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Uniqueness Results Related To Value Distribution of Entire And Meromorphic Functions Concerning Difference Polynomials

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Abstract. ***In this article, we prove some uniqueness results concerning with the difference polynomials of entire and meromorphic functions. The results obtained in this article improves and generalizes the earlier works of Jian Li and Kai Liu*** [4].

Keywords: Difference polynomial, Entire function, Uniqueness, Meromorphic Functions etc.,

I. Introduction

In this research article, we adopt the fundamental concepts of Nevanlinna Theory. We also use standard notations of Value Distribution Theory such as $m(r,f)$, $N(r,f)$, $T(r,f)$ and $\overline{N}(r,f)$ and so on (see [1], [2], [3]). The logarithmic density of the set E is defined by

$$\overline{\lim_{r\to\infty}}\frac{1}{log\ r}\int_{[1,r]\cap E}\frac{dt}{t}.$$

Denote by $S(r,f)$ a quantity of $o\{T(r,f)\}$ as $r\to\infty$ outside a possible exceptional set E of logarithmic density 0. Let k be a non-negative integer or infinity and $a\in\mathbb{C}\cup\{\infty\}$. Set $E(a,f)=\{z: f(z)-a=0\}$, where a zero with multiplicity k is counted k times. If the zeros are counted only once, then we denote the set by $\overline{E}(a,f)$. Let f and g be two non-constant meromorphic functions. If $E(a,f)=E(a,g)$, then we say that f and g share the value a CM (counting multiplicities). If $\overline{E}(a,f)=\overline{E}(a,g)$, then we say that f and g share the value a IM (ignoring multiplicity). We denote by $E_{k)}(a,f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. Also, we denote by $\overline{E}_{k)}(a,f)$ the set of distinct a-points of f with multiplicities not exceedding k. Denote the quantity,

$$L_c(f)=\prod_{j=1}^{s}f(z+c_j)^{v_j} \tag{1}$$

where c_j $(j=1,\ 2,\ldots,s)$ are constants and $\sigma=v_1+v_2+\ldots+v_s$ are integers.

In 2013, Zhang [5] proved the following result.

Theorem 1. *(see [5]) Let f and g be transcendental entire functions of zero order and let n, m, d be positive integers. If $n \geq m + 5d$ then $f^n(f^m - 1)\prod_{i=1}^{d} f(q_i z)$ and $g^n(g^m - 1)\prod_{i=1}^{d} g(q_i z)$ share 1 CM, then $f \equiv tg$, $t^{n+d} = t^m = 1$.*

In 2015, Zhao and Zhang [6] proved the following result.

Theorem 2. *(see [6]) Let f and g be transcendental zero-order entire functions, and let k be a positive integer. If $n \geq 2k + 6$ and $(f^n f(qz + c))^{(k)}$ and $(g^n g(qz + c))^{(k)}$ share 1 CM, then $f \equiv tg$, where $t^{n+1} = 1$.*

In 2020, Jian Li and Kai Liu [4] improved the conditions in Theorems 1 and 2 and obtained the following results.

Theorem 3. *(see [4]) Let f and g be two transcendental zero-order entire functions, and let m be a positive integer. If $n \geq m + d + 3$ and $f^n(f^m - 1)\prod_{i=1}^{d} f(q_i z + c_i)$ and $g^n(g^m - 1)\prod_{i=1}^{d} g(q_i z + c_i)$ where c_i and $q_i \neq 0$, $(i = 1,\ 2, \ldots d)$ are constants and d is a positive integer share 1 CM, then $f \equiv c_1 g$, $c_1^{n+d} = c_1^m = 1$.*

Theorem 4. *(see [4]) Let f and g be transcendental zero-order meromorphic functions, and let k be a positive integer. If $n \geq 6$ and $(f^n f(qz + c))^{(k)}$ and $(g^n g(qz + c))^{(k)}$ share 1 and ∞ CM, then $f \equiv c_2 g$, where $c_2^{n+1} = 1$.*

Question 1. What happens if we replace 1 CM by weakly weighted and relaxed weighted sharing in Theorem 3 ?

