\
ax &= a^{\oplus}ab(a^{\oplus}ab)^{\oplus} \\
(ax)^* &= (a^{\oplus}abb^{\oplus}a^{\oplus}a)^* \\
&= (a^{\oplus}a)^*(bb^{\oplus})^*(a^{\oplus}a)^* \\
&= a^{\oplus}ab(a^{\oplus}ab)^{\oplus} \\
xa^2 &= (a^{\oplus}ab)^{\oplus}(a^{\oplus}ab)^2 \\
&= (a^{\oplus}ab)^{\oplus}(a^{\oplus}ab)(a^{\oplus}ab) \\
&= b^{\oplus}a^{\oplus}aa^{\oplus}ab(a^{\oplus}ab) \\
&= b^{\oplus}a^{\oplus}ab(a^{\oplus}ab) \\
&= (ab)^{\oplus}ab(a^{\oplus}ab) && \text{(using core invertible)} \\
&= a^{\oplus}ab \\
&= a \\
ax^2 &= (a^{\oplus}ab)((a^{\oplus}ab)^{\oplus})^2 \\
&= (a^{\oplus}ab)(a^{\oplus}ab)^{\oplus}(a^{\oplus}ab)^{\oplus} \\
&= (a^{\oplus}ab)^{\oplus} && \text{(using core invertible)} \\
&= x
\end{aligned}$$

Hence, $a^{\oplus}ab \in \mathscr{A}^{\oplus}$ and $(a^{\oplus}ab)^{\oplus} = b^{\oplus}a^{\oplus}a$.
Then we have

$$(ab)^{\oplus} = b^{\oplus}a^{\oplus} = (b^{\oplus}a^{\oplus}a)a^{\oplus} = (a^{\oplus}ab)^{\oplus}a^{\oplus}.$$

$(ii) \Rightarrow (iii)$: From $(ab)^{\oplus} = (a^{\oplus}ab)^{\oplus}a^{\oplus}$ and $(a^{\oplus}ab)^{\oplus} = b^{\oplus}a^{\oplus}a$, we get

$$\begin{aligned}
(ab)^{\oplus} &= (a^{\oplus}ab)^{\oplus}a^{\oplus} \\
&= b^{\oplus}a^{\oplus}aa^{\oplus} \\
&= b^{\oplus}a^{\oplus}.
\end{aligned}$$

Further, Let $a = abb^{\oplus}$, $x = (abb^{\oplus})^{\oplus}$

$$\begin{aligned}
axa &= abb^{\oplus}(bb^{\oplus}a^{\oplus})abb^{\oplus} \\
&= (abb^{\oplus}a^{\oplus}ab)b^{\oplus} && \text{(using (1) inverse)} \\
&= abb^{\oplus} \\
&= a \\
xax &= (abb^{\oplus})^{\oplus}(abb^{\oplus})(abb^{\oplus})^{\oplus} \\
&= bb^{\oplus}a^{\oplus}(abb^{\oplus})bb^{\oplus}a^{\oplus} \\
&= b(b^{\oplus}a^{\oplus}abb^{\oplus}a^{\oplus}) && \text{(using (2) inverse)} \\
&= bb^{\oplus}a^{\oplus} \\
&= x
\end{aligned}$$

$$\begin{aligned}
ax &= (abb^{\oplus})(abb^{\oplus})^{\oplus} \\
&= abb^{\oplus}bb^{\oplus}a^{\oplus} \\
&= abb^{\oplus}a^{\oplus} \\
(ab)^* &= (abb^{\oplus}bb^{\oplus}a^{\oplus})^* \\
&= (abb^{\oplus}a^{\oplus})^* \\
&= abb^{\oplus}a^{\oplus} \\
xa^2 &= (abb^{\oplus})^{\oplus}(abb^{\oplus})^2 \\
&= (abb^{\oplus})^{\oplus}(abb^{\oplus})(abb^{\oplus}) && \text{(using core invertible)} \\
&= abb^{\oplus} \\
&= a \\
ax^2 &= (abb^{\oplus})((abb^{\oplus})^{\oplus})^2 \\
&= (abb^{\oplus})(abb^{\oplus})^{\oplus}(abb^{\oplus})^{\oplus} \\
&= abb^{\oplus}bb^{\oplus}a^{\oplus}(abb^{\oplus})^{\oplus} \\
&= abb^{\oplus}a^{\oplus}(abb^{\oplus})^{\oplus} \\
&= ab(ab)^{\oplus}(abb^{\oplus})^{\oplus} && \text{(using (ii) inverse)} \\
&= (abb^{\oplus})^{\oplus} \\
&= x
\end{aligned}$$

