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Original Article

A FM/M/1/∞ Stochastic Feedback Arrival Model With Performance Measures

S.Gayathri[1], G.Rani[2]

[1]*Research Scholar, PG and Research Department of Mathematics, Seethalakshmi Ramaswami College(Affiliated to Bharathidasan University), Tiruchirappalli-620 002, TamilNadu, India.*

[2]*Associate Professor, PG and Research Department of Mathematics, Seethalakshmi Ramaswami College(Affiliated to Bharathidasan University), Tiruchirappalli-620 002, TamilNadu, India.*

Abstract *- In this paper the FM/M/1/∞ Queueing model with feedback arrival rates and single service rate is considered. The steady state solution and system characteristics are derived for this model. The analytical results are numerically illustrated and the effect of the nodal parameters on the system characteristics are derived and relevant conclusions are presented.*

Keywords - *feedback arrival rates, single server with service rate, infinite capacity, system characteristics.*

I. INTRODUCTION

Queueing systems present the concrete framework for design and analysis of practical applications. Queueing models provide the predictions of behaviour of systems such as waiting times, the average number of waiting customers and so forth. It is used in academic programs of Industrial Engineering, Computer Engineering etc., as well as in programs of Telecommunication and Computer Science. These predictions help us to anticipate situations to take appropriate measures to shorten the queues. Now a days queues with feedback occur in production systems subject to rework, computer network, telecommunication system, super markets, banking, industries, hospital management etc. Several authors have investigated queueing systems subject to feedback. Kalyanaraman and Ranganathan have studied a vacation queueing models with instantaneous Bernoulli feedback [2]. Kalyanaraman and Sumathy have studied a feedback queue with multiple servers and batch service [3]. Thangaraj et al studied a queue with a Markovian feedback [10]. Thangaraj and Vanitha have analysed on the analysis of M/M/1 feedback queues with catastrophes using continued fraction approach [11]. In this paper, the Mathematical model for FM/M/1/∞ single server with different service rate and feedback arrival rates are described, the steady-state equations and the steady-state probabilities are obtained. Moreover, the analytical expressions for the queueing model and certain performance measures are derived. The analytical results are numerically illustrated and relevant conclusions are presented.

A. Feedback arrivals possibility

Feedback arrivals are possible only in some of the places like the product quality of the industries and shops, customers satisfaction etc. In general, feedback arrivals are biased one.

II. DESCRIPTION OF THE MODEL

Consider a single server infinite capacity queueing model with feedback arrival rates. Customers arrive at the service station one by one according to a feedback Poisson process with feedback arrival rate $p\lambda(\geq 0)$. Single server who provides service to all the feedback arriving customers. Service times are independently and identically distributed random variables and exponential distribution with service rate μ. After the completion of each service, the customers can either join at the end of the queue with probability p or they can leave the system with probability $q, p+q=1$, the customers both newly arrived and those opted for feedback are served in the order in which they join the tail of the original queue. The customers are served according to the First come First served (FCFS) rule. The feedback arrival rate $p\lambda$ and the service rate μ of the system are following Poisson process given by

$$Pr\{X_1(t) = px_1, X_2(t) = x_2\} = e^{-(p\lambda_i+\mu_i)t} \sum_{j=0}^{\min(px_1,x_2)} \frac{(p\lambda_i t)^{px_1-j}(\mu_1 t)^{x_2-j}}{j!(px_1-j)!(x_2-j)!}$$

Suppose that in an interval of length h each member has a probability $p\lambda_n h + O(h)$ of giving birth to a new member. If n individuals are present at time t, the probability that there will be one birth between t and $t+h$ is $np\lambda h + O(h)$, then the mean arrival rate when the system is of size infinity is,

$$p\lambda_n = \begin{cases}(p\lambda)n \\ n(p\lambda)\end{cases} n \geq 1$$
$$p\lambda_0 = 0, \quad n = 0$$

Suppose that an individual cannot give birth to a new individual and the probability of death of an individual in $(t, t+h)$ is $\mu_n h + O(h)$. If n individuals are present at time t, the probability of one death in $(t, t+h)$ is $n\mu h + O(h)$, then the mean service rate is $\mu_n = n\mu n \geq 1$.