Question 2. What happens if we replace the function $(f^n f(qz + c))^{(k)}$ by $\left[f^n(f^m - 1)L_c(f)\right]^{(k)}$ in Theorem 4 ?

For the above two questions we have answered affirmatively and we obtained three results which extends the Theorem 3 and Theorem 4 respectively.

II. Main Results

Theorem 5. *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0, \infty)$ be a small function of both f and g with finitely many zeros. Let $L_c(f)$ be defined in equation (1) such that $n \geq 2k + m + \sigma + 5$ where k (≥ 0). If $\left[f^n(z)(f^m(z) - 1)L_c(f)\right]^{(k)}$ and $\left[g^n(z)(g^m(z) - 1)L_c(g)\right]^{(k)}$ share "$(\alpha(z), 2)$" then $f \equiv tg$ where $t^{m+\sigma} = 1$.*

Theorem 6. *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0, \infty)$ be a small function of both f and g with finitely many zeros. Let $L_c(f)$ be defined in equation (1) such that $n \geq 3k + 2m + 2\sigma + 6$ where k (≥ 0). If $\left[f^n(z)(f^m(z) - 1)L_c(f)\right]^{(k)}$ and $\left[g^n(z)(g^m(z) - 1)L_c(g)\right]^{(k)}$ share $(\alpha(z), 2)^*$ then $f \equiv tg$ where $t^{m+\sigma} = 1$.*

Theorem 7. *Let f and g be transcendental zero-order meromorphic functions, and let k be a positive integer. If $n \geq 3\sigma + 3$ and $(f^n(f^m - 1)L_c(f))^{(k)}$ and $(g^n(g^m - 1)L_c(g))^{(k)}$ share 1 and ∞ CM, then $f \equiv c_2 g$, where $c_2^{n+\sigma} = 1$.*

III. Auxiliary Definitions

In this section we state some definitions which are used to prove our main results.

Definition 1. (see [7]) Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f| = 1)$ the counting function of simple a-points of f. For a positive integer k we denote by $N(r, a; f| \leq k)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than k. By $\overline{N}(r, a; f| \leq k)$ we denote the corresponding reduced counting function. Analogously, we can define $N(r, a; f| \geq k)$ and $\overline{N}(r, a; f| \geq k)$.

Definition 2. (see [8]) Let k be positive integer or infinity. We denote by $N_k(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r, a; f) = \overline{N}(r, a : f) + \overline{N}(r, a; f| \geq 2) + \ldots + \overline{N}(r, a; f| \geq k).$$

Its is clear that $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 3. (see [9]) Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N_E(r, a; f, g)(\overline{N}_E(r, a; f, g))$ by the counting function (reduced counting function) of all common zeros of $f - a$ and $g - a$ with the same multiplicities and by $N_0(r, a; f, g)(\overline{N}_0(r, a; f, g))$ the counting function (reduced counting function) of all common zeros of $f - a$ and $g - a$ IM. If

$$\overline{N}(r, a; f) + \overline{N}(r, a; g) - 2\overline{N}_E(r, a; f, g) = S(r, f) + S(r, g)$$

then we say that f and g share the value a CM. If

$$\overline{N}(r, a; f) + \overline{N}(r, a; g) - 2\overline{N}_0(r, a; f, g) = S(r, f) + S(r, g)$$

then we say that f and g share the value a IM.

Definition 4. (see [10]) Let f and g share the value a IM and k be a positive integer or infinity. Then $\overline{N}^E_{k)}(r, a; f, g)$ denotes the reduced counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, and both of their multiplicities are not greater than k. $\overline{N}^0_{(k}(r, a; f, g)$ denotes the reduced counting function of those a- points of f which are a-points of g, and both of their multiplicities are not less than k.

In 2006, authors S. H Lin and W. C Lin [10] introduced the following definitions of weakly weighted sharing which is a scaling between sharing IM and CM.