so, $abb^{\oplus} \in \mathscr{A}^{\oplus}$ and $(abb^{\oplus})^{\oplus} = bb^{\oplus}a^{\oplus}$.
Now $(ab)^{\oplus} = b^{\oplus}a^{\oplus} = b^{\oplus}(bb^{\oplus}a^{\oplus}) = b^{\oplus}(abb^{\oplus})^{\oplus}$.
$(iii) \Rightarrow (iv)$: If $(ab)^{\oplus} = b^{\oplus}(abb^{\oplus})^{\oplus}$ and $(abb^{\oplus})^{\oplus} = bb^{\oplus}a^{\oplus}$,
we deduce

$$(ab)^{\oplus} = b^{\oplus}(abb^{\oplus})^{\oplus} = b^{\oplus}bb^{\oplus}a^{\oplus} = b^{\oplus}a^{\oplus}$$

By Theorem 2.1, we obtain

$$\begin{aligned}
axa &= a^*ab(b^{\oplus}a^{\oplus}(a^{\oplus})^*)a^*ab \\
&= a^*abb^{\oplus}a^{\oplus}aa^{\oplus}ab \\
&= a^*(abb^{\oplus}a^{\oplus}ab) \\
&= a^*ab \\
&= a \\
xax &= (a^*ab)^{\oplus}(a^*ab)(a^*ab)^{\oplus} \\
&= b^{\oplus}a^{\oplus}(a^{\oplus})^*(a^*ab)b^{\oplus}a^{\oplus}(a^{\oplus})^* \\
&= (b^{\oplus}a^{\oplus}abb^{\oplus}b^{\oplus}a^{\oplus})(a^{\oplus})^* \\
&= b^{\oplus}a^{\oplus}(a^{\oplus})^*, \\
&= x \\
ax &= a^*ab(a^*ab)^{\oplus} \\
&= a^*abb^{\oplus}a^{\oplus}(a^*)^{\oplus} \\
&= a^*abb^{\oplus}a^{\oplus}(a^{\oplus})^* \\
&= a^*ab(ab)^{\oplus}(a^{\oplus})^* && \text{(using (ii))} \\
&= a^*(a^{\oplus})^* \\
&= (a^{\oplus}a)^* \\
&= a^{\oplus}a \\
(ab)^* &= (a^*ab(a^*ab)^{\oplus})^* \\
&= (a^*abb^{\oplus}a^{\oplus}(a^{\oplus})^*)^* \\
&= (a^*abb^{\oplus}a^{\oplus}(a^{\oplus})^*)^* \\
&= (a^*ab(ab)^{\oplus}(a^{\oplus})^*)^* && \text{(using (ii))}
\end{aligned}$$

$$
\begin{aligned}
&= (a^*(a^{\oplus})^*)^* \\
&= a^{\oplus}a
\end{aligned}
$$

Therefore $\quad (ax)^* = ax$

$$
\begin{aligned}
xa^2 &= (a^*ab)^{\oplus}(a^*ab)^2 \\
&= (a^*ab)^{\oplus}(a^*ab)(a^*ab) \\
&= b^{\oplus}a^{\oplus}(a^*)^{\oplus}a^*ab(a^*ab) \\
&= b^{\oplus}a^{\oplus}(a^{\oplus})^*a^*ab(a^*ab) \\
&= b^{\oplus}a^{\oplus}(aa^{\oplus})^*ab(a^*ab) \\
&= b^{\oplus}a^{\oplus}aa^{\oplus}ab(a^*ab) && (\text{since } \ aa^{\oplus}a = a) \\
&= b^{\oplus}a^{\oplus}ab(a^*ab) && (\text{ using (i)}) \\
&= (ab)^{\oplus}(ab)(a^*ab) \\
&= a^*ab \\
&= x \\
ax^2 &= (a^*ab)((a^*ab)^{\oplus})^2 \\
&= (a^*ab)(a^*ab)^{\oplus}(a^*ab)^{\oplus} \\
&= \ a^*abb^{\oplus}a^{\oplus}(a^*)^{\oplus}(a^*ab)^{\oplus} \\
&= a^*(ab)(ab)^{\oplus}(a^*)^{\oplus}(a^*ab)^{\oplus} \\
&= a^*(a^*)^{\oplus}(a^*ab)^{\oplus} \\
&= (a^{\oplus}a)^*(a^*ab)^{\oplus} \\
&= a^{\oplus}a(a^*ab)^{\oplus} \\
&= (a^*ab)^{\oplus}
\end{aligned}
$$

