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Ramanujan Summation for Classic Combinatorial Problem

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Abstract - *The concept of Ramanujan Summation has been dealt with several forms in recent decades. In this paper, I will define Ramanujan summation evaluated through a definite integral and using this, I had computed the Ramanujan summation for the divergent series whose terms represent the maximum number of regions formed by considering n points in the circumference of a circle which are joined by chords. This classic geometric problem along with Ramanujan summation method has produced an interesting and new result which is derived in detail in this paper.*

Keywords - *Ramanujan Summation, Regions in a Circle, Binomial Coefficients, Newton's Forward Interpolation Formula, Pascal's Identity*

I. INTRODUCTION

Ever since the concept of Ramanujan summation was introduced by one of the great Indian mathematician Srinivasa Ramanujan in connection with Riemann zeta function, several summation methods and generalizations have emerged. In this paper, after describing the maximum number of regions formed in circle by chords, I had determined the Ramanujan summation for the divergent series representing such terms. Geometric illustration was provided to understand the new result derived in this paper.

II. DEFINITION

Let $\sum_{n=1}^{\infty} a_n$ represent a divergent series of real numbers. The Ramanujan Summation (see **[1]**) abbreviated as *RS* of $\sum_{n=1}^{\infty} a_n$ is defined by $(RS)\left(\sum_{n=1}^{\infty} a_n\right) = \int_{n=-1}^{0}\left(\sum_{k=1}^{n} a_k\right)dn \quad (2.1)$

III. MAXIMUM NUMBER OF REGIONS IN A CIRCLE

Let us consider the following geometric problem

In a circle if we consider *n* distinct points in its circumference, and if those n points are joined by chords then what would be the maximum number of regions formed in the interior of the circle?

This classic combinatorial problem is well known in recreational mathematics and is often stated for not believing initial patterns of numbers. I will present the solution to this problem in a novel way often given in several recreational mathematics textbooks. I will discuss the simplest cases first and try to generalize it further.

If $n = 1$, then there will be only one point on the circumference of the circle and so there cannot be any chord that can be drawn with this single point on the circumference, since a chord should meet the circle at two distinct points and we have only one point on the circumference. In this case, the interior part of the circle is the only possible region that we can have. Figure 1 provided below portrays this situation.

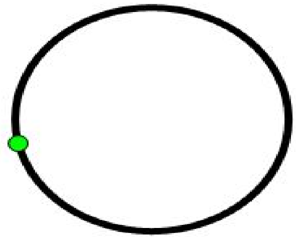

Figure 1: **One point on the circumference of a circle**

Thus, if $n = 1$, we have only one possible region inside the circle.

Now if we consider $n = 2$, then a single chord can be drawn in the circle and this chord divides the circle in two possible regions as shown in Figure 2.

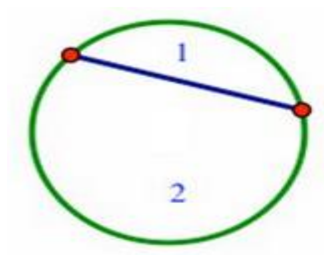

Figure 2: Two points on the circumference of a circle

Thus, if $n = 2$, we have two possible regions inside the circle.

If $n = 3$, then three chords can be drawn in the circle and they split the circle in to four possible regions as shown in Figure 3.

Similarly, for $n = 4$, six possible chords can be drawn in the circle which split the circle in to eight possible regions as shown in Figure 4.

For $n = 5$, fifteen possible chords can be drawn in the circle which split the circle in to sixteen possible regions as shown in Figure 5.

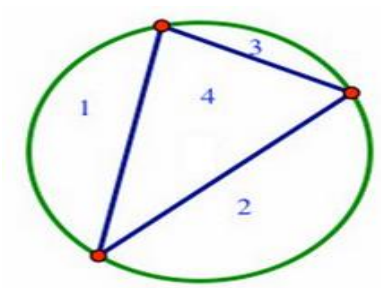

Figure 3

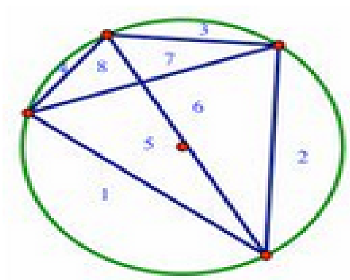

Figure 4

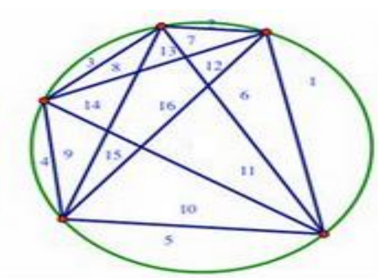

Figure 5

We notice that (from Figures 1 to 5) for $n = 1, 2, 3, 4, 5$ the possible regions formed by drawing chords in the circle are respectively 1, 2, 4, 8, 16. These numbers seems to form a geometric progression whose common ratio is 2, since each is twice the previous number. In this sense, we would expect the possible number of regions for $n = 6$ points on the circumference of a circle to be 32 since it is twice 16. But when we actually take six points on the circumference of a circle and mark the number of regions we would get the following.