Definition 5. (see [10]) Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer or infinity. If

$$\overline{N}(r,a;f| \leq k) - \overline{N}^{E}_{k)}(r,a;f,g) = S(r,f).$$

$$\overline{N}(r,a;g| \leq k) - \overline{N}^{E}_{k)}(r,a;f,g) = S(r,g).$$

$$\overline{N}(r,a;f| \geq k+1) - \overline{N}^{0}_{(k+1}(r,a;f,g) = S(r,f).$$

$$\overline{N}(r,a;g| \geq k+1) - \overline{N}^{0}_{(k+1}(r,a;f,g) = S(r,g).$$

or if $k = 0$ and

$$\overline{N}(r,a;f) - \overline{N}_0(r,a;f,g) = S(r,f).$$

$$\overline{N}(r,a;g) - \overline{N}_0(r,a;f,g) = S(r,g).$$

then we say that f and g share the value a weakly with weight k and we write f and g share "(a,k)".

In 2007, A. Banerjee and S. Mukherjee [11] introduced a new type of sharing known as relaxed weighted sharing, weaker than weakly weighted sharing and is defined as follows.

Definition 6. (see [11]) We denote by $\overline{N}(r,a;f| = p; g| = q)$ the reduced counting function of common a-points of f and g with multiplicities p and q respectively.

Definition 7. (see [11]) Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer or infinity. Suppose that f and g share the value a IM. If for $p \neq q$,

$$\sum_{p,q \leq k} \overline{N}(r,a;f| = p; g| = q) = S(r),$$

then we say that f and g share the value a with weight k in a relaxed manner and in that case we write f and g share $(a,k)^*$.

IV. Some Lemmas

The following sequence of Lemmas will be helpful to prove our main results.

We denote H by the following function.

$$H = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1}.$$

Lemma 1. *(see [15]) Let f be a meromorphic function and let c be a non-zero complex constant. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r,f).$$

Lemma 2. *(see [11]) Let F and G be two non-constant meromorphic functions that share "$(1,2)$" and $H \not\equiv 0$. Then*

$$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) - \sum_{p=3}^{\infty} \overline{N}\Big(r,0;\frac{G}{G'}\Big| \geq p\Big) + S(r,F) + S(r,G).$$

and the same inequality holds for $T(r,G)$ also.

Lemma 3. *(see [11]) Let F and G be two non-constant meromorphic functions that share $(1,2)^*$ and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r,F) \leq\, & N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + \overline{N}(r,0;F) + \overline{N}(r,\infty;F) \\ & - m(r,1;G) + S(r,F) + S(r,G). \end{aligned}$$

and the same inequality holds for $T(r,G)$ also.

Lemma 4. *(see [12]) Let H be defined as above. If $H \equiv 0$ and*

$$\overline{\lim_{r\to\infty}} \frac{\overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G)}{T(r)} < 1, r \in I.$$

where $T(r) = max\{T(r,F), T(r,G)\}$ and I is a set with linear measure, Then $F \equiv G$ or $FG \equiv 1$.

Lemma 5. *(see [13]) Let f be a non-constant meromorphic function, and let $p,\ k$ be two positive integers. Then*

$$N_p\Big(r,\frac{1}{f^{(k)}}\Big) \leq T(r,f^{(k)}) - T(r,f) + N_{p+k}\Big(r,\frac{1}{f}\Big) + S(r,f). \tag{2}$$

$$N_p\Big(r,\frac{1}{f^{(k)}}\Big) \leq k\overline{N}(r,f) + N_{p+k}\Big(r,\frac{1}{f}\Big) + S(r,f). \tag{3}$$

Clearly, $\overline{N}\Big(r,\frac{1}{f^{(k)}}\Big) = N_1\Big(r,\frac{1}{f^{(k)}}\Big)$.

Lemma 6. *Let f be an entire function of finite order and $F = f^n(f^m-1)L_c(f)$, then*

$$T(r,F) = (n+m+\sigma)T(r,f) + S(r,f).$$

Proof. Set $F = f^n(f^m-1)L_c(f)$. By standard Valiron-Mohonko's Theorem and using Lemma 1

$$\begin{aligned} (n+m+\sigma)T(r,f) &= T(r,f^{n+\sigma}(f^m-1)) + S(r,f) \\ &= m(r,f^{n+\sigma}(f^m-1)) + S(r,f) \\ &\leq m\left(r,\frac{f^{n+\sigma}(f^m-1)}{f^n(f^m-1)L_c(f)}\right) + m(r,F) + S(r,f) \\ &\leq m\Big(r,\frac{f^\sigma}{L_c(f)}\Big) + m(r,F) + S(r,F) \\ &\leq T(r,F) + S(r,F). \end{aligned} \tag{4}$$