Postulates of the Model

1. The probability that there is no feedback arrival and no departure in $(t, t+\Delta t\ )$ is
$$P(A_{00}) = P_n(t)\big(1 - p\lambda_n\Delta t + O(\Delta t)\big)\big(1 - \mu_n\Delta t + O(\Delta t)\big)$$
$$= P_n(t)(1 - (p\lambda_n + \mu_n)\Delta t + O(\Delta t))$$
2. The probability that there is one feedback arrival and no departure in $(t, t+\Delta t)$ is
$$P(A_{10}) = P_{n-1}(t)\big(p\lambda_{n-1}\Delta t + O(\Delta t)\big)\big(1 - \mu_{n-1}\Delta t + O(\Delta t)\big)$$
$$= P_{n-1}(t)p\lambda_{n-1}\Delta t + O(\Delta t)$$
3. The probability that there is no feedback arrival and one departure in $(t, t+\Delta t)$ is
$$P(A_{01}) = P_{n+1}(t)\big(1 - p\lambda_n\Delta t + O(\Delta t)\big)\big(\mu_{n+1}\Delta t + O(\Delta t)\big)$$
$$= P_{n+1}(t)\mu_{n+1}\Delta t + O(\Delta t)$$
4. The probability that there is one feedback arrival and one departure in $(t, t+\Delta t)$ is
$$P(A_{11}) = P_n(t)\big(p\lambda_n\Delta t + O(\Delta t)\big)\big(\mu_n\Delta t + O(\Delta t)\big)$$
$$= O(\Delta t)$$

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -(p\lambda_n + \mu_n)P_n(t) + p\lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t) + \frac{O(\Delta t)}{\Delta t}$$
$$\frac{O(\Delta t)}{\Delta t} \to 0, \quad \text{when} \quad \Delta t \to 0$$
$$P_n'(t) = -(p\lambda_n + \mu_n)P_n(t) + p\lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t)$$

For $n = 0$,

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -p\lambda_0 P_0(t) + \mu_1 P_1(t) + \frac{O(\Delta t)}{\Delta t}$$
$$P_0'(t) = -p\lambda_0 P_0(t) + \mu_1 P_1(t)$$

III. STEADY-STATE EQUATIONS

The steady-state equations for the feedback arriving systems are

$$0 = -(p\lambda_n + \mu_n)P_n + p\lambda_{n-1}P_{n-1} + \mu_{n+1}P_{n+1}, n \geq 1 \quad (1)$$
$$0 = -p\lambda_0 P_0 + \mu_1 P_1, n = 1 \quad (2)$$

IV. SINGLE-SERVER QUEUEING MODEL WITH POISSON FEEDBACK ARRIVALS AND EXPONENTIAL SERVICE TIMES (FM/M/1)

The model adopted in this paper is single channel queueing system in which single server is available to handle feedback arriving customers. Let us assume that the feedback customers waiting for service form a single line and then proceed to the server. For this queueing system, it is assumed that the feedback arrivals follow a Poisson probability distribution with rate $\frac{1}{p\lambda}$. The channel has an independent and identical exponential time distribution with mean $\frac{1}{\mu}$.

Equations for FM/M/1 Queueing Model:

Utilization factor,

$$\rho = \frac{p\lambda}{\mu} \quad (3)$$

The probability that there are n number of customers in the system is

$$P_1 = \frac{p\lambda_0}{\mu_1} P_0 \tag{4}$$
$$P_2 = \frac{p^2\lambda_0\lambda_1}{\mu_1\mu_2} P_0 \tag{5}$$
$$\cdots \tag{6}$$
$$P_n = \frac{p^n\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\ldots\ldots n} P_0 \tag{7}$$
$$P_n = \left(\frac{p\lambda}{\mu}\right)^n P_0 \tag{8}$$

The probability that there are zero customers in the system is

$$P_0 = \left(1 + \sum_{n=1}^{\infty} \frac{p^n\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\right)^{-1} \tag{9}$$
$$P_0 = \left(1 + \sum_{n=1}^{\infty} \left(\frac{p\lambda}{\mu}\right)^n\right)^{-1} \tag{10}$$