Thus, $a^*ab \in \mathscr{A}^{\oplus}$ and $(a^*ab)^{\oplus} = b^{\oplus}a^{\oplus}(a^{\oplus})^* = b^{\oplus}(a^*a)^{\oplus}$, by Theorem 2.1

$$ab^{\oplus} = b^{\oplus}a^{\oplus} = b^{\oplus}a^{\oplus}aa^{\oplus} = (b^{\oplus}a^{\oplus}(a^{\oplus})^*)a^* = a^*ab^{\oplus}a^*$$

$(iv) \Rightarrow (v)$: Suppose that (iv) holds. By Theorem 2.1, we get
$(ab)^{\oplus} = (a^*ab)^{\oplus}a^* = b^{\oplus}(a^*a)^{\oplus}a^* = b^{\oplus}a^{\oplus}$,
Let $a = abb^*$, $\ x = (abb^*)^{\oplus}$, we get

$$
\begin{aligned}
axa &= abb^*((b^{\oplus})^*b^{\oplus}a^{\oplus})abb^* \\
&= abb^{\oplus}bb^{\oplus}a^{\oplus}abb^* \\
&= (abb^{\oplus}a^{\oplus}ab)b^* \\
&= abb^* \\
&= a \\
xax &= (abb^*)^{\oplus}(abb^*)(abb^*)^{\oplus} \\
&= (b^{\oplus})^*b^{\oplus}a^{\oplus}(abb^*)(b^{\oplus})^*b^{\oplus}a^{\oplus} \\
&= (b^{\oplus})^*(b^{\oplus}a^{\oplus}abb^{\oplus}a^{\oplus}) \\
&= (b^{\oplus})^*b^{\oplus}a^{\oplus} \\
&= x \\
ax &= (abb^*)(abb^*)^{\oplus} \\
&= abb^*(b^*)^{\oplus}b^{\oplus}a^{\oplus} \\
&= abb^*(b^{\oplus})^*b^{\oplus}a^{\oplus} \\
&= ab(bb^{\oplus})^*b^{\oplus}a^{\oplus} \\
&= abb^{\oplus}a^{\oplus} \\
(ax)^* &= (abb^*(b^{\oplus})^*b^{\oplus}a^{\oplus})^* \\
&= (abb^{\oplus}a^{\oplus})^* \\
&= abb^{\oplus}a^{\oplus}
\end{aligned}
$$

$$\begin{aligned}
\text{Therefore} \quad (ax)^* &= ax \\
xa^2 &= (abb^*)^{\oplus}(abb^*)^2 \\
&= (abb^*)^{\oplus}(abb^*)(abb^*) \\
&= (b^*)^{\oplus}b^{\oplus}a^{\oplus}abb^*(abb^*) \\
&= (b^*)^{\oplus}(ab)^{\oplus}(ab)b^*(abb^*) \\
&= (b^*)^{\oplus}b^*(abb^*) \\
&= abb^* \\
ax^2 &= (abb^*)((abb^*)^{\oplus})^2 \\
&= (abb^*)(abb^*)^{\oplus}(abb^*)^{\oplus} \\
&= abb^*(b^*)^{\oplus}b^{\oplus}a^{\oplus}(abb^*)^{\oplus} \\
&= abb^{\oplus}a^{\oplus}(abb^*)^{\oplus} \\
&= ab(ab)^{\oplus}(abb^*)^{\oplus} \\
&= (abb^*)^{\oplus}
\end{aligned}$$

i.e. $abb^* \in \mathscr{A}^{\oplus}$ and $(abb^*)^{\oplus} = (b^{\oplus})^*b^{\oplus}a^{\oplus} = (bb^*)^{\oplus}a^{\oplus}$.
Therefore,

$$(ab)^{\oplus} = b^{\oplus}a^{\oplus} = b^{\oplus}bb^{\oplus}a^{\oplus} = b^*((b^{\oplus})^*b^{\oplus}a^{\oplus}) = b^*(abb^*)^{\oplus}.$$