On the other hand,

$$T(r,F) \le T(r, f^n(f^m-1)) + T(r, L_c(f)) + S(r,f) \tag{5}$$
$$\le (n+m+\sigma)T(r,f) + S(r,f).$$

Combining the inequalities (4) and (5) we get,

$$T(r,F) = (n+m+\sigma)T(r,f) + S(r,f).$$

□

Lemma 7. *(see [14]) Let f be a meromorphic function of finite order and c be a non-zero complex constant. Then*

$$T(r, f(z+c)) = T(r,f) + S(r,f).$$

Lemma 8. *Let f and g be entire functions, $n \ge 1$, $m \ge 1$, $k \ge 0$ be three integers and let us define $F = \left[f^n(f^m-1)L_c(f)\right]^{(k)}$ and $G = \left[g^n(g^m-1)L_c(g)\right]^{(k)}$, where $L_c(f)$ is defined in equation (1) If there exists non-zero constants c_1 and c_2 such that $\overline{N}(r,c_1;F) = \overline{N}(r,0;G)$ and $\overline{N}(r,c_2;G) = \overline{N}(r,0;F)$ then $n \le 2k+m+\sigma+2$.*

Proof. Put $F_1 = f^n(f^m-1)L_c(f)$ and $G_1 = g^n(g^m-1)L_c(g)$. By Nevanlinna's Second main theorem we have,

$$T(r,F) \le \overline{N}(r,0;F) + \overline{N}(r,c_1;F) + S(r,F) \tag{6}$$
$$\le \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F).$$

Using the inequalities (2), (3), (6) and Lemma 7 and Lemma 6 we obtain

$$\begin{aligned}
(n+m+\sigma)T(r,f) &\le T(r,F) - \overline{N}(r,0;F) + N_{k+1}(r,0;F_1) + S(r,f)\\
&\le \overline{N}(r,0;G) + N_{k+1}(r,0;F_1) + S(r,f)\\
&\le N_{k+1}(r,0;F_1) + N_{k+1}(r,0;G_1) + S(r,f) + S(r,g)\\
&\le (k+1)\{\overline{N}(r,0;f) + \overline{N}(r,0;g)\} + N(r,1;f^m) + N(r,1;g^m) + N(r,0;L_c(f))\\
&+ N(r,0;L_c(g)) + S(r,f) + S(r,g)\\
&\le (k+m+\sigma+1)T(r,f) + T(r,g) + S(r,f) + S(r,g).
\end{aligned}$$

Therefore,

$$(n+m+\sigma)T(r,f) \le (k+m+\sigma+1)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g). \tag{7}$$

On the similar lines, we can get,

$$(n+m+\sigma)T(r,g) \le (k+m+\sigma+1)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g). \tag{8}$$

Adding the inequalities (7) and (8) we get,

$$(n+m+\sigma-2k-2m-2\sigma-2)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g).$$

which gives $n \leq 2k+m+\sigma+2$. Hence the proof of the Lemma. □

Lemma 9. *Let f and g be a meromorphic function of finite order. If $n \geq m+\sigma+5$, where $n,\ m$ are positive integers and $\sigma = v_1+v_2+\ldots+v_s$ and*

$$f^n(f^m-1)L_c(f) = g^n(g^m-1)L_c(g) \tag{9}$$

then $f = tg$, where $t^{m+\sigma} = 1$.