The average number of feedback customers in the system is

$$L_S = E(N)$$
$$= \sum_{n=0}^{\infty} n\left(\left(\frac{p\lambda}{\mu}\right)^n \left(1 - \frac{p\lambda}{\mu}\right)\right)$$
$$= \rho(1-\rho)\sum_{n=1}^{\infty} n\rho^{n-1}$$
$$= \frac{\rho}{1-\rho}$$
$$= \frac{p\lambda}{\mu - p\lambda} \tag{11}$$

The average number of feedback customers in the queue is

$$L_q = E(N-1)$$
$$= \sum_{n=1}^{\infty} (n-1)\left(\frac{p\lambda}{\mu}\right)^n \left(1 - \frac{p\lambda}{\mu}\right)$$

Put $m = n - 1$

$$L_q = \rho^2(1-\rho)\sum_{m=1}^{\infty} m\rho^{m-1}$$
$$= \frac{\rho^2}{1-\rho}$$
$$= \frac{(p\lambda)^2}{\mu(\mu - p\lambda)} \tag{12}$$

The average time a feedback customer spends in the queue is

$$W_q = E(T_q)$$
$$= 0 \times f_q(0) + \int_0^{\infty} w f_q(w)dw$$
$$= \int_0^{\infty} w\frac{p\lambda}{\mu}(\mu - p\lambda)e^{-(\mu - p\lambda)w} dw where, f_q(w) = \sum_{n=1}^{\infty} f_q(w/n)P_n$$
$$f_q(w/n) = \frac{\mu^n}{(n-1)!} e^{-\mu w} w^{n-1}$$
$$f_q(w) = \sum_{n=1}^{\infty} \frac{\mu^n}{(n-1)!} e^{-\mu w} w^{n-1} \left(\frac{p\lambda}{\mu}\right)^n \left(1 - \frac{p\lambda}{\mu}\right)$$
$$= \frac{p\lambda}{\mu} e^{-(\mu - p\lambda)w}(\mu - p\lambda)$$

$$W_q = \frac{p\lambda}{\mu}\int_0^{\infty} e^{-(\mu - p\lambda)w} dw = \frac{p\lambda}{\mu(\mu - p\lambda)} \tag{13}$$

The average time a feedback customer spends in the system is

$$W_S = \frac{W_q}{P(T_q > 0)}$$
$$= \frac{W_q}{1 - P(T_q = 0)}$$

where,$P(T_q = 0) = P$(No customer in thequeue)= $P_0 = 1 - \rho = 1 - \frac{p\lambda}{\mu}$

$$W_S = \frac{1}{\mu - p\lambda} \tag{14}$$

V. SOME OTHER PERFORMANCE MEASURES

The expected number of feedback customers in the system is

$$\begin{aligned} E\{N\} &= \sum_{n=0}^{\infty} nP_n \\ &= \sum_{n=0}^{\infty} n\left(1-\frac{p\lambda}{\mu}\right)\left(\frac{p\lambda}{\mu}\right)^n \\ &= \rho(1-\rho)\sum_{n=1}^{\infty} n\rho^{n-1} \\ &= \frac{\rho(1-\rho)}{(1-\rho)^2} \\ &= \frac{\rho}{1-\rho} \end{aligned} \tag{15}$$

$$\begin{aligned} E\{N^2\} &= \sum_{n=0}^{\infty} n^2 P_n \\ &= \sum_{n=1}^{\infty} n^2\left(1-\frac{p\lambda}{\mu}\right)\left(\frac{p\lambda}{\mu}\right)^n \\ &= (1-\rho)\sum_{n=1}^{\infty} (n^2-n+n)\rho^n \\ &= \frac{2\rho^2}{(1-\rho)^2}+\frac{\rho}{1-\rho} \\ &= \frac{\rho^2+\rho}{(1-\rho)^2} \end{aligned} \tag{16}$$