$(v) \Rightarrow (vi)$: From the conditions in (v) and Theorem 2.1, we obtain

$$(ab)^{\oplus} = b^*(abb^*)^{\oplus} = b^*(bb^*)^{\oplus}a^{\oplus} = b^{\oplus}a^{\oplus}.$$

using the equality and Theorem 2.1, we get

$$\begin{aligned}
axa &= a^*abb^*((b^{\oplus})^*a^{\oplus}a^{\oplus}(a^{\oplus})^*)a^*abb^* \\
&= a^*abb^{\oplus}bb^{\oplus}a^{\oplus}aa^{\oplus}abb^* \\
&= a^*(abb^{\oplus}a^{\oplus}ab)b^* \\
&= a^*abb^* \\
&= a \\
xax &= (a^*abb^*)^{\oplus}(a^*abb^*)(a^*abb^*)^{\oplus} \\
&= (b^{\oplus})^*b^{\oplus}a^{\oplus}(a^{\oplus})^*(a^*abb^*)(b^{\oplus})^*b^{\oplus}a^{\oplus}(a^{\oplus})^* \\
&= (b^{\oplus})^*(b^{\oplus}a^{\oplus}abb^{\oplus}a^{\oplus})(a^{\oplus})^* \\
&= (b^{\oplus})^*b^{\oplus}a^{\oplus}(a^{\oplus})^* \\
&= x
\end{aligned}$$

i.e., $(bb^*)^{\oplus}(a^*a)^{\oplus} = (b^{\oplus})^*b^{\oplus}a^{\oplus}(a^{\oplus})^* \in (a^*abb^*)\{1,2\}$.
Since the elements $a^{\oplus}abb^{\oplus}a^{\oplus}a$, $bb^{\oplus}a^{\oplus}abb^{\oplus}$ are self adjoint and

$$\begin{aligned}
ax &= (a^*abb^*)(a^*abb^*)^{\oplus} \\
&= a^*abb^*(b^*)^{\oplus}b^{\oplus}a^{\oplus}(a^*)^{\oplus} \\
&= a^*abb^*(b^*)^{\oplus}b^{\oplus}a^{\oplus}(a^{\oplus})^* \\
&= a^*ab(b^{\oplus}b)^*b^{\oplus}a^{\oplus}(a^{\oplus})^* \\
&= a^*abb^{\oplus}bb^{\oplus}a^{\oplus}(a^{\oplus})^* \\
&= a^*abb^{\oplus}a^{\oplus}(a^{\oplus})^* \\
(ax)^* &= ((a^*abb^*)(a^*abb^*)^{\oplus})^* \\
&= (a^*abb^*core(b^*)b^{\oplus}a^{\oplus}(a^*)^{\oplus})^* \\
&= (a^*abb^*(b^*)^{\oplus}b^{\oplus}a^{\oplus}(a^{\oplus})^*)^* \\
&= (a^*ab(b^{\oplus}b)^*b^{\oplus}a^{\oplus}(a^{\oplus})^*)^* \\
&= (a^*abb^{\oplus}bb^{\oplus}a^{\oplus}(a^{\oplus})^*)^* && (\text{since } bb^{\oplus}b = b) \\
&= (a^*abb^{\oplus}a^{\oplus}(a^{\oplus})^*)^*
\end{aligned}$$

$$
\begin{aligned}
&= a^*abb^{\oplus}a^{\oplus}(a^{\oplus})^* \\
\text{Therefore} \qquad (ax)^* &= ax \\
ax^2 &= (a^*abb^*)((a^*abb^*)^{\oplus})^2 \\
&= (a^*abb^*)\ (a^*abb^*)^{\oplus}(a^*abb^*)^{\oplus} && \text{(using core invertible)} \\
&= (a^*abb^*)^{\oplus} \\
xa^2 &= (a^*abb^*)^{\oplus}(a^*abb^*)^2 \\
&= (a^*abb^*)^{\oplus}(a^*abb^*)(a^*abb^*) && \text{(using core invertible)} \\
&= a^*abb^*
\end{aligned}
$$

we conclude that the element $a^*abb^*(b^{\oplus})^*b^{\oplus}a^{\oplus}(a^{\oplus})^*, (b^{\oplus})^*b^{\oplus}a^{\oplus}(a^{\oplus})^*a^*bb^*$ are self-adjoint, too. Hence $a^*abb^* \in \mathscr{R}^{\oplus}$ and $(a^*abb^*)^{\oplus} = (b^{\oplus})^*b^{\oplus}a^{\oplus}((a)^{\oplus})^* = (bb^*)^{\oplus}(a^*a)^{\oplus}$. Now