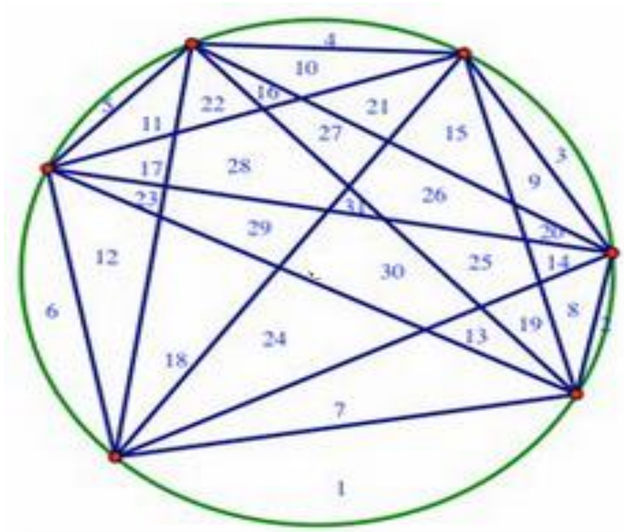

Figure 6: **Six points on the circumference of a circle**

We observe from Figure 6, that the number of regions formed by chords upon taking six points on the circumference of a circle is 31 but not 32 as expected. I now provide a compact expression for determining the number of regions that we can get by taking *n* points on the circumference of a circle.

A. Theorem 1

The number of regions in the interior of a circle formed by the chords joining *n* distinct points on its circumference is $\binom{n}{0}+\binom{n}{2}+\binom{n}{4}$ (3.1)

Proof: Considering the answers for first six values of *n* as shown in Figures 1 to 6, we find that the number of regions obtained for *n* = 1, 2, 3, 4, 5 and 6 are 1, 2, 4, 8, 16, 31 respectively. Let $f(n)$ be the number of regions formed with *n* points in the circumference of the circle. Computing successive forward difference values for the number of regions formed we get the following forward difference table shown in Figure 7.

Number of Points in the Circle *n*	Number of Regions Formed *f* (*n*)	First Difference Values	Second Difference Values	Third Difference Values	Fourth Difference Values
1	**1**				
2	2	**1**			
3	4	2	**1**		
4	8	4	2	**1**	
5	16	8	4	2	**1**
6	31	15	7	3	1

Figure 7: Forward Difference Table for Number of Regions

Now using Newton's Forward Interpolation formula, we get

$$f(n)=1+\binom{n-1}{1}(1)+\binom{n-1}{2}(1)+\binom{n-1}{3}(1)+\binom{n-1}{4}(1)=1+\binom{n-1}{1}+\binom{n-1}{2}+\binom{n-1}{3}+\binom{n-1}{4}$$

Using the Pascal's Identity $\binom{n-1}{r-1}+\binom{n-1}{r}=\binom{n}{r}$ we get

$$f(n)=1+\binom{n}{2}+\binom{n}{4}=\binom{n}{0}+\binom{n}{2}+\binom{n}{4}$$

This completes the proof.

B. Regions from Pascal's Triangle

Since $\binom{n}{r}$ is the binomial coefficient for all $0\le r\le n$ and the numbers in Pascal's triangles are binomial coefficients for particular values of *n* and *r*, from (3.1) we can determine the number of regions formed by chords in the interior of circle directly by adding the pink colored numbers displayed in the Pascal's triangle in Figure 8.

1

1 1

1 2 **1**

1 3 **3** 1

1 4 **6** 4 **1**

1 5 **10** 10 **5** 1

1 6 **15** 20 **15** 6 1

1 7 **21** 35 **35** 21 7 1

1 8 **28** 56 **70** 56 28 8 1

1 9 **36** 84 **126** 126 84 36 9 1

1 10 **45** 120 **210** 252 210 120 45 10 1

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Figure 8: Pascal's Triangle

IV. RAMANUJAN SUMMATION FOR NUMBER OF REGIONS FORMED BY CHORDS IN A CIRCLE

In this section, I will determine the Ramanujan summation value for the divergent series whose terms are the number of regions made by chords in the interior of a circle.

A. Theorem 2

$$(RS)(1+2+4+8+16+31+57+99+163+256+\cdots)=-\frac{211}{480} \quad (4.1)$$

Proof: Considering the number of regions formed in the interior of a circle by joining chords with *n* points in its circumference, according to (3.1) (as well as from Figure 8), we get 1, 2, 4, 8,16, 31, 57, 99, 163, 256, . . . Adding these terms, we notice that the series $1+2+4+8+16+31+57+99+163+256+\cdots$ is divergent, since the *n*th term $\binom{n}{0}+\binom{n}{2}+\binom{n}{4}$ doesn't converges to 0.