The variance for the feedback customers in the system is obtained by substituting equations (15) and (16)

$$\begin{aligned} Var\{N\} &= E\{N^2\}-[E\{N\}]^2 \\ &= \frac{\rho^2+\rho}{(1-\rho)^2}-\frac{\rho^2}{(1-\rho)^2} \\ &= \frac{\rho}{(1-\rho)^2} \end{aligned} \tag{17}$$

Standard deviation for the feedback customers in the system is

$$\text{Standard deviation} = \sqrt{var\{N\}} = \frac{\sqrt{\rho}}{1-\rho} \tag{18}$$

The expected number of feedback customers in the system is

$$\begin{aligned} E\{N^3\} &= \sum_{n=0}^{\infty} n^3 P_n \\ &= \sum_{n=0}^{\infty} (n(n-1)(n-2)+3n(n-1)+n)P_n \\ &= \sum_{n=0}^{\infty} [n(n-1)(n-2)+3n(n-1)+n]\left(1-\frac{p\lambda}{\mu}\right)\left(\frac{p\lambda}{\mu}\right)^n \\ &= (1-\rho)\left[\frac{6\rho^3}{(1-\rho)^4}+\frac{6\rho^2}{(1-\rho)^3}+\frac{\rho}{(1-\rho)^2}\right] \\ &= \frac{\rho^3+4\rho^2+\rho}{(1-\rho)^3} \end{aligned} \tag{19}$$

$$\begin{aligned} E\{N^4\} &= \sum_{n=0}^{\infty} n^4 P_n \\ &= \sum_{n=0}^{\infty} [n(n-1)(n-2)(n-3)+6n(n-1)(n-2)+7n(n-1)+n]P_n \\ &= \sum_{n=0}^{\infty} [n(n-1)(n-2)(n-3)+6n(n-1)(n-2)+7n(n-1)+n]\left(1-\frac{p\lambda}{\mu}\right)\left(\frac{p\lambda}{\mu}\right)^n \\ &= (1-\rho)\left[\frac{24\rho^4}{(1-\rho)^5}+\frac{36\rho^3}{(1-\rho)^4}+\frac{14\rho^2}{(1-\rho)^3}+\frac{\rho}{(1-\rho)^2}\right] \\ &= \frac{\rho^4+11\rho^3+11\rho^2+\rho}{(1-\rho)^4} \end{aligned} \tag{20}$$

VI. PARTICLUAR CASE

If the value of $p = 1$ in FM/M/1 queueing model, then it becomes $M/M/1$ queueing model.

VII. NUMERICAL ILLUSTRATIONS

For various values of λ, μ and fixed value $p = 0.5$, the values of $P_0, P_n, L_s, L_q, W_s, W_q$ values for FM/M/1 model are calculated.

Table 1

λ	μ	P_0	P_n	L_s	L_q	W_s	W_q
1	2	0.75	$n = 1,0.188$	0.33	0.083	0.67	0.167
2	3	0.67	$n = 2,0.073$	0.50	0.167	0.50	0.167
3	4	0.62	$n = 3,0.034$	0.60	0.150	0.40	0.150
4	5	0.6	$n = 4,0.016$	0.67	0.133	0.333	0.133

Also, the values of moments at the origin, variance and standard deviation for FM/M/1 model are calculated.

Table 2

λ	μ	$E\{N\}$	$E\{N^2\}$	$Var\{N\}$	Standard deviation	$E\{N^3\}$	$E\{N^4\}$
1	2	0.333	0.56	0.0004	0.667	1.223	3.525
2	3	0.50	0.998	0.745	0.865	2.741	9.950
3	4	0.60	1.320	0.391	0.970	4.061	16.484
4	5	0.667	1.556	1.11	1.050	5.111	22.231

For various values of λ and μ, the values of $P_0, P_n, L_s, L_q, W_s, W_q$ values for $M/M/1$ model are calculated.