Proof. Let $h = \frac{f}{g}$. If $h^{m+n}L_c(h) \neq 1$ then from (9) we have

$$g^n h^n(g^m h^m-1)L_c(h)L_c(g) = g^n(g^m-1)L_c(g).$$

$$h^n(g^m h^m-1)L_c(h) = g^m-1.$$

$$h^{m+n}L_c(h)g^m - h^nL_c(h) - g^m + 1 = 0.$$

$$g^m(h^{m+n}L_c(h)-1) = h^nL_c(h)-1.$$

or

$$g^m = \frac{h^nL_c(h)-1}{h^{m+n}L_c(h)-1}. \tag{10}$$

If 1 is a Picard exceptional value of $h^{m+n}L_c(h)$, applying Nevanlinna second main theorem with Lemma 7, we get

$$\begin{aligned} T(r,h^{n+m}L_c(h)) &\leq \overline{N}(r,h^{n+m}L_c(h)) + \overline{N}\Big(r, \frac{1}{N(r,h^{n+m}L_c(h))}\Big) \\ &\quad + \overline{N}\Big(r, \frac{1}{N(r,h^{n+m}L_c(h))-1}\Big) + S(r,h) \\ &\leq 2T(r,h) + 2\sigma T(r,h) + S(r,h) \\ &\leq 2(1+\sigma)T(r,h) + S(r,h). \end{aligned} \tag{11}$$

On the other hand, combining with the standard Valiron-Mohon'ko theorem with (11) and Lemma 7, we get

$$\begin{aligned} (n+m)T(r,h) &= T(r,h^{n+m}) \\ &\leq T(r,h^{m+n}L_c(h)) + T(r,L_c(h)) + S(r,h) \\ &\leq (4+\sigma)T(r,h) + S(r,h) \end{aligned} \tag{12}$$

or

$$(n+m+\sigma-4)T(r,h) \leq S(r,h) \tag{13}$$

which contradicts the hypothesis that $n \geq m + \sigma + 5$. Therefore 1 is not a picard exceptional value of $h^{n+m}L_c(h)$. Thus there exists z_0 such that $h(z_0)^{m+n}L_c(h) = 1$ then by (10), we have $h(z_0)^n L_c(h) = 1$. Hence $h(z_0)^m = 1$ and

$$\overline{N}\left(r, \frac{1}{h^{m+n}L_c(h) - 1}\right) \leq \overline{N}\left(r, \frac{1}{h^m - 1}\right) \leq mT(r,h) + O(1). \tag{14}$$

Denote

$$H(z) = h^{m+n}L_c(h) \tag{15}$$

We have $T(r,H) \leq (n+m+\sigma)T(r,h) + S(r,h)$. Applying second main theorem to H and using Lemma 7 and (14), we get

$$\begin{aligned} T(r,H) &\leq \overline{N}(r,H) + \overline{N}(r,0;H) + \overline{N}(r,1;H) + S(r,H) \\ &\leq \overline{N}(r,H) + \overline{N}(r,0;H) + mT(r,H) + S(r,h) \\ &\leq (m+\sigma+3)T(r,h) + S(r,h) \end{aligned}$$

Therefore, we have

$$T(r,H) \leq (m+\sigma+3)T(r,h) + S(r,h) \tag{16}$$

On the other hand using (15) and (16) we get

$$\begin{aligned} (n+m)T(r,h) &\leq T(r,h^{n+m}) + S(r,h) \\ &\leq T(r,H) + T(r,L_c(h)) + S(r,h) \\ &\leq (m+\sigma+4)T(r,h) + S(r,h) \end{aligned}$$

or

$$(n-\sigma-4)T(r,h) \leq S(r,h)$$

which contradicts with the hypothesis that $n \geq m + \sigma + 5$. Therefore, $h^{m+n}L_c(h) \equiv 1$ and $h^n L_c(h) \equiv 1$. Thus $h^m = 1$. Hence, we get $f = tg$ where $t^{m+\sigma} = 1$. □

V. Proof of Main Results

Proof of Theorem 5.

Proof. Keeping $F = \frac{F_1^{(k)}}{\alpha(z)}$ and $G = \frac{G_1^{(k)}}{\alpha(z)}$ where $F_1 = f^n(f^m - 1)L_c(f)$ and $G_1 = g^n(g^m - 1)L_c(g)$. Then F and G are transcendental meromorphic functions that share "(1, 2)" except the zeros and poles of $\alpha(z)$. From Lemma 6, we see that

$$T(r,F_1) = (n+m+\sigma)T(r,f) + S(r,f). \tag{17}$$

$$T(r,G_1) = (n+m+\sigma)T(r,g) + S(r,g). \tag{18}$$

If possible, assume that $H \not\equiv 0$. Using the inequality (2), (17) and Lemma 6 we get,