Let S_n denote the sum of first *n* terms of the series $1+2+4+8+16+31+57+99+163+256+\cdots$

$$S_n = \sum_{k=1}^{n}\left[\binom{k}{0}+\binom{k}{2}+\binom{k}{4}\right] = \sum_{k=1}^{n}\left[1+\frac{k^2}{2}-\frac{k}{2}+\frac{k^4}{24}-\frac{k^3}{4}+\frac{11k^2}{24}-\frac{k}{4}\right]$$

$$= n+\sum_{k=1}^{n}\left[-\frac{3k}{4}+\frac{23k^2}{24}-\frac{k^3}{4}+\frac{k^4}{24}\right]$$

$$= n-\frac{3}{4}\left(\frac{n^2}{2}+\frac{n}{2}\right)+\frac{23}{24}\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)-\frac{1}{4}\left(\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}\right)+\frac{1}{24}\left(\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}\right)$$

$$= \frac{n^5}{120}-\frac{n^4}{24}+\frac{5n^3}{24}+\frac{n^2}{24}+\frac{47n}{60} = \frac{n^5-5n^4+25n^3+5n^2+94n}{120}$$

Thus the Ramanujan summation of the series $1+2+4+8+16+31+57+99+163+256+\cdots$ using (2.1) we get

$$(RS)(1+2+4+8+16+31+57+99+\cdots) = \int_{n=-1}^{0} S_n\,dn = \int_{n=-1}^{0}\left(\frac{n^5}{120}-\frac{n^4}{24}+\frac{5n^3}{24}+\frac{n^2}{24}+\frac{47n}{60}\right)dn$$

$$= -\frac{1}{720}-\frac{1}{120}-\frac{5}{96}+\frac{1}{72}-\frac{47}{120} = -\frac{211}{480}$$

This proves (4.1) and completes the proof.

B. Geometric Meaning

In this section, I will demonstrate the reason behind the answer obtained in (4.2) of theorem 2 geometrically.

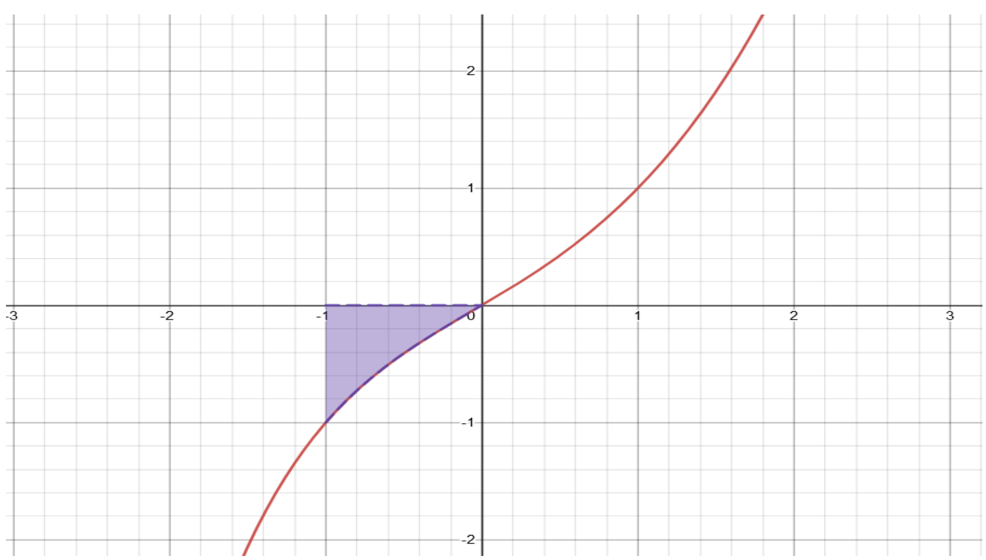

Figure 9: Area bounded by S_n between X – axis in $[-1,0]$

We notice from shaded portion of Figure 9, that the region representing the area of S_n between X – axis in the interval $[-1, 0]$ lies below X – axis. In fact, the signed area of graph of S_n between X – axis in the interval $[-1, 0]$ turns out to be the value $-\frac{211}{480}$ which we obtained in (4.1) of theorem 2.

V. CONCLUSION

Considering the classic combinatorial problem of finding the number of regions formed in the interior of a circle containing n points in its circumference and joined by chords, I had obtained a nice and compact expression in (3.1) of theorem 1. Though this value is known in literature abundantly, the expression as in (3.1) presented in this paper is new and equivalent to the known values. By adding the pink colored numbers in each through the Pascal's triangle provided in Figure 8, we can immediately determine the number of required regions. This is a new and novel way of doing so and appearing for the first time in this paper.

In theorem 2, I had determined the Ramanujan summation for the number of regions formed by chords in the interior of a circle by integrating the sum up to first n terms of the number of regions formed. In doing so, I had proved that the Ramanujan summation for the series representing number of regions formed in the interior of a circle as in theorem 2. This new result will add more understanding and behavior of Ramanujan summation methods applied to various scenarios.

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