Table 3

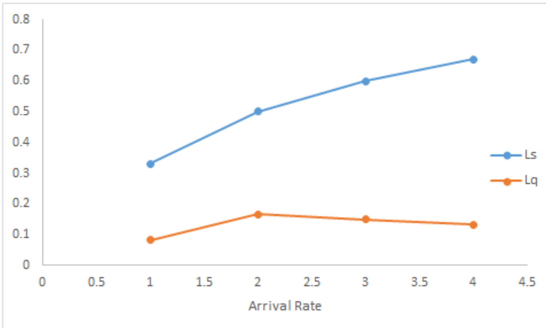
λ	μ	P_0	P_n	L_s	L_q	W_s	W_q
1	2	0.50	$n = 1,0.25$	1	0.50	1	0.50
2	3	0.333	$n = 2,0.148$	2	1.333	1	0.667
3	4	0.250	$n = 3,0.106$	3	2.25	1	0.75
4	5	0.20	$n = 4,0.102$	4	3.20	1	0.80

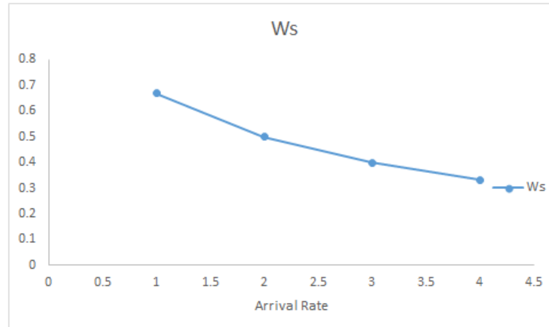
Also, the values of moments at the origin, variance and standard deviation for M/M/1 model are calculated.

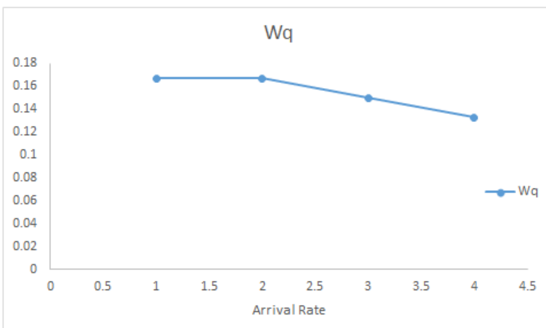
Table 4

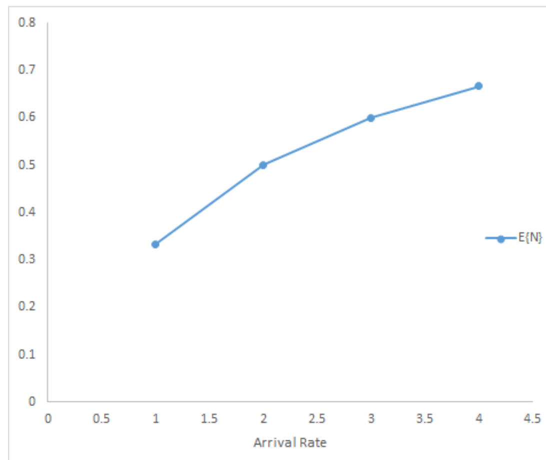
λ	μ	$E\{N\}$	$E\{N^2\}$	$Var\{N\}$	Standard deviation	$E\{N^3\}$	$E\{N^4\}$
1	2	1	3	2	1.414	13	74.413
2	3	2	10.02	6.01	2.453	76.865	751.92
3	4	3	20.841	11.905	3.464	213.875	2,973.75
4	5	4	36	20	4.470	484	8,676.25