$$
\begin{aligned}
(ab)^{\oplus} &= b^{\oplus}a^{\oplus} \\
&= b^{\oplus}bb^{\oplus}a^{\oplus}aa^{\oplus} \\
&= b^*((b^{\oplus})^*b^{\oplus}a^{\oplus}(a^{\oplus})^*))a^* \\
&= b^*(a^*abb^*)^{\oplus}a^*
\end{aligned}
$$

$(vi) \Rightarrow (vii)$: By the hypothesis (vi) and Theorem 2.1, we have

$$({}^{\oplus}ab) = b^*(a^*abb^*)^{\oplus}a^* = b^*(bb^*)^{\oplus}(a^*a)^{\oplus}a^* = b^{\oplus}a^{\oplus}.$$

Let $x = aa^*abb^*b$ and $y = b^{\oplus}(b^{\oplus})^*b^{\oplus}a^{\oplus}(a^{\oplus})^*a^{\oplus}$, we get

$$
\begin{aligned}
xyx &= aa^*abb^*b(b^{\oplus}(b^{\oplus})^*b^{\oplus}a^{\oplus}(a^{\oplus})^*a^{\oplus})aa^*abb^*b \\
&= aa^*(abb^{\oplus}a^{\oplus}ab)b^*b \\
&= aa^*abb^*b \\
&= x \\
yxy &= b^{\oplus}(b^{\oplus})^*b^{\oplus}a^{\oplus}(a^{\oplus})^*a^{\oplus}(aa^*abb^*b)b^{\oplus}(b^{\oplus})^*b^{\oplus}a^{\oplus}(a^{\oplus})^*a^{\oplus} \\
&= b^{\oplus}(b^{\oplus})^*(b^{\oplus}a^{\oplus}abb^{\oplus}a^{\oplus})(a^{\oplus})^*a^{\oplus} \\
&= b^{\oplus}(b^{\oplus})^*b^{\oplus}a^{\oplus}(a^{\oplus})^*a^{\oplus} \\
&= y
\end{aligned}
$$

Hence, $y \in x\{1,2\}$, from the equalities

$$
\begin{aligned}
xy &= abb^{\oplus}a^{\oplus} \\
(xy)^* &= (aa^*abb^*bb^{\oplus}(b^{\oplus})^*b^{\oplus}a^{\oplus}(a^{\oplus})^*a^{\oplus})^* \\
&= (aa^*abb^{\oplus}a^{\oplus}(a^{\oplus})^*a^{\oplus})^* \\
&= (a^{\oplus})^*a^{\oplus}abb^{\oplus}a^{\oplus}aa^* \\
&= (a^{\oplus})bb^{\oplus}a^* \\
&= (abb^{\oplus}a^{\oplus})^* \\
&= abb^{\oplus}a^{\oplus} \\
yx^2 &= (aa^*abb^*b)^{\oplus}(aa^*abb^*b)^2 \\
&= (aa^*abb^*b)^{\oplus}(aa^*abb^*b)(aa^*abb^*b) && \text{(using core invertible)} \\
\text{R} \qquad &= aa^*abb^*b \\
xy^2 &= (aa^*abb^*b)((aa^*abb^*b)^{\oplus})^2 \\
&= (aa^*abb^*b)(aa^*abb^*b)^{\oplus}(aa^*abb^*b)^{\oplus} && \text{(using core invertible)} \\
&= aa^*abb^*b
\end{aligned}
$$

we have $y \in x\{3,6,7\}$. Hence, $x \in \mathscr{R}^{\oplus}$ and $x^{\oplus} = y$.
By lemma 1.1, it follows $(aa^*abb^*b)^{\oplus} = (bb^*b)^{\oplus}(aa^*a)^{\oplus}$.
Then

$$
\begin{aligned}
(ab)^{\oplus} &= b^{\oplus}a^{\oplus} \\
&= b^{\oplus}bb^{\oplus}a^{\oplus}aa^{\oplus}
\end{aligned}
$$

$$\begin{aligned} &= b^*b(b^{\oplus}(b^{\oplus})^*b^{\oplus}a^{\oplus}(a^{\oplus})^*a^{\oplus})aa^* \\ &= b^*b(aa^*abb^*b)^{\oplus}aa^* \end{aligned}$$