$$\begin{aligned} N_2(r,0;F) &\le N_2(r,0;F_1^{(k)}) \\ &\le T(r,F_1^{(k)}) - T(r,F_1) + N_{k+2}(r,0;F_1) + S(r,f) \\ &\le T(r,F) - (n+m+\sigma)T(r,f) + N_{k+2}(r,0;F_1) + S(r,f). \end{aligned}$$

which gives

$$(n+m+\sigma)T(r,f) \le T(r,F) - N_2(r,0;F) + N_{k+2}(r,0;F_1) + S(r,f). \tag{19}$$

Also, by (3) we obtain,

$$\begin{aligned} N_2(r,0;F) &\le N_2(r,0;F_1^{(k)}) + S(r,f) \\ &\le N_{k+2}(r,0;F_1) + S(r,f). \end{aligned} \tag{20}$$

Similarly,

$$N_2(r,0;G) \le N_{k+2}(r,0;G_1) + S(r,g). \tag{21}$$

By using the inequalities (20) and (21) and Lemma 7 and Lemma 2 we get,

$$\begin{aligned} (n+m+\sigma)T(r,f) &\le N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + N_{k+2}(r,0;F_1) + S(r,f) + S(r,g) \\ &\le N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + S(r,f) + S(r,g) \\ &\le (k+2)\{\overline{N}(r,0;F_1) + \overline{N}(r,0;G_1)\} + N(r,1;f^m) + N(r,1;g^m) + N(r,0;L_c(f)) \\ &+ N(r,0;L_c(g)) + S(r,f) + S(r,g) \\ &\le (k+m+\sigma+2)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \end{aligned}$$

Therefore,

$$(n+m+\sigma)T(r,f) \le (k+m+\sigma+2)\{T(r,f) + T(r,g)\} + +S(r,f) + S(r,g). \tag{22}$$

Similarly,

$$(n+m+\sigma)T(r,g) \le (k+m+\sigma+2)\{T(r,f) + T(r,g)\} + +S(r,f) + S(r,g). \tag{23}$$

Adding the inequalities (22) and (23), we get,

$$(n+m+\sigma-2k-2m-2\sigma-4)\{T(r,f) + T(r,g)\} \le S(r,f) + S(r,g).$$

which is obviously a contradiction as $n \ge 2k+m+\sigma+5$. Therefore $H \equiv 0$. Consider the case when $H \equiv 0$. i.e.,

$$H = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1} \equiv 0$$

Integrating the above equation, we get,

$$\frac{1}{F-1}=\frac{P}{G-1}+Q \tag{24}$$

where $P\neq 0$ and Q are integrating constants. From the equation (24) it is clear that F and G share 1 CM and hence they share "$(1,2)$". Therefore $n\geq 2k+m+\sigma+5$. Upon considering the some of the cases separately, we obtain as follows.

Case 1. Suppose $Q\neq 0$ and $P=Q$ then from equation (24), we get,

$$\frac{1}{F-1}=\frac{QG}{G-1}. \tag{25}$$

If $Q=-1$ then from equation (25), we get, $FG\equiv 1$.
i.e., $\left[f^n(f-1)(f^{m-1}+\ldots+1)\prod_{j=1}^{s}f(z+c_j)\right]^{(k)}\left[g^n(f-1)(g^{m-1}+\ldots+1)\prod_{j=1}^{s}g(z+c_j)\right]^{(k)}\equiv\alpha^2.$
It can be easily verified from above that, $N(r,0;f)=S(r,f)$ and $N(r,1;f)=S(r,f)$. Thus

$$\delta(0,f)+\delta(1,f)+\delta(\infty,f)=3.$$

which is not possible.
If $Q=-1$ from equation (25), we have, $\frac{1}{F}=\frac{QG}{(1+Q)G-1}$ and so $\overline{N}\Big(r,\frac{1}{1+Q};G\Big)=\overline{N}(r,0;F)$. Using the equations (2), (3) and (18) and Second main theorem of Nevanlinna, we get

$$\begin{aligned}T(r,G)&\leq\overline{N}(r,0;G)+\overline{N}\Big(r,\frac{1}{1+Q};G\Big)+\overline{N}(r,\infty;G)+S(r,G)\\&\leq\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+S(r,G)\\&\leq N_{k+1}(r,0;F_1)+T(r,G)+N_{k+1}(r,0;G_1)-(n+m+\sigma)T(r,g)+S(r,g).\end{aligned}$$