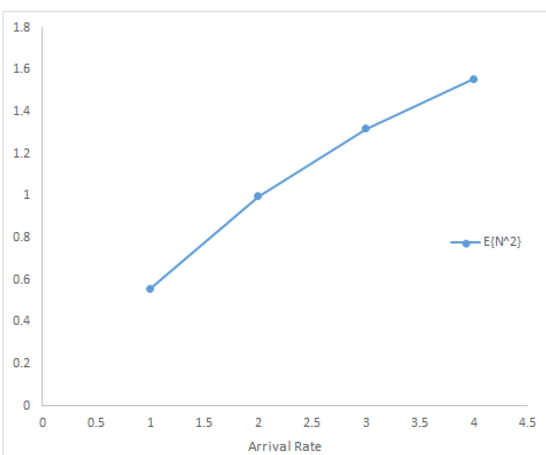
GRAPH REPRESENTATION FOR FM/M/1 MODEL

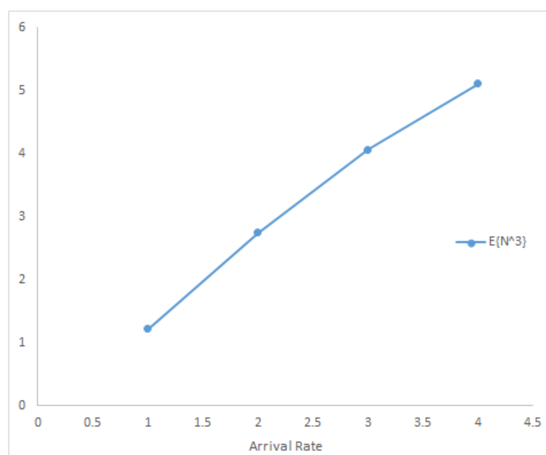


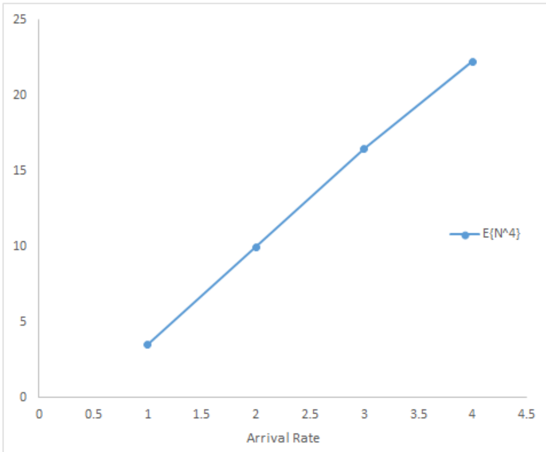


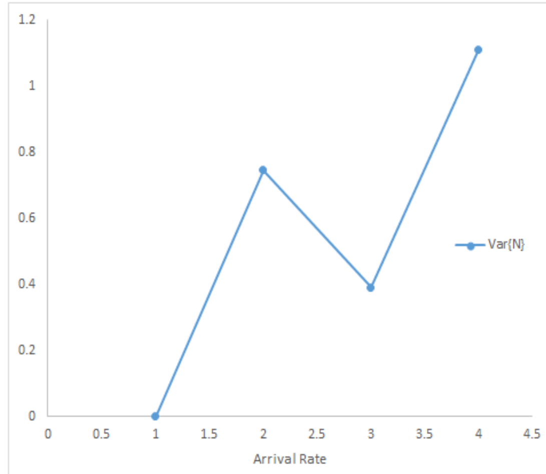


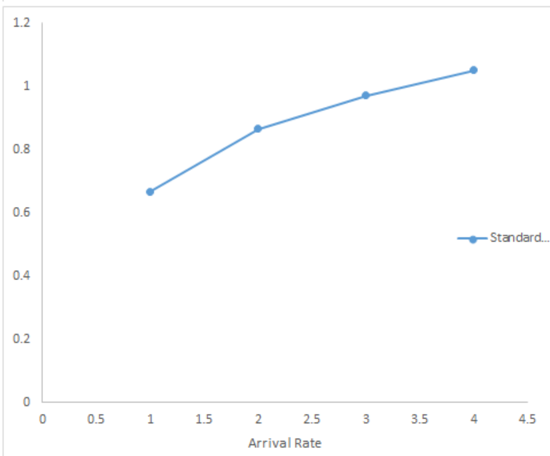












VIII. CONCLUSION

This research offers a feedback arrival process for service process, from the Tables 1 and 2 the values of $P_0, P_n, L_s, L_q, W_s W_q$ and the moment generating function are obtained for the FM/M/1 model, also from the Tables 3 and 4 the values of $P_0, P_n, L_s L_q, W_s, W_q$ and the moment generating function are obtained for the M/M/1 model.

Comparing FM/M/1 model with M/M/1 model by having the same arrival rates and service rates, all the values of performance measures in FM/M/1 model are decreasing than the values of M/M/1 model. In FM/M/1 model, the values are decreasing due to feedback arrivals with the probability values of $p = 1/2$ and $q = 1/2$.

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