$(vii) \Rightarrow (viii)$: From the conditions in (vii) and lemma 2.1, we obtain

$$\begin{aligned} (ab)^{\oplus} &= b^*b(bb^*b)^{\oplus}(aa^*a)^{\oplus}aa^* \\ &= b^*bb^{\oplus}(b^{\oplus})^*b^{\oplus}a^{\oplus}(a^{\oplus})^*a^{\oplus}aa^* \\ &= b^{\oplus}a^{\oplus}. \end{aligned}$$

Let $x = (a^*a)^2(bb^*)^2$ and $y = [(b^*)^{\oplus}b^{\oplus}]^2[a^{\oplus}(a^*)^{\oplus}]^2$ then,

$$\begin{aligned} xyx &= a^*aa^*abb^*(b^*)^{\oplus}b^{\oplus}(b^*)^{\oplus}b^{\oplus}a^{\oplus}(a^*)^{\oplus}a^{\oplus}(a^*)^{\oplus}a^*aa^*abb^*bb^* \\ &= a^*aa^*abb^*bb^{\oplus}(b^*)^{\oplus}b^{\oplus}a^{\oplus}(a^*)^{\oplus}a^{\oplus}a^*aa^*abb^*bb^* \\ &= a^*aa^*(abb^{\oplus}a^{\oplus}ab)b^*bb^* \\ &= a^*aa^*abb^*bb^* \\ &= x \\ yxy &= (b^*)^{\oplus}b^{\oplus}(b^*)^{\oplus}b^{\oplus}a^{\oplus}(a^*)^{\oplus}a^{\oplus}(a^*)^{\oplus}a^*aa^*abb^*bb^* \\ &\qquad (b^*)^{\oplus}b^{\oplus}(b^*)^{\oplus}b^{\oplus}a^{\oplus}(a^*)^{\oplus}a^{\oplus}(a^*)^{\oplus} \\ &= (b^*)^{\oplus}b^{\oplus}(b^*)^{\oplus}b^{\oplus}a^{\oplus}(a^*)^{\oplus}aa^*abb^*bb^*(b^*)^{\oplus}b^{\oplus}(b^*)^{\oplus}b^{\oplus}a^{\oplus}(a^*)^{\oplus}a^{\oplus} \\ &= (b^*)^{\oplus}b^{\oplus}(b^*)^{\oplus}(b^{\oplus}a^{\oplus}abb^{\oplus}a^{\oplus})(a^*)^{\oplus}a^{\oplus}(a^*)^{\oplus} \\ &= (b^*)^{\oplus}b^{\oplus}(b^*)^{\oplus}b^{\oplus}a^{\oplus}(a^*)^{\oplus}a^{\oplus}(a^*)^{\oplus} \end{aligned}$$

Thus, $y \in x\{1,2\}$.

From the hypothesis the elements $abb^{\oplus}a^{\oplus}$, $b^{\oplus}a^{\oplus}ab$ are self-adjoint and then

$$\begin{aligned} xy &= a^{\oplus}abb^{\oplus}a^{\oplus}a \\ (xy)^* &= (a^*aa^*abb^*(b^*)^{\oplus}b^{\oplus}(b^*)^{\oplus}b^{\oplus}a^{\oplus}(a^*)^{\oplus}a^{\oplus}(a^*)^{\oplus})^* \\ &= (a^*aa^*(abb^{\oplus}a^{\oplus})(a^*)^{\oplus}a^{\oplus}(a^*)^{\oplus})^* \\ &= a^{\oplus}(a^*)^{\oplus}a^{\oplus}abb^{\oplus}a^{\oplus}aa^*a \\ &= a^{\oplus}(abb^{\oplus}a^{\oplus})^*a \\ &= a^{\oplus}abb^{\oplus}a^{\oplus}a \end{aligned}$$