Therefore,

$$(n+m+\sigma)T(r,g)\leq(k+m+\sigma+1)\{T(r,f)+T(r,g)\}+S(r,g). \tag{26}$$

Likewise, we also get,

$$(n+m+\sigma)T(r,f)\leq(k+m+\sigma+1)\{T(r,f)+T(r,g)\}+S(r,f). \tag{27}$$

From the inequalities (26) and (27) we obtain a contradiction as $n\geq 2k+m+\sigma+5$.

Case 2. Let $Q\neq 0$ and $P\neq Q$, then from equation (24) we get,

$$F=\frac{(Q+1)G-(Q-P+1)}{QG+(P-Q)}$$

and so $\overline{N}\Big(r,\frac{Q-P+1}{Q+1};G\Big)=\overline{N}(r,0;F)$. By providing the same argument as in case 1, we obviously get a contradiction.

Case 3. If $Q = 0$ and $P \neq 0$ then from equation (24) we get $F = \frac{G+P-1}{P}$ and $G = PF - (P-1)$. If $P \neq 1$, it follows that $\overline{N}\left(r, \frac{P-1}{P}; F\right) = \overline{N}(r, 0; G)$ and $N(r, 1-P; G) = N(r, 0; F)$. Now by using Lemma 8, it can be shown that $n \leq 2k + m + \sigma + 2$, a contradiction. Thus $P = 1$ and then $F \equiv G$ i.e.,

$$\left[f^n(f^m - 1)L_c(f)\right]^{(k)} \equiv \left[g^n(g^m - 1)L_c(g)\right]^{(k)}.$$

Anti-Differentiate the above equation, we get,

$$\left[f^n(f^m - 1)L_c(f)\right]^{(k-1)} \equiv \left[g^n(g^m - 1)L_c(f)\right]^{(k-1)} + E_{k-1}.$$

where E_{k-1} is a constant. If $E_{k-1} \neq 0$, using Lemma 9 it follows that $n \leq 2k+m+\sigma$, which is a contradiction. Hence $E_{k-1} = 0$. Repeating the above process k times we get

$$\left[f^n(f^m - 1)L_c(f)\right] \equiv \left[g^n(g^m - 1)L_c(g)\right]$$

which by Lemma 9 gives $f = tg$, where t is a constant satisfying $t^{m+\sigma} = 1$. This completes the proof of Theorem 5.

□

Proof of Theorem 6.

Proof. Let F, G, F_1 and G_1 be defined as in the proof of Theorem 5. Then F and G are transcendental meromorphic functions that share $(1, 2)^*$ except the zeros and poles of $\alpha(z)$. Let $H \not\equiv 0$. Then by using Lemma 3, Lemma 5 and Lemma 8 we get,

$$\begin{aligned}(n + m + \sigma)T(r, f) &\leq N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G)\\ &\quad + \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + N_{k+2}(r, 0; F_1) + S(r, f) + S(r, g)\\ &\leq N_{k+2}(r, 0; F_1) + N_{k+2}(r, 0; G_1) + N_{k+1}(r, 0; F_1) + S(r, f) + S(r, g)\\ &\leq (2k + 2m + 2\sigma + 3)T(r, f) + (k + m + \sigma + 2)T(r, g) + S(r, f) + S(r, g).\end{aligned}$$

Therefore,

$$(n + m + \sigma)T(r, f) \leq (2k + 2m + 2\sigma + 3)T(r, f) + (k + m + \sigma + 2)T(r, g) + S(r, f) + S(r, g). \tag{28}$$

Likewise,

$$(n + m + \sigma)T(r, g) \leq (2k + 2m + 2\sigma + 3)T(r, g) + (k + m + \sigma + 2)T(r, f) + S(r, f) + S(r, g). \tag{29}$$

Adding the inequalities (28) and (29) we get,

$$(n + m + \sigma)\{T(r, f) + T(r, g)\} \leq (3k + 3m + 3\sigma + 5)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g).$$

which is a contradiction as $n \geq 3k + 2m + 2\sigma + 6$. Thus $H \equiv 0$. Proceeding similarly as done in Theorem 5 we get the proof of Theorem 6. $\square$

Proof of Theorem 7.