Therefore $(xy)^* = xy$

$$\begin{aligned} yx^2 &= (a^*aa^*abb^*bb^*)^{\oplus}(a^*aa^*abb^*bb^*)^2 \\ &= (a^*aa^*abb^*bb^*)^{\oplus}(a^*aa^*abb^*bb^*)(a^*aa^*abb^*bb^*) \\ &= ((bb^*)^2)^{\oplus}((a^*a)^2)^{\oplus}(a^*a)^2(bb^*)^2(a^*a)^2(bb^*)^2 \\ &= (((bb^*)^2)^{\oplus}(bb^*)^2((a^*a)^2(bb^*)^2) \\ &= a^*aa^*abb^*bb^* \\ xy^2 &= (a^*aa^*abb^*bb^*)((a^*aa^*abb^*bb^*)^{\oplus})^2 \\ &= (a^*aa^*abb^*bb^*)(a^*aa^*abb^*bb^*)^{\oplus}(a^*aa^*abb^*bb^*)^{\oplus} \\ &= (a^*a)^2(bb^*)^2((bb^*)^2)^{\oplus}((a^*a)^2)^{\oplus}((bb^*)^2)^{\oplus}((a^*a)^2)^{\oplus} \\ &= ((a^*a)^2)^{\oplus}(a^*a)^2((a^*a)^2)^{\oplus}((bb^*)^2)^{\oplus} \\ &= ((a^*a)^2)^{\oplus}((bb^*)^2)^{\oplus} \end{aligned}$$

By these equalities, since the elements $a^{\oplus}abb^{\oplus}a^{\oplus}a, bb^{\oplus}a^{\oplus}abb^{\oplus}$ are self-adjoint.
we conclude that $y \in x\{3,6,7\}$.
Thus, we have $x \in \mathscr{A}^{\oplus}$, $x^{\oplus} = y$ and, by lemma 2.2.
$x = [(a^*a)^2(bb^*)^2]^{\oplus} = [(bb^*)^2]^{\oplus}[(a^*a)^2]^{\oplus}$. Now we get

$$\begin{aligned} (ab)^{\oplus} &= b^{\oplus}a^{\oplus} \\ &= b^*bb^*((b^{\oplus})^*b^{\oplus}(b^{\oplus})^*b^{\oplus}a^{\oplus}(a^{\oplus})^*a^{\oplus}(a^{\oplus})^*)a^*aa^* \\ &= b^*bb^*[(a^*a)^2(bb^*)^2]^{\oplus}a^*aa^*. \end{aligned}$$

$(viii) \Rightarrow (ix)$: By (viii) and lemma 2.2, we get

$$(ab)^{\oplus} = b^*bb^*[(bb^*)^2]^{\oplus}[(a^*a)^2]^{\oplus}a^*aa^*$$

$$\begin{aligned} &= b^*bb^*[(bb^*)^{\oplus}]^2[(a^*a)^{\oplus}]^2a^*aa^* \\ &= b^{\oplus}a^{\oplus} \end{aligned}$$

Let $a = (a^{\oplus})^*b,\ x = ((a^{\oplus})^*b)^{\oplus}$. we get

$$\begin{aligned} axa &= (a^{\oplus})^*bb^{\oplus}a^*(a^{\oplus})^*b \\ &= (a^{\oplus})^*a^{\oplus}(abb^{\oplus}a^{\oplus}ab) \\ &= (a^{\oplus})^*a^{\oplus}ab \\ &= (a^{\oplus})^*b \\ xax &= b^{\oplus}a^*(a^{\oplus})^*bb^{\oplus}a^* \\ &= (b^{\oplus}a^{\oplus}abb^{\oplus}a^{\oplus})aa^* \\ &= b^{\oplus}a^{\oplus}aa^* \\ &= b^{\oplus}a^{\oplus} \\ ax &= (a^{\oplus})^*bb^{\oplus}a^* \\ &= abb^{\oplus}a^{\oplus} \\ (ax)^* &= ((a^{\oplus})^*bb^{\oplus}a^*)^* \\ &= abb^{\oplus}a^{\oplus} \\ \text{Therefore} \qquad (ax)^* &= ax \\ ba^2 &= b^{\oplus}a^*((a^{\oplus})^*b)^2 \\ &= b^{\oplus}a^*(a^{\oplus})^*b(a^{\oplus})^*b \\ &= b^{\oplus}(a^{\oplus}a)^*b(a^{\oplus})^*b \\ &= b^{\oplus}a^{\oplus}ab(a^{\oplus})^*b \\ &= (ab)^{\oplus}ab(a^{\oplus})^*b \\ &= (a^{\oplus})^*b \\ ab^2 &= (a^{\oplus})^*b)(b^{\oplus}a^*)^2 \\ &= (a^{\oplus})^*bb^{\oplus}a^*b^{\oplus}a^* \\ &= (a^{\oplus})^*a^*b^{\oplus}a^* \\ &= (aa^{\oplus})^*b^{\oplus}a^* \\ &= b^{\oplus}a^* \end{aligned}$$

Hence, $(a^{\oplus})^*b \in \mathscr{R}^{\oplus}$ and $[(a^{\oplus})^*b]^{\oplus} = b^{\oplus}a^*$.

$(x) \Rightarrow (i)$: Using (x), we get the following:

$$\begin{aligned} abb^{\oplus}a^{\oplus}ab &= aa^{\oplus}abb^{\oplus}a^*(a^{\oplus})^*b \\ &= aa^*((a^{\oplus})^*bb^{\oplus}a^*(a^{\oplus})^*b) \\ &= aa^*(a^{\oplus})^*b \\ &= ab, \\ (abb^{\oplus}a^{\oplus})^* &= (a^{\oplus})^*bb^{\oplus}a \text{ is self-adjoint,} \end{aligned}$$

Thus, $ab \in \mathscr{R}^{\oplus}$ and $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$.

$(viii) \Rightarrow (x) \Rightarrow (i)$: This part can be proved in the same way as $(viii) \Rightarrow (ix) \Rightarrow (i)$.

Corollary 3.2. Let $\mathscr{A}$ be a ring with involution, and let $a, b, ab \in \mathscr{R}^{\oplus}$. Then the following starements are equivalent;

(i) $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$;

(ii) $(ab)^{\oplus} = b^{\oplus}a^{\oplus}abb^{\oplus}a^{\oplus}$.

Applying Theorem 3.1, for $a = b$, necessary and sufficient condition for to be bi-dagger, that is $(a^2)^{\oplus} = (a^{\oplus})^2$, follow as a corollary.

Corollary 3.3. Let $\mathscr{A}$ be a ring with involution, and let $a \in \mathscr{R}^{\oplus}$. Then the following condition are equivalent:

(i) $a^2 \in \mathscr{A}$ and $(a^2)^{\oplus} = (a^{\oplus})^2$
(ii) $a^2,\ a^{\oplus}a^2 \in \mathscr{A},\ (a^2)^{\oplus} = (a^{\oplus}a^2)^{\oplus}a^{\oplus};$
(iii) $a^2,\ a^2a^{\oplus} \in \mathscr{A},\ (a^2)^{\oplus} = a^{\oplus}(a^2a^{\oplus})^{\oplus}$ and $(a^{\oplus}a^2)^{\oplus} = (a^{\oplus})^2a$
(iv) $a^2,\ a^*a^2 \in \mathscr{A},\ (a^2)^{\oplus} = (a^*a^2)^{\oplus}a^*$ and $(a^*a^2)^{\oplus} = a^{\oplus}(a^*a)^{\oplus};$
(v) $a^2,\ a^2a^* \in \mathscr{A},\ (a^2)^{\oplus} = a^*(a^2a^*)^{\oplus}$ and $(a^2a^*)^{\oplus} = (aa^*)^{\oplus}a^{\oplus};$
(vi) $a^2,\ a^*a^2a^* \in \mathscr{A},\ (a^2)^{\oplus} = a^*(a^*a^2a^*)^{\oplus}a^*$ and $(a^*a^2a^*)^{\oplus} = (aa^*)^{\oplus}(a^*a)^{\oplus}$
(vii) $a^2,\ (aa^*a)^2 \in \mathscr{A},\ (a^2)^{\oplus} = a^*a[(aa^*a)^2]^{\oplus}aa^*$ and $[(aa^*a)^2]^{\oplus} = [(aa^*a)^{\oplus}]^2$
(viii) $a^2,\ (a^*a)^2(aa^*)^2 \in \mathscr{A},\ (a^2)^{\oplus} = a^*aa^*[(a^*a)^2(aa^*)^2]^{\oplus}a^*aa^*$ and
$[(a^*a)^2(aa^*)^2]^{\oplus} = [(aa^*)^2]^{\oplus}[(aa^*)^2]^{\oplus};$
(ix) $(a^{\oplus})^*a,\ \in \mathscr{A}$ and $[(a^{\oplus})^*a]^{\oplus} = a^{\oplus}a^*$
(x) $a(a^{\oplus})^*,\ \in \mathscr{A}$ and $[a(a^{\oplus})^*]^{\oplus} = a^*a^{\oplus}$

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