Proof. Let $F = f^n(f^m - 1)L_c(f)$ and $G = g^n(g^m - 1)L_c(g)$. From the condition in Theorem 7 we know that $F^{(k)}$ and $G^{(k)}$ share 1 and ∞ CM, so

$$\frac{F^{(k)} - 1}{G^{(k)} - 1} = C,$$

where C is non-zero constant, that is,

$$F^{(k)} = CG^{(k)} - C + 1 \tag{30}$$

Integrating both sides of (30) we get

$$F = CG + \frac{1 - C}{k!} z^k + p_1(z) \tag{31}$$

where $p_1(z)$ is a polynomial of degree atmost $k - 1$. Denote $\frac{1-C}{k!} z^k + p_1(z)$ by $p(z)$. If $p(z) \not\equiv 0$, then by second fundamental theorem of Nevanlinna and from Lemma 7 and 31 we get

$$\begin{aligned} T(r, F) &\leq \overline{N}(r, F) + \overline{N}(r, 0; F) + \overline{N}\Big(r, \frac{1}{F - p}\Big) + S(r, f) \\ &\leq \overline{N}(r, f) + \overline{N}(r, 0; f) + \overline{N}(r, 0; G) + \overline{N}_1(r) + \overline{N}_0(r) + S(r, f) \\ &\leq (1 + \sigma)T(r, f) + (1 + \sigma)T(r, g) + S(r, f) + S(r, g) \end{aligned} \tag{32}$$

where $\overline{N}_0(r)$ denotes the counting function ignoring multiplicities of the common zeros of $F(z)$ and $L_c(f)$ and $\overline{N}_1(r)$ denotes the counting function ignoring multiplicities of the common poles of $F(z)$ and $L_c(f)$. On the other hand

$$nm(r, f) = m(r, f^n) \leq m(r, F) + m\Big(r, \frac{1}{L_c(f)}\Big) \tag{33}$$

$$\begin{aligned} nN(r, f) &= N(r, f^n) + N\Big(r, \frac{F(z)}{L_c(f)}\Big) \\ &\leq N(r, F) + N\Big(r, \frac{1}{L_c(f)}\Big) - N_1(r) - N_0(r) \end{aligned} \tag{34}$$

From (33), (34) and Lemma 7 we have

$$(n - \sigma) \leq T(r, F) - \overline{N}_1(r) - N_0(r) + O(1). \tag{35}$$

Substituting (32) in to (35) we obtain that

$$(n - 2\sigma - 1)T(r, f) \leq (1 + \sigma)T(r, g) + S(r, f) + S(r, g). \tag{36}$$

Similarly

$$(n - 2\sigma - 1)T(r, g) \leq (1 + \sigma)T(r, f) + S(r, f) + S(r, g). \tag{37}$$

Combining the inequalities (36) and (37) we get,

$$(n-3\sigma-2)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g). \tag{38}$$

which is a contradiction to $n > 3\sigma+3$, an thus $p(z) \equiv 0$. Since the degree of $p_1(z)$ is atmost $k-1$, we have $C = 1$ and $p(z) \equiv 0$. From (31) we get

$$f^n(f^m-1)L_c(f) = g^n(g^m-1)L_c(g).$$

Assume that $h = \frac{f}{g}$. Then $L_c(h)h^n = 1$, that is $h^n = \frac{1}{L_c(h)}$ and from Lemma 7 we have

$$(n+\sigma)T(r,h) = T(r, L_c(h)) \leq T(r,h) + S(r,h),$$

which is a contradiction to $n > 3\sigma+3$, so $h(z)$ is non-zero constant, say c_2, so $f \equiv c_2 g$ and $c_2^{n+\sigma} = 1$. This completes the proof of Theorem 7. □

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